ONLINE APPENDIX

Optimal Pricing for a Multinomial Logit Choice Model with Network Effects

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A Uniqueness and Stability of Optimal Sales Levels

In our developments, we have used $\boldsymbol{q}$ as the decision variable in the seller's revenue optimization problem. However, as mentioned in Section 2, given a price vector $\boldsymbol{p}$, there could be multiple $\boldsymbol{q}$ that satisfy the equilibrium condition (4), which we write here as $\boldsymbol{q} = F(\boldsymbol{p}, \boldsymbol{q})$ to show the dependence upon $\boldsymbol{p}$. We effectively assumed that the seller can select sales quantities $\boldsymbol{q}$ by using prices $\boldsymbol{p}(\boldsymbol{q}) = (p_1(\boldsymbol{q}), \ldots, p_n(\boldsymbol{q}))$ given by (5). In this section, we provide some justification for this assumption.

Observe first that for any $(\boldsymbol{p}, \boldsymbol{q})$ pair that satisfies $\boldsymbol{q} = F(\boldsymbol{p}, \boldsymbol{q})$, it must be that $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{q})$ because as noted in Section 2, given a $\boldsymbol{q}$, there is a unique $\boldsymbol{p}$ for which $\boldsymbol{q} = F(\boldsymbol{p}, \boldsymbol{q})$. Consequently, if a $(\boldsymbol{p}, \boldsymbol{q})$ pair satisfies $\boldsymbol{q} = F(\boldsymbol{p}, \boldsymbol{q})$, then $\sum_j q_j p_j = \sum_j q_j p_j(\boldsymbol{q}) \leq \sum_j q_j^* p_j(\boldsymbol{q}^*) = \sum_j q_j^* p_j^*$. That is, if a $(\boldsymbol{p}, \boldsymbol{q})$ pair is an equilibrium, then the revenue accrued at that equilibrium is no greater than that accrued at the equilibrium sales quantities and prices identified in Theorems 3.1 and 3.2.

The seller will implement prices $\boldsymbol{p}^* = \boldsymbol{p}(\boldsymbol{q}^*)$, so it is of particular interest to look at the possibility that for the optimal $\boldsymbol{q}^*$, there could be some other $\boldsymbol{q}'$ that also satisfies the equilibrium condition for $\boldsymbol{p}^* = \boldsymbol{p}(\boldsymbol{q}^*)$, i.e., $\boldsymbol{q}' = F(\boldsymbol{p}^*, \boldsymbol{q}')$. By the preceding argument, we see that the revenue associated with $(\boldsymbol{q}', \boldsymbol{p}^*)$ cannot exceed that of $(\boldsymbol{q}^*, \boldsymbol{p}^*)$. Hence, if there are multiple equilibria associated with prices $\boldsymbol{p}(\boldsymbol{q}^*)$, then we may view the seller as picking the best one that can arise from those prices.

Proposition A.1 below provides a sufficient condition that ensures that for each $\boldsymbol{p}$, there exists a unique $\boldsymbol{q}$ that satisfies (4), i.e., that satisfies $\boldsymbol{q} = F(\boldsymbol{p}, \boldsymbol{q})$. Consequently, for problems that satisfy the sufficient condition, if the seller implements prices $\boldsymbol{p}(\boldsymbol{q}^*)$, then the only sales levels that satisfy (4) are $\boldsymbol{q}^*$ and hence the issue of multiple equilibria is not present. The proposition is proved in Miyao and Shapiro (1981) for a model more general than ours and is specialized to the network choice model (4) in Wang and Wang (2014).

Proposition A.1. *For any values of $\{(\alpha_i, \gamma_i, y_i) : i \in \mathcal{N}\}$ and any $\boldsymbol{p}$, there exists at least one solution to (4). Moreover, if $\alpha_i \leq 2$ for all $i \in \mathcal{N}$, then for any $\{(\gamma_i, y_i) : i \in \mathcal{N}\}$ and any $\boldsymbol{p}$, the solution to (4) is unique.*

One important question about the choice model we consider is whether sales levels $\boldsymbol{q}$ will converge to an equilibrium if they are initially out of equilibrium and customers repeatedly adjust their purchase decisions according to market conditions. To address this, we consider the following dynamics of the sales quantities:

$$q_i^t = \frac{\exp(y_i - \gamma_i p_i + \alpha_i q_i^{t-1})}{1 + \sum_{j=1}^n \exp(y_j - \gamma_j p_j + \alpha_j q_j^{t-1})}. \tag{22}$$

The following result is proved in Miyao and Shapiro (1981) and Wang and Wang (2014):

Proposition A.2. *Suppose* $|\alpha_i| \leq 2$ *for all* $i \in \mathcal{N}$*. Fix any* $\boldsymbol{p}$ *and any* $\boldsymbol{q}^0 = (q_1^0, \ldots, q_n^0)$ *and consider* $\{\boldsymbol{q}^t = (q_1^t, \ldots, q_n^t)\}$ *in (22). Then* $\{\boldsymbol{q}^t\}$ *converges to the unique solution to (4).*

In Wang and Wang (2014), the authors use a set of DVD purchase data from a major online retailer to calibrate the network MNL choice model, and find the optimal fit of the coefficient α (they assume homogeneous α_i across products) is 0.998 (statistically significant). Therefore, in that case, the conditions in Propositions A.1 and A.2 hold, the equilibrium is unique for any prices, and dynamic customer adjustments will give us convergence to that equilibrium.

In general, if the α_i do not satisfy the conditions in the above propositions, then it is possible that there are multiple equilibria for (4) and the above dynamic adjustments (22) may converge to different equilibria depending on the starting point (see Section 5.1 for further discussion). In fact, such phenomenon of multiple equilibria is quite common in models that incorporate network effects; see, for example, Galeotti et al. (2010), Jackson and Yariv (2007), Sundararajan (2007), Economides (1996), Katz and Shapiro (1985), Dybvig and Spatt (1983), and Rohlfs (1974), among many others. There is evidently not a single "right" answer to the question of what will happen in the presence of multiple equilibria. However, one notion that has been used to explain why a particular equilibrium might arise while another might not is that of (local) stability.

We will argue next that $\boldsymbol{q}^*$ is a stable equilibrium under prices $\boldsymbol{p}(\boldsymbol{q}^*)$. For the ensuing discussion we say that $\boldsymbol{q}^*$ is a stable equilibrium if all the eigenvalues of the Jacobian matrix of $F(\boldsymbol{q})$ at $\boldsymbol{q}^*$ have real part less than 1 (see Chapter 4 in Merkin 1997 for a reference). Such stability ensures that there exists a neighborhood of $\boldsymbol{q}^*$ such that the differential equation system $\frac{\partial \boldsymbol{q}(t)}{\partial t} = F(\boldsymbol{q}(t)) - \boldsymbol{q}(t)$ with starting point $\boldsymbol{q}(0)$ in a neighborhood of $\boldsymbol{q}^*$ will converge to $\boldsymbol{q}^*$ as t goes to infinity. In practice, the seller may first guide the customers to a neighborhood of $\boldsymbol{q}^*$ (e.g., by posting expected sales) and then the dynamics of the system will make the sales levels converge to $\boldsymbol{q}^*$, thereby justifying the choice of $\boldsymbol{q}^*$. Related notions of equilibrium stability are discussed in, e.g., Jackson and Yariv (2007) and Economides (1996).

Proposition A.3. *Consider the homogeneous case with $\alpha_i = \alpha \le \hat{\alpha}$ ($\hat{\alpha}$ is defined in Theorem 3.1), $\gamma_i = 1$, and $y_i = y$ for all $i \in \mathcal{N}$. The optimal $\boldsymbol{q}^*$ is a stable equilibrium.*

Proof. Consider the Jacobian matrix $J = \frac{\partial F}{\partial \boldsymbol{q}}$. We have

$$\frac{\partial F_i}{\partial q_i} = \frac{\alpha_i B_i(1+\sum_{k=1}^n B_k) - \alpha_i B_i^2}{(1+\sum_{k=1}^n B_k)^2} = \alpha_i q_i(1-q_i),$$
$$\frac{\partial F_i}{\partial q_j} = \frac{-\alpha_j B_i B_j}{(1+\sum_{k=1}^n B_k)^2} = -\alpha_j q_i q_j, \quad i \ne j.$$

where $B_i = \exp(y_i - \gamma_i p_i + \alpha_i q_i)$. When $\alpha_i = \alpha \le \hat{\alpha}$, $\gamma_i = 1$, and $y_i = y$, the Jacobian matrix J can be written as:

$$J = \left[\frac{\partial F_i}{\partial q_j}\right]_{(i,j)} = \alpha \begin{bmatrix} q_1(1-q_1) & -q_1q_2 & \cdots & -q_1q_n \\ -q_1q_2 & q_2(1-q_2) & & -q_2q_n \\ \vdots & & \ddots & \vdots \\ -q_1q_n & -q_2q_n & \cdots & q_n(1-q_n) \end{bmatrix}.$$

By Theorem 3.1(a), $q_1^* = \cdots = q_n^* = q^*$ when $\alpha \le \hat{\alpha}$. In this case, $J = \alpha q^*[I - q^*\mathbf{e}\mathbf{e}^T]$, where I is an identity matrix. The largest eigenvalue of J is αq^*. To see this, note that the eigenvalues of $\mathbf{e}\mathbf{e}^T$ are $0, \ldots, 0, n$ and therefore the eigenvalues of $[I - q^*\mathbf{e}\mathbf{e}^T]$ are $1, \ldots, 1, 1 - nq^*$ and the eigenvalues of J are $\alpha q^*, \ldots, \alpha q^*, \alpha q^*(1-nq^*)$. By Lemma C.1, we have $\frac{\partial^2 \tilde{\pi}}{\partial s_i^2}(\boldsymbol{s}^*) = 2\alpha - \frac{1}{q^*} \le 0$ in this case. Thus we have proved that $\boldsymbol{q}^*$ must be a stable fixed point when $\alpha \le \hat{\alpha}$. □

Unfortunately, we have not been able to prove the stability of the optimal $\boldsymbol{q}^*$ in the general case. However, we have conducted a large number of numerical experiments (with both homogeneous and heterogeneous parameters) and in all experiments, the optimal $\boldsymbol{q}^*$ is stable.

B Three Extensions

In this section, we consider three extensions to our basic pricing model. In Section B.1, we allow inter-product network effects. In Section B.2, we consider the possibility that network effects may enter customers' expected utility in a non-linear fashion. In Section B.3, we consider network effects for the no-purchase option.

B.1 Inter-product Network Effects

In this section, we incorporate inter-product network effects into consumers' utilities. More precisely, we replace the customer's utility function (1) with

$$v_i = y_i - \gamma_i p_i + \alpha_i x_i + \beta_i x_{-i},$$

where $\beta_i \geq 0$ is the inter-product network sensitivity parameter, and $x_{-i} = \sum_{j \in \mathcal{N} \setminus \{i\}} x_j$ is the total consumption of all products other than i. Hence, the new term $\beta_i x_{-i}$ represents the additional expected utility for product i from the market's consumption of products other than i. We assume $\alpha_i > \beta_i$, which means that the within-product sensitivity is stronger than the inter-product sensitivity.

In the following, we consider a homogeneous case in which $y_i = y, \gamma_i = 1, \alpha_i = \alpha$, and $\beta_i = \beta$ for all $i \in \mathcal{N}$. Now the expression (7) becomes

$$p_i(\boldsymbol{q}) = \alpha q_i + \beta q_{-i} - \log q_i + \log\left(1 - \sum_{j=1}^{n} q_j\right) + y,$$

and the revenue function (8) becomes

$$\pi(\boldsymbol{q}) = (\alpha - \beta)\sum_{j=1}^{n} q_j^2 + \sum_{j=1}^{n} q_j \left(y + \beta \sum_{j=1}^{n} q_j + \log\left(1 - \sum_{j=1}^{n} q_j\right)\right) - \sum_{j=1}^{n} q_j \log q_j.$$

As in (P1), we may assume $q_1 \geq \cdots \geq q_n$ without loss of optimality. With the same means of selecting a particular optimal solution as in (9), it turns out that most of the results in Section 3 for (P1) still hold in this new setting.

In particular, Theorem 3.1 holds with the minor modification that the threshold $\hat{\alpha}$ (which now depends upon y, n, and β) satisfies $\beta + 1/2 < \hat{\alpha} \leq \alpha^{\bar{R}}$ where $\alpha^{\bar{R}}$ is the smallest solution to $\bar{R}(\alpha) = 0$ where

$$\bar{R}(\alpha) = y + \log(2\alpha - 2\beta - n) - \frac{n}{2\alpha - 2\beta - n} + \frac{n\beta}{\alpha - \beta}.$$

Theorem 3.2 carries over without modification to this setting. The proofs of these results follow almost exactly as the proofs of Theorems 3.1 and 3.2 with only minor changes. A key step is using Proposition 3.3, but with α replaced by $\alpha - \beta$ and $C(\sigma)$ redefined as $\frac{1}{1-\sigma} - \log(1-\sigma) - y - 2\beta\sigma$. (We note that we opted to consider the case without inter-product effects in the main body of the paper to help keep arguments as transparent as possible.)

B.2 Non-linear Network Effects

In this section, we consider a more general form of network effects. More precisely, we assume the customer's utility function (1) is replaced by

$$v_i = y_i - \gamma_i p_i + f_i(x_i),$$

where $f_i(x_i)$ is the expected utility gained from the network effects. We assume that $f_i(0) = 0$ and that $f_i(\cdot)$ is increasing. For simplicity, we also assume $f_i(\cdot)$ is differentiable. With the same normalization as in (3), the expression for prices (5) becomes

$$p_i(\boldsymbol{q}) = \frac{1}{\gamma_i}\left(f_i(q_i) - \log q_i + \log\left(1 - \sum_{j=1}^{n} q_j\right) + y_i\right),$$

and the revenue function (6) becomes

$$\pi(\boldsymbol{q}) = \sum_{j=1}^{n} \frac{q_j f_j(q_j)}{\gamma_j} + \sum_{j=1}^{n} \frac{q_j}{\gamma_j} \log\left(1 - \sum_{j=1}^{n} q_j\right) + \sum_{j=1}^{n} \frac{q_j}{\gamma_j}\left(y_j - \log q_j\right). \tag{23}$$

Using an analysis as in Section 4, we see that the optimal $\boldsymbol{q}^\dagger$ must satisfy the first-order necessary conditions, and thus (16) becomes

$$q_i^\dagger f_i'(q_i^\dagger) + f_i(q_i^\dagger) - \log q_i^\dagger = C_i(\boldsymbol{q}^\dagger) \quad \text{for all } i \in \mathcal{N},$$

where $C_i(\boldsymbol{q})$ is as defined in Lemma 4.1. Note that if $f_i(q) = \alpha_i q$, then $q_i^\dagger f_i'(q_i^\dagger) + f_i(q_i^\dagger) = 2\alpha_i q_i^\dagger$ in which case the preceding condition simplifies to (15).

Define $H_i(q) = q f_i'(q) + f_i(q) - \log q$. Below, we assume $H_i''(q) > 0$ for $q > 0$. In that case, for $c \geq \min_q H_i(q)$, we re-define $\underline{q}_i^c \leq \overline{q}_i^c$ to be the solutions $H_i(q) = c$. (If there is only solution to $H_i(q) = c$, then we let both $\underline{q}_i^c$ and $\overline{q}_i^c$ be that solution.) The original definition (for problems with $f_i(q) = \alpha_i q$) appears in the paragraph that follows Lemma 4.2.

Proposition B.1. *Suppose $H_i''(q) > 0$ for all $q > 0$ for each $i \in \mathcal{N}$. Then Proposition 4.3 holds in the more general setting of* (23)*, with $\underline{q}_i^c$ and $\overline{q}_i^c$ redefined as above.*

Examples of functions $f_i(\cdot)$ such that $H_i''(q) > 0$ for all $q > 0$ are $f_i(q) = \log(1 + q)$ and $f_i(q) = \alpha q^\theta$ where $\alpha > 0$ and $\theta \geq 1$. The proof of Proposition B.1 follows almost exactly as the proofs of Lemma 4.2 and Proposition 4.3. As long as the network effect term in the utility function satisfies the conditions in Proposition B.1, the optimal strategy remains to either maintain a semblance of balance among all products or else to boost the sales of just one product.

B.3 Network Effects for the No-Purchase Option

Here, we consider an extension of (P0) in which an individual customer's utility from not purchasing any product depends on the fraction of customers that do not purchase. This setting captures the idea that the customers who do not choose the products offered by our seller could buy a product from someone else. In this case, (1) remains unchanged but v_0 becomes

$$v_0 = \alpha_0 x_0,$$

where x_0 is the number of customers that do not buy any product offered by this firm. With the same normalization as in (3), the expression (5) becomes

$$p_i(\boldsymbol{q}) = \frac{1}{\gamma_i}\left(\alpha_i q_i - \log q_i + \log\left(1 - \sum_{j=1}^{n} q_j\right) + y_i - \alpha_0\left(1 - \sum_{j=1}^{n} q_j\right)\right),$$

and the revenue function (6) becomes

$$\pi(\boldsymbol{q}) = \sum_{j=1}^{n} q_j p_j(\boldsymbol{q}) = \sum_{j=1}^{n} \frac{\alpha_j}{\gamma_j} q_j^2 + \sum_{j=1}^{n} \frac{q_j}{\gamma_j}\left(y_j + \log\left(1 - \sum_{j=1}^{n} q_j\right) - \log q_j - \alpha_0\left(1 - \sum_{j=1}^{n} q_j\right)\right).$$

As in Section 4, the optimal $\boldsymbol{q}^\dagger$ must satisfy the first-order necessary conditions

$$2\alpha_i q_i^\dagger - \log q_i^\dagger = D_i(\boldsymbol{q}^\dagger) \quad \text{for all } i \in \mathcal{N}, \tag{24}$$

where $D_i(\boldsymbol{q}) = C_i(\boldsymbol{q}) + \alpha_0\left(1 - \sum_{j=1}^{n} q_j\right) - \alpha_0\gamma_i \sum_{j=1}^{n} q_j/\gamma_j$ and $C_i(\boldsymbol{q})$ is defined in Lemma 4.1.

Since the left hand side of (24) remains the same as in (15), Lemma 4.2 still applies and the results of Proposition 4.3 still hold in this setting. The proofs follow exactly as before. Therefore even if the non-purchase option is sensitive to network effects, the optimal strategy follows the same structure as in our basic model.

C Proofs for Section 3

To prove the theorems and propositions in Section 3, we need the following lemma.

Lemma C.1. *For any optimal solution $\hat{\boldsymbol{s}}$ to (P2), we must have $\frac{\partial \tilde{\pi}}{\partial s_i}(\hat{\boldsymbol{s}}) = 0$ and $\frac{\partial^2 \tilde{\pi}}{\partial s_i^2}(\hat{\boldsymbol{s}}) \le 0$ for all $i \in \mathcal{N}$. Similarly, for any optimal solution $\hat{\boldsymbol{q}}$ to (P1), we must have $\frac{\partial \pi}{\partial q_i}(\hat{\boldsymbol{q}}) = 0$ and $\frac{\partial^2 \pi}{\partial q_i^2}(\hat{\boldsymbol{q}}) \le 0$ for all $i \in \mathcal{N}$.*

Proof. Consider (P0) with homogeneous parameters. If we substitute for $\boldsymbol{q}$ in terms of $\boldsymbol{s}$, then we get a problem (P0′) that is the same as (P2) except the feasible region is $\{\boldsymbol{s} | 1/\sqrt{n} \ge s_1 \ge 0, q_i \ge 0\}$ where the q_i are given by (13). Next we show that any optimal solution $\tilde{\boldsymbol{s}}$ to (P0′) must be an interior point in the feasible region. For this, it suffices to show that $0 < \tilde{s}_1 < 1/\sqrt{n}$ and $\tilde{q}_i > 0$. The KKT condition for s_1 at optimality is

$$2\alpha s_1 + \sqrt{n}\left(y + \log(1 - \sqrt{n}s_1)\right) - \frac{\sqrt{n}}{1 - s_1\sqrt{n}} - \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\log q_j + \nu_1 - \sum_{j=1}^{n}\lambda_j\frac{1}{\sqrt{n}} = 0,$$

where ν_1 and λ_j are Lagrange multipliers satisfying $\nu_1 \geq 0, \lambda_j \geq 0$. From the KKT condition, it can be seen that at optimality, we must have $s_1 < 1/\sqrt{n}$ and $q_i > 0$ for all $i \in \mathcal{N}$. It then follows that $s_1 > 0$ by (12). Thus, any optimal $\tilde{\boldsymbol{s}}$ must be in the interior, and therefore $\frac{\partial\tilde{\pi}}{\partial s_i}(\tilde{\boldsymbol{s}}) = 0$ and $\frac{\partial^2\tilde{\pi}}{\partial s_i^2}(\tilde{\boldsymbol{s}}) \leq 0$ must hold for any such $\tilde{\boldsymbol{s}}$. Recall that $\hat{\boldsymbol{s}}$ is an optimal solution to (P2), and $A^{-1}\hat{\boldsymbol{s}}$ is an optimal solution to (P1). Thus $A^{-1}\hat{\boldsymbol{s}}$ is also optimal for (P0). It follows that $\hat{\boldsymbol{s}}$ is also optimal to (P0′). Therefore, we must have $\frac{\partial\tilde{\pi}}{\partial s_i}(\hat{\boldsymbol{s}}) = 0$ and $\frac{\partial^2\tilde{\pi}}{\partial s_i^2}(\hat{\boldsymbol{s}}) \leq 0$. The second half of the lemma follows immediately, because $\frac{\partial\pi}{\partial\boldsymbol{q}} = A\frac{\partial\tilde{\pi}}{\partial\boldsymbol{s}}$ and $\frac{\partial^2\pi}{\partial\boldsymbol{q}^2} = A^T\frac{\partial^2\tilde{\pi}}{\partial\boldsymbol{s}^2}A$. □

We are now ready to prove Propositions 3.3 and 3.4.

Proof of Proposition 3.3. By Lemma C.1, we must have $\frac{\partial\pi}{\partial q_i}(\boldsymbol{q}^*) = 0$, thus (11) follows. Note that $2\alpha x - \log x$ is strictly convex in x, and hence for any σ, $2\alpha q - \log q = C(\sigma)$ has at most two different solutions. Therefore $\boldsymbol{q}^*$ has at most two distinct entries. □

Proof of Proposition 3.4. To prove that $\tilde{\pi}(s_1, \ldots, s_n)$ is supermodular in $(\boldsymbol{s}, \alpha)$, it suffices to show that all the cross partial derivatives are non-negative. We have

$$\frac{\partial^2\tilde{\pi}}{\partial\alpha\partial s_i} = 2s_i \quad \text{for all } i$$

$$\frac{\partial^2\tilde{\pi}}{\partial s_i\partial s_j} = \begin{cases} \frac{1}{\sqrt{nj(j-1)}}\sum_{k=1}^{j-1}(\frac{1}{q_j} - \frac{1}{q_k}), & j > i = 1 \\ \frac{1}{\sqrt{ij(i-1)(j-1)}}\sum_{k=1}^{i-1}(\frac{1}{q_i} - \frac{1}{q_k}), & j > i \geq 2. \end{cases}$$

By our assumption, $q_1 \geq \cdots \geq q_n$. Therefore, all the cross partials are non-negative and $\tilde{\pi}(s_1, \ldots, s_n)$ is supermodular in $(\boldsymbol{s}, \alpha)$.

Next we prove that the feasible set is a sublattice on $\mathbb{R}^n$. This is true because all the constraints $1/\sqrt{n} \geq s_1 \geq \sqrt{n-1}s_n, s_2 \geq 0, s_{i+1} \geq \sqrt{(i-1)/(i+1)}s_i$ are bimonotone linear inequalities, i.e., for each inequality there are at most two non-zero coefficients that are of opposite signs. The rest of the lemma follows directly from Theorem 2.8.2 of Topkis (1998). □

Lemma C.2. *If $\boldsymbol{q}^*$ has two distinct entries q_H and q_L with $d = q_H - q_L > 0$, then we must have $q_H = q_H(d)$ and $q_L = q_L(d)$ where*

$$q_H(d) = \frac{de^{2\alpha d}}{e^{2\alpha d} - 1}, \quad q_L(d) = \frac{d}{e^{2\alpha d} - 1}. \tag{25}$$

Moreover, $q_H(d)$ is increasing and $q_L(d)$ is decreasing in d with $q_H(d) > \frac{1}{2\alpha} > q_L(d)$ for all $d > 0$.

Proof. If $\boldsymbol{q}^*$ has two distinct entries q_H, q_L with $d = q_H - q_L > 0$, then (11) implies that $\log q_H - \log q_L = 2\alpha(q_H - q_L)$. From this we can solve for q_H and q_L to obtain (25).

Define $q_H(0) = \lim_{d\to 0} q_H(d) = 1/2\alpha$ and $q_L(0) = \lim_{d\to 0} q_L(d) = 1/2\alpha$. First consider $q_L(d)$. We have

$$q_L'(d) = \frac{e^{2\alpha d} - 2\alpha d e^{2\alpha d} - 1}{(e^{2\alpha d} - 1)^2}.$$

Define $x = 2\alpha d$ and $g_1(x) = e^x - xe^x - 1$. To prove $q_L(d)$ is decreasing, it suffices to prove $g_1(x) < 0$ on $x > 0$. The latter condition is indeed true, because $g_1'(x) = -xe^x < 0$ and $g_1(0) = 0$. Thus $q_L(d)$ is decreasing in $d > 0$ and $q_L(d) < q_L(0) = 1/2\alpha$.

Similarly, we have

$$q_H'(d) = \frac{e^{4\alpha d} - e^{2\alpha d} - 2\alpha d e^{2\alpha d}}{(e^{2\alpha d} - 1)^2}.$$

To show $q_H(d)$ is increasing, we again take $x = 2\alpha d$. It suffices to prove $g_2(x) = e^{2x} - e^x - xe^x > 0$ on $x > 0$. Because $e^x > 1 + x$ on $x > 0$, we have $g_2(x) = e^x(e^x - 1 - x) > 0$. Therefore $q_H(d)$ is increasing in $d > 0$ with $q_H(d) > q_H(0) = 1/2\alpha$. □

Proof of Theorem 3.1. First we prove that $\boldsymbol{q}^*$ must be of the form $q_1^* \geq q_2^* = \cdots = q_n^*$. By Proposition 3.3, entries of $\boldsymbol{q}^*$ can take at most two distinct values. Because of the constraint $q_1 \geq \cdots \geq q_n$, we can assume

$$q_i^* = \begin{cases} q_H & i = 1, \ldots, k \\ q_L & i = k+1, \ldots, n, \end{cases}$$

where $q_H > q_L$ for some k (the case $q_1 = \cdots = q_n$ corresponds to $k = 0$). Next we will prove $k \leq 1$ by contradiction.

If $k \geq 2$, we have by (12) and (14),

$$\frac{\partial^2 \tilde{\pi}}{\partial s_k^2}(\boldsymbol{s}^*) = 2\alpha - \sum_{j=1}^{n} \frac{1}{q_j}\left(\frac{\partial q_j}{\partial s_k}\right)^2 = 2\alpha - \sum_{j=1}^{k-1} \frac{1}{k(k-1)q_j^*} - \frac{k-1}{k}\frac{1}{q_k^*} = 2\alpha - \frac{1}{q_H}.$$

By Lemma C.2, $q_H > \frac{1}{2\alpha}$ and thus $\frac{\partial^2 \tilde{\pi}}{\partial s_i^2}(\boldsymbol{s}^*) > 0$, which contradicts Lemma C.1. Therefore $\boldsymbol{q}^*$ must be of the form $q_1^* \geq q_2^* = \cdots = q_n^*$.

Now by Proposition 3.4, we know that $s_2^* = (q_1^* - q_2^*)/\sqrt{2}$ increases in α. Therefore, there is a threshold $\hat{\alpha}$ for α below which $q_1^* = \cdots = q_n^*$, and above which $q_1^* > q_2^* = \cdots = q_n^*$. Furthermore, for the case when $q_1^* = \cdots = q_n^*$, by Proposition 3.4 we have that $s_1^* = \sqrt{n} q_1^*$ increases in α. Therefore q_i^* increases in α for all $i \in \mathcal{N}$ in this case. Part (a) is thus proved.

When $q_1^* > q_2^* = \cdots = q_n^*$, by Lemma C.2, $q_L(d)$ decreases in d and $d = q_1^* - q_2^* = \sqrt{2} s_2^*$ increases in α by Proposition 3.4. Furthermore, with d fixed, $q_L = \frac{d}{e^{2\alpha d} - 1}$ decreases in α. Thus q_2^* decreases in α. Moreover, by Proposition 3.4, $s_1^* = \sum_{i=1}^n q_i^*/\sqrt{n} = (q_H + (n-1)q_L)/\sqrt{n}$ increases in α, and hence $q_1^* = q_H$ must increase in α.

As α goes to infinity, we have $q_L < 1/2\alpha$ by Lemma C.2. Therefore $q_i = q_L$ goes to 0 in the limit for all $i \geq 2$. Furthermore, by Lemma C.1, we have

$$\frac{\partial \pi}{\partial q_1}(\boldsymbol{q}^*) = 2\alpha q_1^* + \log(1 - \sqrt{n} s_1^*) + y - \frac{1}{1 - \sqrt{n} s_1^*} - \log q_1^* = 0,$$

and thus $\log(1 - \sqrt{n} s_1^*) - \frac{1}{1 - \sqrt{n} s_1^*} = -2\alpha q_1^* - y + \log q_1^*$. Now we have proved that q_1^* increases in α. Thus, $-2\alpha q_1^* - y + \log q_1^*$ goes to $-\infty$ as α goes to infinity. Therefore s_1^* goes to $1/\sqrt{n}$ in the limit, which also means q_1^* goes to 1. Hence, part (b) is also proved.

Proposition C.3 below completes the proof of the theorem. □

Proposition C.3. *Let $R(\alpha) = y + \log(2\alpha - n) - \frac{n}{2\alpha - n}$. There is a unique solution α^R to $R(\alpha) = 0$. Moreover, $1/2 < \hat{\alpha} \leq \alpha^R$. If $n = 2$, then $\hat{\alpha} = \alpha^R$.*

Proof. We first prove the uniqueness of α^R. First, we note that $R(\alpha) \to -\infty$ as $\alpha \downarrow n/2$ and $R(\alpha) \to \infty$ as $\alpha \to \infty$. Moreover, $R(\alpha)$ is continuous and increasing in α on $(n/2, \infty)$, so there is a unique solution α^R to $R(\alpha) = 0$. And we have

$$R(\alpha) > 0 \quad \text{for } \alpha > \alpha^R. \tag{26}$$

To show that $\hat{\alpha} > 1/2$, observe that when $\alpha \leq 1/2$, we must have $q_1^* = \cdots = q_n^*$. Otherwise, Lemma C.2 implies that $q_H > \frac{1}{2\alpha} \geq 1$, which is a contradiction. Hence, $\hat{\alpha} > 1/2$.

Next, we will prove that $q_1^* > q_2^* = \cdots = q_n^*$ when $\alpha > \alpha^R$, from which it follows that $\hat{\alpha} \leq \alpha^R$. By the results we have proved, it suffices to rule out the possibility $q_1^* = \cdots = q_n^*$. In the ensuing argument we shall use the expressions (27)–(29) for derivatives of $\tilde{\pi}(\cdot)$, which follow from (12)–(14):

$$\frac{\partial \tilde{\pi}}{\partial s_1} = 2\alpha s_1 + \sqrt{n}(y + \log(1-\sqrt{n}s_1)) - \frac{ns_1}{1-\sqrt{n}s_1} - \sum_{j=1}^{n} \frac{1}{\sqrt{n}}(\log q_j + 1) \tag{27}$$

$$\frac{\partial^2 \tilde{\pi}}{\partial s_1^2} = 2\alpha - \frac{n}{1-\sqrt{n}s_1} - \frac{n}{(1-\sqrt{n}s_1)^2} - \sum_{j=1}^{n} \frac{1}{nq_j} \tag{28}$$

$$\frac{\partial^2 \tilde{\pi}}{\partial s_i^2} = 2\alpha - \sum_{j=1}^{i-1} \frac{1}{(i-1)iq_j} - \frac{i-1}{iq_i} \qquad \text{for } i = 2, \ldots, n. \tag{29}$$

Consider $\alpha > \alpha^R$ and suppose for a contradiction that $q_1^* = \cdots = q_n^* = q^*$. By (12) we have that $\boldsymbol{s}^*$ is given by $s_1^* = \sqrt{n}q^*$ and $s_2^* = \cdots = s_n^* = 0$. By Lemma C.1 and (29), we have $0 \geq \frac{\partial^2 \tilde{\pi}}{\partial s_i^2}(\boldsymbol{s}^*) = 2\alpha - \frac{1}{q^*}$ for $i = 2, \ldots, n$. Therefore, $q^* \leq \frac{1}{2\alpha}$.

Next we show $q^* > 1/3n$. From (27) we obtain

$$\frac{\partial \tilde{\pi}}{\partial s_1}(\boldsymbol{s}^*) = \sqrt{n}\left(2\alpha q^* + y + \log(1-nq^*) - \frac{1}{1-nq^*} - \log q^*\right) = \sqrt{n}\varphi(q^*), \tag{30}$$

where we define $\varphi(q) = 2\alpha q + y + \log(1-nq) - \frac{1}{1-nq} - \log q$. Note that α^R must satisfy $2\alpha^R - n > 0$, and therefore $2\alpha > n$ because we are considering an $\alpha > \alpha^R$. Thus

$$\frac{\partial \tilde{\pi}}{\partial s_1}(\boldsymbol{s}^*) > \sqrt{n}\left(nq^* + \log(1-nq^*) - \frac{1}{1-nq^*} - \log q^*\right) = \sqrt{n}\widehat{\varphi}(q^*),$$

where we define $\widehat{\varphi}(q) = nq + \log(1-nq) - \frac{1}{1-nq} - \log q$. We claim $\widehat{\varphi}(q) > 0$ when $q \leq 1/3n$. This is true because $\widehat{\varphi}(1/3n) = \log 2n - 7/6 > 0$ and $\widehat{\varphi}'(q) = n - 1/q(1-nq)^2$ is negative for $0 < q < 1/3n$. Therefore, by Lemma C.1, we must have $q^* > 1/3n$.

Consider now the second derivative of $\tilde{\pi}$ with respect to s_1 evaluated at $\boldsymbol{s}^*$. By (28) and Lemma C.1 we have

$$0 \;\geq\; \frac{\partial^2 \tilde{\pi}}{\partial s_1^2}(\boldsymbol{s}^*) \;=\; 2\alpha - \frac{1}{q^*(1-nq^*)^2} \;=\; \psi(q^*),$$

where we define $\psi(q) = 2\alpha - \frac{1}{q(1-nq)^2}$. It is easy to verify that $\psi(q)$ decreases in q for $q \geq 1/3n$, thus $\psi(q) \leq 0$ for all $q \in [q^*, \frac{1}{2\alpha}]$. Note also that $\psi(q) = \varphi'(q)$, and hence $\varphi'(q) \leq 0$ for all $q \in [q^*, \frac{1}{2\alpha}]$. It follows immediately that $\varphi(q^*) \geq \varphi(1/2\alpha)$. Therefore,

$$\sqrt{n}\varphi(q^*) \geq \sqrt{n}\varphi(1/2\alpha) = \sqrt{n}\left(y + \log(2\alpha - n) - \frac{n}{2\alpha - n}\right) = \sqrt{n}R(\alpha) > 0,$$

where the final inequality follows from (26). Combining the preceding with (30), we see that $\frac{\partial \tilde{\pi}}{\partial s_1}(\boldsymbol{s}^*) > 0$, which is a contradiction with Lemma C.1. Thus we have proved that $q_1^* > q_2^* = \cdots = q_n^*$ for $\alpha > \alpha^R$. Hence, $\hat{\alpha} \leq \alpha^R$.

To complete the proof it remains only to establish that $\hat{\alpha} = \alpha^R$ when $n = 2$. We just proved that $\hat{\alpha} \leq \alpha^R$ for any n. Hence, it suffices to show that for $n = 2$, $q_1^* = q_2^*$ at $\alpha = \alpha^R$.

Suppose $n = 2$ and $\alpha = \alpha^R$. By (12) and (27), the first-order condition for s_1 is

$$\frac{\partial \tilde{\pi}}{\partial s_1} = \frac{1}{\sqrt{2}}\left(2\alpha s + 2y + 2\log 2 + 2\log(1-s) - \frac{2}{1-s} - \log(s+d) - \log(s-d)\right) = 0 \qquad (31)$$

where $s = q_1 + q_2$ and $d = q_1 - q_2$. The feasible region for (P2) is $0 \leq d \leq s \leq 1$. From the definition of α^R, we have $2y + 2\log 2 = -2\log(\alpha - 1) + \frac{2}{\alpha-1}$ at $\alpha = \alpha^R$. By Proposition 3.3, if $q_1^* - q_2^* = d > 0$, then $s = s(d)$ where $s(d)$ is defined as $s(d) = q_H(d) + q_L(d) = \frac{d\exp(2\alpha d)+d}{\exp(2\alpha d)-1}$. We also let $s(0) = \lim_{d\to 0} s(d) = 1/\alpha$. With this we can re-write the first order condition (31) without the leading $1/\sqrt{2}$ as $f(d) = 0$ where

$$f(d) = 2\alpha s(d) - 2\log(\alpha-1) + \frac{2}{\alpha-1} + 2\log(1-s(d)) - \frac{2}{1-s(d)} - \log(s(d)+d) - \log(s(d)-d).$$

Note that $f(0) = 0$. To establish that $q_1^* = q_2^*$, it is sufficient to verify that $f(d) \neq 0$ on $(0, \bar{d}]$ where $\bar{d}$ is such that $s(\bar{d}) = 1$.

With Lemma C.4 below, we know that $f(d) \neq 0$ on $(0, \bar{d}]$ and thus the proposition holds. □

Lemma C.4. *$f(d)$ defined in the proof of Proposition C.3 is strictly decreasing on $[0, \bar{d})$.*

Proof. We first remove the terms that do not depend upon d in $f(d)$ and define

$$g(d) = 2\alpha s(d) + 2\log(1-s(d)) - \frac{2}{1-s(d)} - \log(s(d)^2 - d^2).$$

Now it suffices to prove $g(d)$ decreases on $[0, \bar{d})$. To do so, we write $g(d) = 2g_1(d) + g_2(d)$ where

$$g_1(d) = \alpha s(d) - \frac{1}{1-s(d)}, \quad g_2(d) = 2\log(1-s(d)) - \log(s(d)^2 - d^2).$$

It suffices to prove that both $g_1(d)$ and $g_2(d)$ are decreasing on $[0, \bar{d})$.

We first consider $g_1(d)$. We have $g_1'(d) = (\alpha - 1/(1-s(d))^2)s'(d)$. We claim that $s(d)$ is increasing on $d > 0$. To prove this, we define $x = \exp(2\alpha d)$. Then $s(d) = \bar{s}(x) = \frac{(x+1)\log x}{2\alpha(x-1)}$. Differentiating, we get $\bar{s}'(x) = \frac{x - 1/x - 2\log x}{2\alpha(x-1)^2}$. The numerator $x - 1/x - 2\log x$ is 0 at $x = 1$; taking the derivative of this expression yields $1 + 1/x^2 - 2/x = (1 - 1/x)^2$. Thus the numerator of $\bar{s}'(x)$ is zero at $x = 1$ and strictly positive for $x > 1$, and thus the claim is proved. Therefore, $s(d) > s(0) = 1/\alpha$ when $d > 0$. For $d \in (0, \bar{d})$ we have $0 < 1 - s(d) < 1 - 1/\alpha$ and therefore

$$g_1'(d) < (\alpha - 1/(1-1/\alpha)^2)s'(d) = (\alpha - \alpha^2/(\alpha-1)^2)s'(d) = \frac{\alpha s'(d)}{(1-\alpha)^2}(\alpha^2 - 3\alpha + 1).$$

From the above, $g_1'(d) < 0$ if $\alpha^R \in (1, \frac{3+\sqrt{5}}{2})$. (Recall we have taken $\alpha = \alpha^R$.) It is evident that $\alpha^R > 1$ from the definition of $R(\cdot)$. Moreover $R(\cdot)$ increases in y, so $\alpha^R < \alpha_0 = 2.18 < \frac{3+\sqrt{5}}{2}$ where α_0 satisfies $0 + \log(2\alpha_0 - 2) - \frac{1}{\alpha_0 - 1} = 0$ (i.e., $R(\alpha_0) = 0$ when $y = 0$). Hence, $g_1'(d) < 0$.

Next we consider $g_2(d)$. We have

$$g_2(d) = 2\log(1 - s(d)) - \log(s(d)^2 - d^2) = 2\log\left(\frac{e^{\alpha d} - e^{-\alpha d}}{2d} - \frac{e^{\alpha d} - e^{-\alpha d}}{2}\right) \doteq 2\log h(d).$$

Therefore, in order to prove that $g_2(d)$ is decreasing, it suffices to show that $h(d)$ is decreasing in d on $[0, \bar{d})$. We take the derivative, and we have

$$h'(d) = \frac{1}{2d^2 e^{\alpha d}}\left\{e^{2\alpha d}(-\alpha d^2 + \alpha d - 1) + \alpha d^2 + \alpha d + 1\right\}.$$

Now we want to show that $h'(d) \leq 0$. For this, it suffices to show that the numerator is less than 0. Denote the numerator by $h_1(d)$. We have $h_1(0) = 0$, and $h_1'(d) = \alpha e^{2\alpha d}(-2d - 2\alpha d^2 + 2\alpha d - 1) + 2\alpha d + \alpha$. Thus $h_1'(0) = 0$. Now it suffices to show that $h_1''(d) \leq 0$ for all $d \in [0, \bar{d})$. Taking another derivative, we get $h_1''(d) = \alpha e^{2\alpha d}(-2 - 8\alpha d - 4\alpha^2 d^2 + 4\alpha^2 d) + 2\alpha$. We have $h_1''(0) = 0$ so it suffices to show that $h_1'''(d) \leq 0$ for all $d \in [0, \bar{d})$. Taking yet another derivative gives us $h_1'''(d) = \alpha e^{2\alpha d}(-4\alpha(3 - \alpha) - 8\alpha^2 d(3 - \alpha) - 8\alpha^3 d^2)$. Since $0 < \alpha^R \leq 2.18$ when $n = 2$, $h_1'''(d) < 0$. Thus we have proved that $g_2(d)$ is decreasing, which completes the proof. □

Proof of Theorem 3.2. First we prove part (a). When $\alpha \leq \hat{\alpha}$, we have $q_1^* = \cdots = q_n^* = q^*$ by Theorem 3.1. By (7), the entries of the optimal price vector $\boldsymbol{p}^*$ must also be identical and given by

$$p(q^*) = \alpha q^* - \log q^* + \log(1 - nq^*) + y. \tag{32}$$

Furthermore, with the condition $q_1 = \cdots = q_n = q^*$ and (11), we can see that q^* must satisfy

$$2\alpha q^* + y + \log(1 - nq^*) - \frac{1}{1 - nq^*} - \log q^* = 0.$$

Using the above equation to substitute for αq^* in (32), we have

$$p(q^*) = \frac{1}{2}\left(\frac{1}{1 - nq^*} + \log(1 - nq^*) - \log q^* + y\right) \text{ and } p'(q^*) = \frac{2nq^* - 1}{2q^*(1 - nq^*)^2}.$$

From the preceding expression, it can be seen that the behavior of $p(q^*)$ depends on the sign of $2nq^* - 1$. When $q^* \leq 1/2n$, $p(q^*)$ decreases in α. When $q^* \geq 1/2n$, $p(q^*)$ increases in α.

By Theorem 3.1, q^* monotonically increases in $\alpha \in [0, \hat{\alpha}]$. Thus if $q^* \geq 1/2n$ at $\alpha = 0$, then $p(q^*)$ increases in α; if $q^* \leq 1/2n$ at $\alpha = \hat{\alpha}$, then $p(q^*)$ decreases in α; if $q^* = 1/2n$ at $\alpha \in (0, \hat{\alpha})$, then $p(q^*)$ first decreases and then increases in α. This completes the proof of part (a).

Next we prove part (b). When $\alpha > \hat{\alpha}$, we have $q_1^* > q_2^* = \cdots = q_n^*$ by Theorem 3.1. Also, by (7), we have

$$p_1^* = \alpha q_1^* - \log q_1^* + \log(1-s^*) + y, \quad p_2^* = \alpha q_2^* - \log q_2^* + \log(1-s^*) + y, \tag{33}$$

where $s^* = q_1^* + (n-1)q_2^*$. Thus,

$$p_1^* - p_2^* = \alpha(q_1^* - q_2^*) - (\log q_1^* - \log q_2^*) = -\alpha(q_1^* - q_2^*) < 0,$$

so, $p_1^* < p_2^* = \cdots = p_n^*$. Similar to part (a), by (11), we have

$$2\alpha q_1^* + \log(1-s^*) + y - \frac{1}{1-s^*} - \log q_1^* = 0, \quad 2\alpha q_2^* + \log(1-s^*) + y - \frac{1}{1-s^*} - \log q_2^* = 0.$$

Substituting for αq_1^* and αq_2^* in (33), we obtain

$$p_2^* = \frac{1}{2}\left(\frac{1}{1-s^*} + \log(1-s^*) + y - \log q_2^*\right), \quad p_1^* = \frac{1}{2}\left(\frac{1}{1-s^*} + \log(1-s^*) + y - \log q_1^*\right).$$

It follows that p_2^* increases in α because $\log q_2^*$ decreases in α by Theorem 3.1, s^* increases in α by Proposition 3.4, and $f(s) = \frac{1}{1-s} + \log(1-s)$ increases in s. By Theorem 3.1, $\lim_{\alpha\to\infty} s^* = \lim_{\alpha\to\infty} q_1^* = 1$. Consequently, $\lim_{\alpha\to\infty} p_1^* = \infty$. Therefore, part (b) is proved. □

Proof of Continuity of q^* in α for $n \leq 2$. We prove the $n = 1$ scenario first. When $n = 1$, the problem becomes a one-variable optimization problem. The objective function is

$$\pi(q) = \alpha q^2 + q\big(y + \log(1-q)\big) - q\log(q).$$

The function π is jointly continuous in q and α. By Corollary A4.8 of Kreps (2012), to prove the continuity of q^* in α it is sufficient to establish that for each α there is a unique q that maximizes $\pi(q)$.

For fixed α, observe that $\pi(q)$ is continuous in q and $\pi'(0) > 0$ and $\pi(1) = -\infty$, so $\pi(q)$ has at least one maximizer on $(0, 1)$. Hence, the argument will be complete if we can show that for fixed α, there is at most one maximizer of $\pi(q)$. We do this next by showing that for fixed α, there is at most one local maximizer of $\pi(q)$.

The first order and the second order derivatives of π are:

$$\pi'(q) = 2\alpha q + y + \log(1-q) - \frac{1}{1-q} - \log(q), \quad \pi''(q) = 2\alpha - \frac{1}{q(1-q)^2}.$$

It is easy to prove that $\frac{1}{q(1-q)^2}$ achieves its minimum at $q = 1/3$ and the minimal value is $\frac{27}{4}$. Therefore if $\alpha < 27/8$, then $\pi''(q) < 0$ always holds and $\pi(q)$ is strictly concave. Thus, there is at most one local maximum when $\alpha < 27/8$.

We next consider $\alpha \geq 27/8$. In that case we have (recall that $y \geq 0$)

$$\pi'(q) = 2\alpha q + y + \log(1-q) - \frac{1}{1-q} - \log(q) \geq \frac{27}{4}q + \log(1-q) - \frac{1}{1-q} - \log(q).$$

Let $f(q) = \frac{27}{4}q + \log(1-q) - \frac{1}{1-q} - \log(q)$ denote the expression on the right side of the above inequality. Observe that $f(1/3) = 3/4 + \log 2 > 0$ and $f'(q) = \frac{27}{4} - \frac{1}{q(1-q)^2} \leq 0$ on $(0, 1/3]$. Hence, $f(q) > 0$ for $q \in (0, 1/3]$. Therefore, $\pi'(q) > 0$ for $q \in (0, 1/3]$.

For $q \in [1/3, 1)$, recall from above that $\pi''(q) = 2\alpha - \frac{1}{q(1-q)^2}$. It is easy to prove that $\pi''(q) > 0$ on $[1/3, q')$ and $\pi''(q) < 0$ on $(q', 1)$, where q' is the unique solution to $2\alpha = \frac{1}{q(1-q)^2}$ on $[1/3, 1)$. Therefore $\pi'(q)$ will either first increase and then decrease, or strictly decrease on $[1/3, 1)$. Combining this with the discussion for $q \in (0, 1/3)$, we see that there is at most one point where $\pi'(q) = 0$ on $(0, 1)$. Therefore there is at most one local maximum.

We have established that for fixed α, there is at most one local maximum of $\pi(q)$. Hence, we are done for $n = 1$.

Next we consider the $n = 2$ scenario. By Theorem 3.1, we have $\hat{\alpha} = \alpha^R$. For $\alpha = \hat{\alpha}$, (11) implies that $\boldsymbol{q}^* = (q^*, q^*)$ must satisfy

$$2\hat{\alpha}q^* + \log(1 - 2q^*) + y - \frac{1}{1 - 2q^*} - \log q^* = 0. \tag{34}$$

From the definition of α^R, we have $y = 1/(\hat{\alpha} - 1) - \log(2\hat{\alpha} - 2)$. Now we claim that the unique solution to (34) is $q^* = 1/(2\hat{\alpha})$.

First, it is easy to see that $1/(2\hat{\alpha})$ is indeed a solution to (34). Next we show that the left hand side of (34) is strictly decreasing in q^*, thus the solution must be unique. Let $l(q) = 2\hat{\alpha}q + \log(1 - 2q) - \frac{1}{1-2q} - \log q$. We have

$$l'(q) = 2\hat{\alpha} - \frac{1}{q(1-2q)} - \frac{2}{(1-2q)^2} < 2\hat{\alpha} - \frac{1}{q(1-2q)} \leq 2(\hat{\alpha} - 4)$$

where the last inequality is because $q(1 - 2q) \leq 1/8$.

Now it remains to show that $\hat{\alpha} \leq 4$. We note that given y, the function $R(\alpha)$ is increasing in α, therefore, $\hat{\alpha}$ is decreasing in y. Furthermore, when $y = 0$, $R(4) = \log 6 - 1/3 > 0$, therefore, it must hold for all y that $\hat{\alpha} < 4$.

Finally, by Lemma C.2, $q_H(d) = \frac{de^{2\alpha d}}{e^{2\alpha d} - 1}$ and $q_L(d) = \frac{d}{e^{2\alpha d} - 1}$ when $d > 0$. Note that $\lim_{d \to 0} q_H(d) = \lim_{d \to 0} q_L(d) = 1/(2\alpha)$, which is the same as q^* at $\alpha = \hat{\alpha}$. Therefore the continuity is proved. □

D Proofs for Section 4

Proof of Lemma 4.1. Consider problem (P0), it is easy to see that at optimal $\boldsymbol{q}^\dagger$ we must have $1 - \sum_{j=1}^n q_j^\dagger > 0$ and $q_i^\dagger > 0$ for all $i \in \mathcal{N}$. Therefore $\boldsymbol{q}^\dagger$ must satisfy the first-order necessary condition, i.e.,

$$\frac{\partial \pi}{\partial q_i} = \frac{2\alpha q_i}{\gamma_i} + \frac{1}{\gamma_i}\log\left(1 - \sum_{j=1}^n q_j\right) - \frac{\sum_{j=1}^n q_j/\gamma_j}{1 - \sum_{j=1}^n q_j} + \frac{y_i}{\gamma_i} - \frac{1}{\gamma_i} - \frac{\log q_i}{\gamma_i} = 0.$$

Thus (15) follows. □

Proof of Lemma 4.2. We have $h_\alpha''(q) = 1/q^2 > 0$. Thus $h_\alpha(q)$ is a convex function and achieves its minimal value $h_\alpha(q) = 1 + \log(2\alpha)$ at $q = 1/2\alpha$. Furthermore, we know that $h_\alpha'(q) = 2\alpha - 1/q$. Therefore $h_\alpha(q)$ decreases on $(0, 1/2\alpha)$ and increases on $(1/2\alpha, \infty)$. The lemma is thus proved. □

Proof of Proposition 4.3. We prove the result by contradiction. Suppose there exists an optimal solution $\boldsymbol{q}'$ in which $q_i' = \overline{q}_i^{C_i}$, $q_j' = \overline{q}_j^{C_j}$ for some $i, j \in \mathcal{N}$. Consider $\boldsymbol{q}^\epsilon$ where $q_i^\epsilon = q_i' + \epsilon$ and $q_j^\epsilon = q_j' - \epsilon$, while all the other entries remain the same as $\boldsymbol{q}'$. Define $\Delta(\epsilon) = \pi(\boldsymbol{q}^\epsilon) - \pi(\boldsymbol{q}')$. When ϵ is sufficiently small, $\boldsymbol{q}^\epsilon$ is still feasible. Because $\boldsymbol{q}'$ is optimal, $\epsilon = 0$ should be a local maximizer of $\Delta(\epsilon)$. Thus $\epsilon = 0$ should satisfy the first- and second-order necessary conditions. Taking the second-order derivative of $\Delta(\epsilon)$, we obtain $\Delta''(0) = \frac{1}{\gamma_i}(2\alpha_i - 1/q_i') + \frac{1}{\gamma_j}(2\alpha_j - 1/q_j')$. Since $q_i' = \overline{q}_i^{C_i}$ and $q_j' = \overline{q}_j^{C_j}$, we have $q_i' > 1/2\alpha_i$ and $q_j' > 1/2\alpha_j$ by Lemma 4.2. Hence $\Delta''(0) > 0$, indicating $\boldsymbol{q}'$ is not optimal. Thus we reach a contradiction and the proposition holds. □

Proof of Proposition 4.4. The objective function (17) is symmetric in $(q_1, \ldots, q_n)$ except for the first term $\sum_{j=1}^n \alpha_j q_j^2$. Therefore, $q_1^\dagger > \cdots > q_n^\dagger$ because $\alpha_1 > \cdots > \alpha_n$.

Any optimal solution for (P3) must be an interior point, and hence the first-order optimality conditions are necessary. The first-order conditions are

$$\frac{\partial \pi}{\partial q_i} = 2\alpha_i q_i - \log q_i + y + \log\left(1 - \sum_{j=1}^n q_j\right) - \frac{1}{1 - \sum_{j=1}^n q_j} = 0 \text{ for all } i \in \mathcal{N}$$

and hence (18) follows.

For part 2, Proposition 4.4 part 1 and Lemma 4.2 with $\alpha = \alpha_1$ imply $C \geq 1 + \log(2\alpha_1)$. In addition, recall that $\alpha_1 > \cdots > \alpha_n$. For each $i \geq 2$, we have $\alpha_1 > \alpha_i$ and $h_{\alpha_1}(q) > h_{\alpha_i}(q)$ for all $q > 0$. By part 1, $h_{\alpha_1}(q_1^\dagger) = h_{\alpha_i}(q_i^\dagger) = C$. Suppose for a contradiction that $q_i^\dagger = \overline{q}_i^C$. Then we have $q_i^\dagger > 1/2\alpha_i > 1/2\alpha_1$. By Proposition 4.4 part 1, $q_1^\dagger \geq q_i^\dagger$, and because $h_{\alpha_i}(q)$ increases on $q > 1/2\alpha_i$, then $h_{\alpha_1}(q_1^\dagger) > h_{\alpha_i}(q_1^\dagger) \geq h_{\alpha_i}(q_i^\dagger) = C$ which contradicts $h_{\alpha_1}(q_1^\dagger) = C$. This completes the proof of part 2.

For part 3, from (5), we know that $p_i^\dagger - p_j^\dagger = \alpha_i q_i^\dagger - \alpha_j q_j^\dagger - (\log q_i^\dagger - \log q_j^\dagger)$ for any $i \neq j$. And from Proposition 4.4 part 1, we know that $\log q_i^\dagger - \log q_j^\dagger = 2(\alpha_i q_i^\dagger - \alpha_j q_j^\dagger)$. Thus $p_i^\dagger - p_j^\dagger = \alpha_i q_i^\dagger - \alpha_j q_j^\dagger - 2(\alpha_i q_i^\dagger - \alpha_j q_j^\dagger) = -(\alpha_i q_i^\dagger - \alpha_j q_j^\dagger)$. Since $q_i^\dagger > q_j^\dagger$ and $\alpha_i > \alpha_j$ for $i < j$, it follows that $\alpha_i q_i^\dagger > \alpha_j q_j^\dagger$ and therefore $p_i^\dagger < p_j^\dagger$. □

Proof of Proposition 4.5. Similarly, (20) follows from the first-order condition. Next we prove $q_1^\dagger > q_2^\dagger > \cdots > q_n^\dagger$ by contradiction. Suppose $\boldsymbol{q} = (q_1, q_2, \ldots, q_n)$ is an optimal solution. In the following, we show that $q_1 > q_2$, the rest will follow from exactly the same argument. We consider another solution $\tilde{\boldsymbol{q}}$ such that $\tilde{q}_1 = q_2$, $\tilde{q}_2 = q_1$, and $\tilde{q}_i = q_i$ for $i \geq 3$. Now we consider $\pi(\boldsymbol{q}) - \pi(\tilde{\boldsymbol{q}})$, we have

$$\pi(\boldsymbol{q}) - \pi(\tilde{\boldsymbol{q}}) = \left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)\left(\alpha(q_1^2 - q_2^2) + (q_1 - q_2)\left(y + \log\left(1 - \sum_{j=1}^n q_j\right)\right) + q_2 \log q_2 - q_1 \log q_1\right).$$

Since $\boldsymbol{q}$ is optimal, $\pi(\boldsymbol{q}) - \pi(\tilde{\boldsymbol{q}}) \geq 0$, and therefore,

$$\alpha(q_1^2 - q_2^2) + (q_1 - q_2)\left(y + \log\left(1 - \sum_{j=1}^n q_j\right)\right) + q_2 \log q_2 - q_1 \log q_1 \geq 0.$$

Also by (20), $2\alpha q_1 - \log q_1 < 2\alpha q_2 - \log q_2$, therefore $-q_1 \log q_1 < 2\alpha q_1 q_2 - 2\alpha q_1^2 - q_1 \log q_2$. Thus, we must have

$$\begin{aligned} 0 &\leq \alpha(q_1^2 - q_2^2) + (q_1 - q_2)(y + \log(1 - \sum_{j=1}^n q_j)) + q_2 \log q_2 - q_1 \log q_1 \\ &< -\alpha(q_1 - q_2)^2 + (q_1 - q_2)(y + \log(1 - \sum_{j=1}^n q_j)) + (q_2 - q_1)\log q_2 \\ &= (q_1 - q_2)(-\alpha(q_1 - q_2) + y + \log(1 - \sum_{j=1}^n q_i) - \log q_2). \end{aligned}$$

Again by (18), we have

$$\begin{aligned} \alpha(q_2 - q_1) + y + \log(1 - \sum_{j=1}^n q_j) - \log q_2 &= \alpha(q_2 - q_1) + 1 - 2\alpha q_2 + \frac{\gamma_2 \sum_{j=1}^n q_j/\gamma_j}{1 - \sum_{j=1}^n q_j} \\ &= -\alpha(q_1 + q_2) + 1 + \frac{\gamma_2 \sum_{j=1}^n q_j/\gamma_j}{1 - \sum_{j=1}^n q_j} \\ &\geq -\alpha(q_1 + q_2) + 1 + \frac{q_1 + q_2}{1 - \sum_{j=1}^n q_j} \end{aligned} \quad (35)$$

where the last inequality holds because $\gamma_1 < \gamma_2$.

Now it is easy to see that when $1 - \sum_{j=1}^{n} q_j \leq 1/\alpha$ or $\alpha \leq 1$, the right hand side of (35) is positive. Thus $q_1 > q_2$. Now it remains to consider the case when $1 - \sum_{j=1}^{n} q_j > 1/\alpha$ and $\alpha > 1$.

In this case, we rewrite the difference $\pi(\boldsymbol{q}) - \pi(\tilde{\boldsymbol{q}})$ in the following way:

$$\pi(\boldsymbol{q}) - \pi(\tilde{\boldsymbol{q}}) = \left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)\left(\alpha q_1^2 + q_1(y+C) - q_1 \log q_1 - (\alpha q_2^2 + q_2(y+C) - q_2 \log q_2)\right)$$

where $C = \log\left(1 - \sum_{j=1}^{n} q_j\right) > -\log\alpha$. Now define $f(x) = \alpha x^2 + x(y+C) - x\log x$. Next we show that $f(x)$ is strictly increasing in x on $[0,1]$ for any C. If this is the case, in order for $\pi(\boldsymbol{q}) \geq \pi(\tilde{\boldsymbol{q}})$, we must have $q_1 \geq q_2$. By (20), $q_1 \neq q_2$. Consequently, we must have $q_1 > q_2$.

To show $f(x)$ is increasing, we have $f'(x) = 2\alpha x + y + C - 1 - \log x$. Note that this function is convex and achieves minimum on $[0,1]$ at $x = 1/2\alpha$ (remember in this case, $\alpha > 1$). The minimum value of $f'(x)$ is $C + \log 2\alpha + y \geq \log 2 > 0$. Therefore, $f'(x) > 0$ for all x on $[0,1]$. And thus the part 1 is proved. Part 2 follows exactly the same as in proof of Proposition 4.4 part 2. ☐

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