Modeling of arterial stenosis and its applications to blood diseases

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Abstract

Blood flow in a stenosed tube has been modeled in the present studies. Blood flow is assumed to be represented by a couple stress fluid. Flow parameters such as velocity, resistance to flow, and shear stress distribution have been computed for different suspension concentrations (haematocrit), and for the blood diseases; polycythemia, plasma cell dyscrasias, and for Hb SS (sickle cell). The results have been compared with the case of normal blood and for other theoretical models. The importance of size effects in blood flow studies has been highlighted.

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1. Introduction

One of the leading causes of the deaths in the world is due to heart diseases, and the most commonly heard names among the same are ischemia, atherosclerosis, and angina pectoris. Ischemia is the deficiency of the oxygen in a part of the body, usually temporary. It can be due to a constriction (stenosis) or obstruction in the blood vessel supplying that part. Atherosclerosis is a type of arteriosclerosis. It comes from the Greek words athero (meaning gruel or paste) and sclerosis (hardness). It involves deposits of fatty substances, cholesterol, cellular waste products, calcium and fibrin (a clothing material in the blood) in the inner lining of an artery. The build up
that results is called plaque. Plaque may partially or totally block the blood flow through an artery. Two things that can happen where plaque occurs are (i) bleeding (hemorrhage) into the plaque, and (ii) formation of a blood clot (thrombus) on the plaque’s surface. If either of these occurs and blocks the entire artery, a heart attack or stroke may result. Atherosclerosis affects large and medium sized arteries. The type of artery where the plaque develops varies with each person. A symptom complex of ischemic heart disease characterized by paroxysmal attacks of chest pain, usually substernal or pre-cordial is referred as angina pectoris. Usually high-grade stenosis with acute coronary changes results in sudden cardiac arrest (or death) which strikes 300,000–400,000 persons annually around the globe. Owing to its serious concern, a major research work is being done over all parts of the world for early detection and prevention from being affected by the cardiac attack and the development of the art therapies for the diagnosis of the heart diseases.

In cardiac-related problems, the affected arteries get hardened as a result of accumulation of fatty substances inside the lumen or because of formation of plaques as a result of hemorrhage. As the disease gets progressed, the arteries/arteriole gets constricted. The flow behavior in the stenosed artery is quite different than one in the normal arteries. Also, stresses and resistance to flow are much higher in stenosed arteries in comparison to the normal ones. Having knowledge on flow parameters, such as velocity, flow rate, pressure drop will aid bio-medical engineers in developing bio-medical instruments for treatment (surgical) modalities. Hence fluid mechanics aspect of arterial stenosis have received a considerable attention in the recent past. In view of its importance, a fluid mechanics aspect of arterial stenosis has been undertaken in the present studies.

While modeling blood flow in a stenosed tube, it was initially assumed that, the flow obeys Newtonian hypothesis and the flow variables have been computed by using basic Navier–Stoke’s equations [1–5]. Later, the model has been extended by assuming that, it obeys non-Newtonian hypothesis and developed the model for either Casson fluid or for Power-law fluids [6,7], and showed that, under low shear rates, the model could be best described by this representation. During constriction (stenosis), the lumen of an artery gets considerably reduced thereby, size effects (particle size (mainly red blood cells) to tube diameter ratio) influences the flow characteristics significantly [8,9]. Eringen [10], Cowin [11], and Stokes [12], have proposed micro-continuum theories for accounting size effects in the fluid flows. Later, Ariman et al. [13], Valanis and Sun [14], Chaturani and Pralhad [15] have applied micro-continuum theories for studying the blood flow models. The advantage of application of micro-continuum theories to blood flow model is that, anomalies of blood flows such as Fahraeus and Lindquist effect (FLE) that is dependence of blood viscosity on tube radius, inverse FLE (increase of blood viscosity with decrease in tube radius (1–5 μm)), and existence of peripheral plasma layer near the tube wall [8,9], can be accounted. In the present model we have accounted micro-continuum fluids proposed by Stokes [12]. These [12] fluids have also been referred as couple stress fluids wherein the parameters \( \bar{\alpha} \), \( \bar{\eta} \) accounts for the size effects in the flow field. Higher value of \( \bar{\alpha} \) implies that the flow is tending towards Newtonian whereas, the lower values of \( \bar{\alpha} \) implies that the flow has dominance of particle size effects. \( \bar{\eta} \) is the parameter which accounts for the effect of local viscosity due to particles in addition to bulk viscosity of the fluid (\( \mu \)). The application of micro-continuum fluids for the stenosed model has been successfully studied by Pralhad and Schultz [16], Sinha and Singh [17] and Srivatsava [18]. The model discussed so far on micro-continuum fluids on stenosis accounts for only viscous effects. Whereas in the flow both the inertia and viscous effects plays an important
role. In view of the same, an effort has been made in the present model, to account for both inertia and viscous terms by assuming blood as a couple stress fluid.

2. Analysis

It is assumed that blood flow is represented by a homogeneous and incompressible couple stress fluid of constant viscosity $\mu$, and density $\rho$. The constitutive equations and equation of motion for couple stress fluid [12] are

$$T_{ij} = \rho \frac{dV_i}{dt}$$  \hspace{1cm} (1)

$$e_{ijk} T_{jk}^A + M_{ij} = 0$$  \hspace{1cm} (2)

$$I_{i,j} = -p \delta_{ij} + 2\mu d_{ij}$$  \hspace{1cm} (3)

$$\mu_{ij} = 4\eta \omega_{j,i} + 4\eta' \omega_{i,j},$$  \hspace{1cm} (4)

where $V_i$ the velocity vector, $I_{ij}$ and $T_{ij}^A$ are the symmetric and antisymmetric part of the stress tensor $T_{ij}$ respectively. $M_{ij}$ is the couple stress tensor, $\mu_{ij}$ is the deviatoric part of $M_{ij}$, $\omega_{ij}$ is the vorticity vector, $d_{ij}$ is the symmetric part of the velocity gradient, $\eta$ and $\eta'$ are the constants associated with the couple stress, $p$ is the pressure and other terms have their normal meaning of the tensor analysis. The appropriate equation from (1)–(4) for the two-dimensional flow subject to the additional conditions

$$Re_0 \frac{2\delta}{L_0} \ll 1, \hspace{0.5cm} \frac{2R_0}{L_0} \approx o(1)$$  \hspace{1cm} (5)

together with the equation of continuity can be written as

$$\rho \left[ v \frac{\partial u}{\partial r} + u \frac{\partial v}{\partial z} \right] + \frac{\partial p}{\partial z} = \nabla^2 (\mu v - \eta \nabla^2 v)$$  \hspace{1cm} (6)

$$\rho \left[ v \frac{\partial v}{\partial r} + u \frac{\partial v}{\partial z} \right] + \frac{\partial p}{\partial r} = \nabla^2 (\mu v - \eta \nabla^2 v)$$  \hspace{1cm} (7)

$$\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} + \frac{v}{r} = 0,$$  \hspace{1cm} (8)

where

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right).$$  \hspace{1cm} (9)

$\Re_0$ is the Reynolds number, $R_0$ is the tube radius of the unobstructed part (see Fig. 1), $\delta_0$ is the maximum height of the stenosis, $L_0$ is the total length of the stenosis, $v$ and $u$ are the radial and axial velocity components in the $r$ and $z$ directions respectively. The geometry of the stenosis which is assumed to be symmetric is given by
\[
R(z) = \begin{cases} 
1 - (\bar{\delta}_s/2)[1 + \cos(2\pi/L_0)\{z - d - (L_0/2)\}], & d \leq z \leq d + L_0, \\
1, & \text{otherwise}
\end{cases}
\] (10)

where
\[
\bar{R}(z) = R(z)/R_0, \quad \bar{\delta}_s = \delta_s/R_0.
\] (11)

Following the order of magnitude analysis of Forrester and Young [19], Eqs. (6) and (7) can be approximated as
\[
u_{oo}u + v_{oo}r = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{1}{\rho} \nabla^2 \psi
\] (12)
\[
\frac{\partial p}{\partial r} = 0,
\] (13)

where
\[
\psi = \mu u - \eta \nabla^2 u.
\] (14)

Eq. (12) can now be integrated across the tube to obtain
\[
\frac{\partial}{\partial z} \int_0^{R(z)} ru^2 \, dr = -\frac{1}{\rho} \frac{\partial p}{\partial z} \frac{R^2}{2} + \frac{R}{\rho} \left( \frac{\partial \psi}{\partial r} \right)
\] (15)

where the boundary condition \( u = v = 0 \) at \( r = R(z) \) has been applied. The integrated form of the continuity equation is
\[
Q = \pi R^2 \bar{U} = \int_0^{R(z)} 2\pi ru \, dr,
\] (16)

where \( \bar{U} \) is the mean velocity at any given cross section with radius \( R(z) \), and \( Q \) is the volumetric flow rate. We now assume that the radial dependence of the axial velocity can be expressed as a fourth order
\[
\frac{u}{\bar{U}} = A\eta_1 + B\eta_2 + C\eta_3 + D\eta_4 + E
\] (17)

polynomial of the form, where \( \eta_1 = 1 - (r/R) \), \( \bar{U} \) is the center line velocity, \( A \) through \( E \) are, as yet, undetermined coefficients. These coefficients are evaluated from the following boundary conditions:
\( (i) \) at \( r = R, \ u = 0 \), \( (ii) \) at \( r = 0, \ \partial u / \partial r = 0 \), \( (iii) \) at \( r = 0, \ u = U \)

\( (iv) \) at \( r = R, \ \partial p / \partial z = \nabla^2 \psi, \ (v) \) at \( r = 0, \ \partial^2 u / \partial r^2 = -(2U/R^2)C^* \),

where

\[
C^* = 1 - [Ha^2/4I_0(a)]
\]

\[
H = [2(1 - \eta)/a^2]/[1 - \{(1 + \eta)I_1(a)/aI_0(a)\}]
\]

\[
a = \bar{R}(z) \cdot \bar{x}, \quad \bar{x} = R_0/l, \quad l = \sqrt{\eta / \mu}, \quad \bar{\eta} = \eta' / \eta.
\]

The last initial condition is obtained by the assumption that, at \( r = 0 \), the velocity profile is not parabolic but of couple stress fluid one. That is

\[
u = U[1 - (r/R)^2 - H\{(1 - (I_0(r/a)/I_0(a))\}].
\]

So that the second derivative of \( \nu \) with respect to \( r \), at \( r = 0 \) can be approximated by the condition (v). Here \( \bar{x} \) and \( \bar{\eta} \) are couple stress parameters. Eq. (17) reduces to

\[
\frac{\nu}{U} = A\eta_1 + (6 - C^* - 3A)\eta_1^2 + (-8 + 2C^* + 3A)\eta_1^3 + (3 - C^* - A)\eta_1^4,
\]

where

\[
A = S_1/S_2 - (a^2\lambda/S_2)
\]

\[
S_1 = (12 - 2C^*)a^2 - 156 + 46C^*
\]

\[
S_2 = 7a^2 - 55
\]

\[
\lambda = (R^2/\mu U) \frac{dp}{dz}.
\]

Eq. (21) can be substituted into Eq. (16) to give

\[
U = \frac{30S_2}{\pi R^2 S_3} \left[ O + (\pi R^4a^2/15\mu S_2) \frac{dp}{dz} \right],
\]

where

\[
S_3 = 108a^2 - 11a^2C^* + 147C^* - 972.
\]

With the substitution of Eq. (20) into the left-hand side of Eq. (15), the integral reduces to

\[
\frac{d}{dz}(4R^2U^2T_1) = -\frac{1}{\rho} \frac{dp}{dz} \frac{R^2}{2} + \frac{R}{\rho} \left( \frac{\partial \psi}{\partial r} \right)_r,
\]

where

\[
T_1 = \frac{1}{6} + H^2 - \frac{H}{2} + \frac{4H}{a^2} - \frac{2H^2}{a} D_1 - \frac{H^2}{2} D_1 - \frac{8H}{a^2} D_1
\]

and

\[
D_1 = \frac{I_1(a)}{I_0(a)}.
\]
Expressions for $\bar{U}$ and $(\frac{\partial \bar{U}}{\partial r})_R$ from Eqs. (16) and (21) are now substituted into Eq. (28) to give an equation, which when combined with Eq. (26) yields the pressure gradient

$$\frac{dp}{dz} = \left( \frac{8\rho Q^2 B_1 B_2}{\pi^2 R^3} \right) \frac{dR}{dz} - \frac{60\mu Q B_3}{\pi R^4},$$

(31)

where $B_1 = S_2 S_3 / S_5$, $B_2 = J_0 - (1/3)$, $B_3 = S_2 S_4 / a^2 S_5$.

$J_0 = H^2 D_2 + H B_4 D_3 + (B_4 / 2a^3) D_4 + (H / a^3) D_5$

$D_2 = -4 + a D_1 + 4 D_2^1 + ((8/a) - a) D_1$

$D_3 = 2 - (4/a) D_1 - D_2^2$

$D_4 = 8a - a^3 - 16 D_1$

$D_5 = a^3 - 24a + 48 D_1 + 8a D_2^2$

$S_4 = -12a^4 + 2C^* a^4 + 12a^2 + 6C^* a^2 + 288C^* - 288$

$S_5 = -588a^4 + 63a^4 C^* - 810a^2 C^* + 5880a^2 + 2475C^* - 9900$

$B_4 = -2(1 - \bar{\eta})a[a^2 - (1 + \bar{\eta})a D_1]^{-2} D_6$

$D_6 = [2a - (1 + \bar{\eta})a(1 - D_1^2) - 2D_1]$.  

Eqs. (26) and (31) are now being substituted into Eq. (21) to give the velocity $u$ as a function of $r$ and $z$.

$$\frac{u}{U_0} = [T_2 \eta_1 + T_3 \eta_1^2 + T_4 \eta_1^3 + T_5 \eta_1^4]\bar{R}(z)]^{-2} + [T_6 \eta_1 + T_7 \eta_1^2 + T_8 \eta_1^3 + T_9 \eta_1^4][\bar{R}(z)]^{-3} R e_0 B_2 \frac{dR}{dz},$$

(33)

where

$T_2 = E_1 E_2 - E_1 E_3 + E_4$, $T_3 = E_2 E_5 - E_3 E_5 - 3E_4$, $T_4 = E_6 E_2 - E_6 E_3 + 3E_4$,

$T_5 = E_7 E_2 - E_7 E_3 - E_4$, $T_6 = E_1 E_8 - E_9$, $T_7 = E_5 E_8 + 3E_9$, $T_8 = E_6 E_8 - 3E_9$,

$T_9 = E_7 E_8 + E_9$, $E_1 = S_1 / S_2$, $E_2 = 30S_2 / S_3$, $E_3 = 120S_2 S_4 / S_3 S_5$,

$E_4 = (a^2 / S_2)(60S_2 S_4 / a^2 S_5)$, $E_5 = (6 - C^* - 3(S_1 / S_2))$, $E_6 = (2C^* - 8 + 3(S_1 / S_2))$,

$E_7 = (3 - C^* - (S_1 / S_2))$, $E_8 = 8a^2 S_2 / S_5$, $E_9 = 4a^2 S_3 / S_5$.

(34)

3. Shear stress

Shear stress has been computed by two different methods.

3.1. Method-I

By using the relation

$$\tau_s = -\frac{R(z)}{2} \frac{dp}{dz},$$

(35)
Substituting for $\frac{d\rho}{dz}$ from Eq. (31), and evaluating the shear stress at the maximum height of the stenosis (i.e., at $z = d + (L_0/2)$), we get

$$\tau_{\text{max}} = \frac{30\mu QB_3}{\pi R^3}.$$  \hspace{1cm} (36)

Let $\tau_0$ represents the shear stress for the Newtonian case (i.e., when $a$ approaches infinity), and in the absence of stenosis ($R(z) = R_0$) Eq. (36) reduces to

$$\tau_0 = \frac{4\mu_1 Q}{\pi R_0^3},$$  \hspace{1cm} (37)

where $\mu_1$ is the viscosity of plasma (Newtonian fluid). Using $\tau_0$ for non-dimensionalizing, we get

$$\tau_R = \frac{15\bar{\mu}B_3}{2} \left[ \frac{R(z)}{R(z)} \right]^{-3},$$  \hspace{1cm} (38)

where

$$\tau_R = \tau_{\text{max}}/\tau_0, \quad \bar{\mu} = \mu/\mu_1.$$  \hspace{1cm} (39)

3.2. Method-II

By using relation (35), and substituting for $\frac{d\rho}{dz}$ from Eq. (31), we get

$$\tau_w = \frac{4S_3S_5}{S_5} \left( \frac{R_0}{R(z)} \right)^4 \left[ -\frac{1}{3} + J_0 \right] \left( \frac{dR}{dz} \right) - \frac{60S_3S_4}{a^2S_5} \left( \frac{R_0}{R(z)} \right)^3 \frac{1}{Re_0}$$  \hspace{1cm} (40)

for separation re-attachment point $\frac{\tau_w}{\rho U_0^2} = 0$, which implies

$$Re_0 \left( \frac{dR}{dz} \right) \left( \frac{R_0}{R} \right) = \frac{15S_4}{a^2S_3 \left[ \left(-\frac{1}{3}\right) + J_0 \right]},$$  \hspace{1cm} (40a)

where

$$R(z) = \begin{cases} R_0 - \left( \frac{a}{2} \right)(1 + \cos(\pi z/z_0)), & 2z_0 = L_0 \\ R_0, & \text{otherwise} \end{cases}.$$  \hspace{1cm} (41)

4. Resistance to flow

Resistance to flow, which is of physiological importance, has also been computed by two methods.

4.1. Method-I

The resistance to flow, $\lambda$, is defined by

$$\lambda = \frac{P_0 - P}{Q},$$  \hspace{1cm} (42)
where \( P_0 \) and \( P \) is the pressure at the entry and exit level respectively. Integrating Eq. (31) using the condition that \( p = p_0 \) at \( z = 0 \) and \( p = P \) at \( z = L \), we have

\[
P_0 - P = \frac{8\rho Q^2}{\pi} \int_0^L \left( \frac{B_1 B_2}{R^5} \right) \left( \frac{dR}{dz} \right) dz - \left( \frac{60\mu Q}{\pi} \right) \int_0^L \left( \frac{B_3}{R^4} \right) dz.
\]

Using Eq. (10) for \( \bar{R}(z) \), for the substitution of \( \frac{d\bar{R}}{dz} \) in Eq. (43), \( \lambda \) is given by

\[
\lambda = \frac{60\mu L}{\pi R_0^3} (1 - \bar{L}_0) B_3^4 + \int_{d+L_0}^{d+L_0} \left( \frac{60\mu B_3}{\pi R^4} - \frac{8\rho Q B_1 B_2}{\pi^2 R^5} \frac{dR}{dz} \right) dz,
\]

where \( B_3^4 = B_3 \) at \( R(z) = R_0 \), and \( \bar{L}_0 = (L_0/L) \).

Let \( \lambda_0 \) be the resistance to flow for the Newtonian case in absence of stenosis. Eq. (44) reduces to

\[
\lambda_0 = \frac{8\mu L}{\pi R_0^4}.
\]

Using \( \lambda_0 \) for non-dimensionnalization with \( \lambda \), Eq. (44) simplifies to

\[
\lambda_R = \frac{15\mu B_3^4}{2} (1 - \bar{L}_0) + \left( \frac{15L_0}{4\pi} \right) \int_{-\pi}^{\pi} \frac{B_3}{\bar{R}(z)^4} d\theta - \frac{Re_0 \mu \bar{R}_0}{4} \int_{-\pi}^{\pi} \frac{B_1 B_2 \sin \theta}{[R(z)]^5} d\theta,
\]

where

\[
\lambda_R = \lambda_0, \quad Re_0 = \frac{2R_0 \rho U_0}{\mu}, \quad \theta = \frac{2\pi}{L_0} \left[ z - d - \left( \frac{L_0}{2} \right) \right], \quad \bar{R}_0 = \frac{R_0}{L}.
\]

4.2. Method-II

Using \( R(z) \) in the form of Eq. (41) and simplifying the integration term, we get

\[
\frac{P_0 - P}{\rho U_0^2} = 2B_1 B_2 [(\bar{R}(z))^{-4} - 1] + \frac{40B_3 Z_0}{A_1 \pi Re_0} A^*\lambda_R, \quad (48)
\]

where

\[
A^* = \frac{5A_2 \delta \sin \theta}{2(\bar{R}(z))^3} + \frac{5\bar{A}_2 \delta \sin \theta}{4A_1 (\bar{R}(z))^2} + \frac{[2.5\bar{A}_2 + \{3\bar{A}_2 + 0.5(\bar{a})^3\} \sin \theta]}{2A_1^2 \bar{R}(z)} + \frac{[6A_2^2 + 2.25(\bar{a})^2] A_2}{\sqrt{A_1^2}} \left[ \tan^{-1}\left(\tan(\theta/2)/\sqrt{A_1}\right) + \frac{\pi}{2} \right]
\]

\[
A_1 = 1 - \bar{a}, \quad A_2 = 1 - (\bar{a}/2), \quad Z_0 = \frac{Z_0}{R_0}, \quad U_0 = \frac{Q}{\pi R_0^2}.
\]

Here ‘a’ is assumed to be of the form \( \frac{\bar{a}}{a} \) in order to compare with the other theoretical models.
5. Comparison to other theoretical models

The present model has been compared with the other theoretical models [17,19]. The present model can be reduced to the model of Forrester and Young [19] by allowing the parameter \( a (= \bar{R}(z) \cdot \bar{x}) \) tending to infinity (which implies the absence of size effects) in the equation for velocity (Eq. (33)), shear stress (Eq. (40)), and in the equation for the resistance to flow (Eq. (48)). The reduced form is given below.

\[
\frac{u}{U_0} = 2(\eta_1 - \eta_1^2)(1/\bar{R}(z))^2 + \left[ -\frac{308}{1575} \eta_1 + \frac{1204}{1575} \eta_1^3 - \frac{4}{5} \eta_1^4 + \frac{4}{15} \eta_1^4 \right] \cdot \frac{dR}{dz} \left( 1/\bar{R}(z) \right)^3. \tag{50}
\]

\[
\frac{\tau_w}{\rho U_0^2} = \left( \frac{616}{1575} \right) (1/\bar{R}(z))^4 \cdot \frac{dR}{dz} - \left( \frac{8}{Re_0} \right) (1/\bar{R}(z))^3. \tag{51}
\]

\[
\frac{P_0 - P}{\rho U_0^2} = \left( \frac{194}{225} \right) \left[ (1/\bar{R}(z))^4 - 1 \right] + \left( \frac{16 \bar{z}_0}{3 \pi Re_0 A_1} \right) A^*. \tag{52}
\]

The results of the present investigation have also been compared with the analysis of Sinha and Singh [17]. The results of their investigations have been modified as per the reasons stated in Ref. [20] and the modified results are given below.

\[
\tau_R^* = \frac{\mu}{(R(z))^3 B_5}. \tag{53}
\]

\[
\lambda_R^* = \frac{\mu (1 - 3 \bar{L}_0)}{2B_5^*} + \frac{\bar{L}_0 \mu}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta}{(R(z))^4 B_5^*}. \tag{54}
\]

\[
B_5 = [1 - (2H/a)(a - 2D_1)], \quad B_5^* = B_5|_{R(z) = R_0}, \quad \tau_R^* = \tau/\tau_0, \quad \lambda_R^* = \lambda/\lambda_0. \tag{55}
\]

While comparing the results of present model with that of Forrester and Young [19], the equation of the tube radius of the present model (Eq. (10)) and that of Forrester and Young [19] (Eq. (41)) have been used independently and while plotting the graph (Figs. 6–8) these changes have been accounted. The \( \theta \) variable which is mentioned in Eq. (47) is a part of the substitution for the computation of the integral in Eq. (46) by the Simpson’s rule. It should not be treated as another form of \( \bar{R}(z) \) rather it is the same form as referred in Eq. (10).

### Table 1

Experimental values of rotational viscosity, \( \mu_R \), for different haematocrit [13]

<table>
<thead>
<tr>
<th>Haematocrit (%)</th>
<th>( \mu_R ) (cP)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.78</td>
</tr>
<tr>
<td>20</td>
<td>0.92</td>
</tr>
<tr>
<td>40</td>
<td>0.98</td>
</tr>
</tbody>
</table>
Table 2
Experimental values of $\mu$ (from [21]), and theoretical values of $\bar{\alpha}$, $\bar{\eta}$ and $\eta$ (from [14]) for different diseases

<table>
<thead>
<tr>
<th>Diseases</th>
<th>Viscosity ($\mu$, cP)</th>
<th>$\bar{\alpha}$</th>
<th>$\bar{\eta}$</th>
<th>$\eta$ (cP cm²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal blood</td>
<td>3.81</td>
<td>2.63</td>
<td>−0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>Polycythemia</td>
<td>6.75</td>
<td>3.50</td>
<td>−0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>Plasma cell dyscrasias</td>
<td>4.99</td>
<td>3.01</td>
<td>−0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>Hb SS (sickle cell)</td>
<td>3.29</td>
<td>2.44</td>
<td>−0.5</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Fig. 2. Variation of shear stress ($\tau_s$) with stenosis height ($\delta_s$) for different $\bar{\alpha}$, $\bar{\eta}$ and concentration.
6. Computation details

The flow variables have been computed for the different suspension concentration of haematocrits (10%, 20%, and for 40%) and for the diseases such as, polycythemia, sickle cell (Hb SS), and for plasma cell dyscrasias. The results have also been compared to the case of normal blood. The value of $\mu$, when computing for different suspension concentrations, is chosen to be of the form $\mu = \mu_1 + \mu_R$ where $\mu_1$ is the viscosity of plasma (Newtonian fluid) and $\mu_R$ is the rotational viscosity due to suspension of blood cells. The value of $\mu_1$ is assumed to 1.2 cP (centi Poise) and $\mu_R$ for different suspension concentrations are taken from Ariman et al. [13] and are shown in Table 1. For the computation of axial velocity, resistance to flow and the shear stress distribution for the different diseases, the values of $\mu$ have been taken from Chien [21]. The value of $\tilde{\eta}$ has been fixed to $-0.5$ for the computation of flow parameters for all the diseases and for the normal blood (Table 2). Resistance to flow for different suspension concentrations (Eqs. (46) and (54)) have been computed by using Simpson’s rule. The values of $\delta_0$ and $L_0$ are varied from 0 to 0.5, and 0.5 to 1.0 respectively. Size effects parameters $\tilde{a}$ and $\tilde{g}$ are varied from 2 to 20 and $-1.0$ to $1.0$ respectively. Lower the value of $\tilde{a}$ more is size effects pronounced in the model, whereas the higher values of $\tilde{a}$ accounts that the model is tending towards the Newtonian results. $\tilde{g}$ values are restricted between $-1$ and $1$ only (Ref. [14], Eq. (41)). Chaturani and Upadhya [22], and Chaturani and Pralhad [23] have discussed in detail on the aspect of choice of parametric values of $\tilde{a}$, $\tilde{g}$ for the blood flow.

Fig. 3. Variation of shear ($\tau_s$) stress with $\tilde{a}$ for different $\eta$ concentrations ($\delta_0 = 0.20$).
modeling. The values of $R_0$ and $z_0$, have been taken to be 0.02 and 4 respectively. The values of $\bar{z} (= z/z_0)$ have been varied from $-1$ to $+1$. It is worthwhile to mention at this stage that, though $\bar{z}$ and $\bar{\eta}$ are related through $\mu$, however their values are chosen separately due to the non-availability of the experimental values on $\eta$ and $\eta'$. The reasons for computing flow variables for the different suspension concentrations and for different diseases are due to the fact that the model can be used for general-purpose reference. Once the volume concentration of blood cells (haematocrit) for a particular disease is known, the flow variables can be computed by making use of Tables 1 and 2 and the method of interpolation and extrapolation.

In comparison of the flow variables, the tube geometry is assumed to be of the size 0.95 cm. The reason for accounting higher tube size is for comparison with the existing Newtonian models [19]. In doing so, it is assumed that the flow is steady and laminar. The assumption is quite acceptable for the reason that, the Reynolds numbers chosen for the computation purpose is ranging from 10

Fig. 4. Variation of resistance to flow ($\lambda_R$) with stenosis height ($\delta_s$) for different $\bar{\eta}$, $Z_0$ and concentrations ($\bar{z} = 3.0$, $Re_0 = 10$).
to 400 only. Once the results are confirmed (after comparing with the other theoretical models) that the present model yields satisfactory results, the model then can be computed for the lesser tube diameter also.

7. Results

The results on shear stress (Eqs. (38) and (53)) and on resistance to flow (Eqs. (46) and (54)) for different suspension have been shown in Figs. 2–5. It is observed that the values of shear stress increases with the increase of stenosis height and decrease with the increase of couple stress parameters $\bar{a}$ and $\bar{g}$. Also, it indicates that the values of the present model are lower in comparison to the values of Sinha and Singh [17] for the parametric values of $\bar{a} > 3$ but for $\bar{a} \leq 2$, the values of the present model are higher in comparison to Ref. [17].

The results on axial velocity (Eqs. (33) and (50)), shear stress (Eqs. (40) and (51)) and resistance to flow (Eqs. (48) and (52)) for different diseases are shown in Figs. 6–8. The results on shear stress

Fig. 5. Variation of resistance to flow ($\lambda_R$) with $\bar{g}$ for different $\bar{g}$ and concentrations ($L_o = 0.5$, and $\bar{d}_c = 0.20$).
and resistance to flow agree well with the quantitative observation of Forrester and Young [19] (shown as dotted lines). The values are lower in case of shear stress distribution, and are higher in case of resistance to flow in comparison to Forrester and Young [19]. The comparison between the present model and that of Sinha and Singh [17] is rather poor (except for the trends) for the set of values computed for the shear stress and the resistance to flow as shown in Figs. 2–5. Whereas the comparison between the present model and that of Forrester and Young [19] are quite good (Figs. 6–8). This is for the reason that the present model, and that of Forrester and Young [19], accounts
for both the inertia and the viscous terms whereas Sinha and Singh [17] model accounts for only viscous terms.

The values of axial velocities appear to be higher in comparison to Ref. [19] for lower values of Reynolds number \( \leq 100 \), but for higher values of Reynolds number, Newtonian values predominate over the present ones. Also, it is observed that, for higher values of Reynolds number, the velocity profiles are tending towards bluntness, thus explaining one of the anomalies associated with the blood flows [8,9,13–16,24,25]. It appears from the present modeling that introduction of both inertia and viscous terms in the model makes the net values finer in computation to the one computed by accounting either only inertia or viscous terms independently.

8. Conclusions

Blood flow in a stenosed tube has been studied in the present analysis. Blood is assumed to be represented by a couple stress fluid (a class of micro-continuum fluids proposed by Stokes [12]). The flow variables such as axial velocity, shear stress distribution, and resistance to flow have been computed for different suspension concentrations (haematocrit) and for the diseases like polycythemia, plasma cell dyscrasias, and for Hb SS (sickle cell). The flow variables have also been computed for normal blood for the comparison purpose. The results are quite encouraging and
are in quantitative agreement with the other theoretical [6,17,19] and other experimental observations on blood flow [8,9].

Fig. 8. Variation of resistance to flow $\left(\frac{P_0-P_i}{\rho U_0^2}\right)$ with stenosis height $\left(\delta_i\right)$ for different $Re_0$ and diseases ((—) present model; (---) Forrester and Young [19]).
Fluid mechanics of arterial stenosis plays an important role in the early detection of any stroke-related problems associated with cerebrovascular or cardiovascular system [26]. Shear stress and Resistance to flow are the two important parameters associated with the mechanical characteristics of vessel wall. Information pertaining to these two flow variables can be used for the assessment of strength of the affected vessel wall. In view of the above, It is hoped that, the present findings could be useful in the design of flow meters in bio-medical instrumentation.

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References