PATHWISE PROPERTIES AND PERFORMANCE BOUNDS FOR A PERISHABLE INVENTORY SYSTEM

WILLIAM L. COOPER

Department of Mechanical Engineering, University of Minnesota, 111 Church Street S.E., Minneapolis, MN 55455
billcoop@me.umn.edu

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We study a perishable inventory system under a fixed-critical number order policy. By using an appropriate transformation of the state vector, we derive several key sample-path relations. We obtain bounds on the limiting distribution of the number of outdates in a period, and we derive families of upper and lower bounds for the long-run number of outdates per unit time. Analysis of the bounds on the expected number of outdates shows that at least one of the new lower bounds is always greater than or equal to previously published lower bounds, whereas the new upper bounds are sometimes lower than and sometimes higher than the existing upper bounds. In addition, using an expected cost criterion, we compare optimal policies and different choices of critical-number policies.

INTRODUCTION

In this paper we study an inventory system for a product with a fixed finite lifetime. An unsold unit of inventory becomes unusable (perishes) if it is still in the system when its lifetime ends. Such units, called "outdates," must be thrown away and cannot be sold. A key performance measure for a perishable inventory system is the number of items that must be discarded in such a manner. Indeed, it is precisely the outdating phenomenon that differentiates perishable inventory systems from nonperishable inventory systems. From a modeling standpoint, the very fact that perishability is explicitly included in a formulation suggests that the outdating process has a significant impact on the inventory system.

The most frequently studied application for perishable inventory models has been the control of inventories of blood products. In this case, the inventory facility is a hospital blood bank. The daily demand for blood is a random quantity that depends on the number of transfusions needed each day. When a patient requires a transfusion, the demand is satisfied from inventory, provided that the blood is there. Thus, it is important, when ordering blood supplies from a regional blood center, that a hospital take into account the key factors that influence its inventory levels. One such influence is the fact that blood products are perishable. Although the lifetime of a unit of inventory depends on the type of blood product in question, a common thread is that such products cannot be used if they remain in stock for too long. The survey paper by Prastacos (1984) provides an overview of the numerous issues involved in blood inventory management. Other examples of perishable inventories include food products such as meat, milk, and produce.

The recursions studied herein can also be used to model a problem faced by commercial airlines. The airlines have the ability to carry packages inside the cargo hold of their airplanes. However, the amount of space available in the hold depends upon the number passengers on the flight. When there are more passengers on a flight, there is more luggage in the hold, and hence there is less space available for cargo. Thus, the shipping capacity is a random quantity. When an airline agrees to ship a parcel, it does not necessarily need to travel on any particular flight. In fact, the airline may agree to ship the package within a specified time window. The length of this window may be measured in integer units, corresponding to the number of scheduled flights between the package's origin and destination within the time window. If the package still has not been shipped by the end of the window, the airline must pay for it to be sent by some other means. Relating this cargo problem to the perishable inventory model, we see that capacity is the analog of demand, the number of flights within the time window is the analog of the product lifetime, and the packages that must be sent by other means are the analog of the outdates. For a different modeling approach and a more detailed description of such cargo problems when there is an entire network of flights, see Kasilingam (1996).

Decision models typically require an expression for the expected number of outdates per unit time to evaluate ordering policies (under some cost structure). However, closed-form expressions for the expected number of outdates have proved to be unobtainable, even for relatively simple ordering policies. Thus, many authors have focused on deriving bounds and approximations. In light of the importance of the number of outdates as a performance measure, we will focus primarily on providing improved bounds on both the expectation and distribution of the number of outdates for a reasonable class of ordering policies.

Optimal dynamic ordering policies in perishable inventory systems are known to be quite difficult to compute, because the state of the system depends not only on the

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amount of inventory being held, but also on the ages of the inventory. For some insights on optimal policies, refer to Fries (1975), Nahmias (1975), and Nandakumar and Morton (1993). The complexity of optimal policies, as well as the difficulties involved in computing them, has led many authors to analyze heuristic methods for controlling perishable inventories. One such method, proposed by Chazan and Gal (1977), Cohen (1976), and Nahmias (1976), is the fixed-critical number order policy, in which orders are placed so that the total amount of inventory is the same at the end of each time period, regardless of the ages of the inventory. Computational studies by Nahmias (1976, 1977) and Nandakumar and Morton (1993) show that under a variety of different assumptions, such fixed-critical-number policies can be quite good when compared with other methods, as well as to optimal policies. In addition, these policies provide a good baseline against which other types of policies can be compared.

For nonperishable inventory systems, it is well known that under fairly general cost structures, there are optimal policies of this type (i.e., order so that inventory is brought up to a fixed number each period). See Porteus (1990) for an overview of such results. As is the case with nonperishable systems, fixed-critical number policies perform best for perishable inventory systems when there are not large fixed per-order costs. This is indeed the case in the studies mentioned above; they consider a situation where ordering costs are a linear function of the number of items ordered. In cases where there are significant charges for placing an order, it may be more appropriate to use an (s, S)-type heuristic (see Nahmias 1978). For more discussion of other types of heuristics and extensive reviews of the perishable inventory literature, see Nahmias (1982) or Prastacos (1984). For some of the earliest work in perishable inventory theory, see Van Zyl (1964).

The above-cited computational studies have analyzed heuristics for choosing the best fixed-critical number policy, and have compared the performance of this policy to that of other types of policies. This paper will have a different emphasis. Fixed-critical number policies have already been shown to be good in many cases; therefore, we will focus primarily on descriptive analyses (although we will also present numerical comparisons among various critical-number policies and optimal policies). We study the pathwise and limiting properties of the Markov chain induced by such a policy. The complications introduced both by the multidimensionality of the state vector and by the perishability of the inventory render the state recursions interesting in their own right. The analysis, in turn, yields a number of bounds for various performance measures, some of which improve previously published bounds.

The basic building blocks of this study are several pathwise relations satisfied by the outdating process and the state vector of inventories. With these, we obtain a new family of bounds on the expected long-run number of outdates per time period when a fixed-critical number order policy is employed. Bounds and approximations for the expected long-run number of outdates per unit time have been used in newsboy-type formulations to choose the best critical level (see Nahmias 1976, 1977; Nandakumar and Morton 1993). At least one of the new lower bounds on the expectation is tighter than lower bounds appearing in the literature, whereas the new upper bounds on the expectation are sometimes better and sometimes worse than those in the literature. In addition, the proof techniques provide insight into the intuition behind some of the existing results. Also, we briefly compare optimal policies and various critical-number policies, where the choice of critical number is based on different expressions for the the number of outdates.

We derive bounds on the limiting distribution of the number of outdates in a period. Using these, we compute an upper bound on the variance of the stationary number of outdates per period. Such bounds could prove useful when trying choose the best possible fixed-critical-number policy subject to a quality-of-service constraint. The works cited above have focused on optimality criteria involving the expected value of the difference between revenues and costs.

The rest of the paper is organized as follows: §1 introduces the model and provides a key state-space transformation; §2 includes a number of pathwise relationships satisfied by the outdate process; §3 contains bounds on the distribution of the age of the oldest and youngest inventory in the system as well as a discussion of the relationship between perishable and nonperishable systems; §4 presents a family of upper and lower bounds on the expected long-run number of outdates per unit time; §5 contains a family of refined bounds on the limiting distribution of outdates, along with corresponding bounds on the expectation and variance; §6 compares the new bounds to previously published results; §7 describes methods for choosing the critical number, and compares the resulting choices to optimal policies; and §8 provides a brief summary.

1. THE MODEL

Throughout we assume that the fixed-critical number-order policy is to order up to a nonnegative integer m, so that at the end of each period there are exactly m units in inventory. In addition, we will take the fixed lifetime of a unit of inventory to be $n \geqslant 2$ time units. A nonnegative, integer-valued amount of demand D_t arrives at the beginning of each time unit. Assume that $\{D_t\}$ is a sequence of independent, identically distributed random variables, independent of the initial state, with distribution G. Let $\overline{G}(d) = 1 - G(d)$. We denote by G^{*k} the k-fold convolution of G, and we define $\overline{G}^{*k}(d) = 1 - G^{*k}(d)$.

We will describe the state of the system at the end of time period t by the n-dimensional vector $X_t = (X_t^1, \dots, X_t^n)$, where X_t^i is the amount of inventory of age i in the system at time t. In each period $t = 1, 2, \dots$, events occur in the following order: (1) Demand D_t arrives and is satisfied from the current inventory, using oldest inventory first.

Any portion of the demand in excess of m cannot be satisfied and is lost. (2) The unsold portion of X_{t-1}^n (if there is any) perishes. This quantity is the number of outdates in period t. (3) An order is placed so that the total level of inventory in the system is brought back to m. The amount ordered is assumed to arrive immediately. (4) All inventories age one time unit. Note that this means the just-ordered inventory is now considered to be age 1.

The vector of inventories at the end of period t is X_t. Formally, X_{t-1} and D_t determine X_t via the recursion

$$X_t^1 = m - \sum_{i=2}^{n} X_t^i, \tag{1}$$

$$X_t^i = \left[X_{t-1}^{i-1} - \left(D_t - \sum_{j=i}^{n} X_{t-1}^j \right)^+ \right]^+, \qquad i = 2, \ldots, n. \tag{2}$$

Here $a^+ = \max\{a, 0\}$. This model was proposed by Chazan and Gal (1977), and an equivalent model with a slightly different state space was studied independently by Cohen (1976). From the recursions, it follows that $X = \{X_t\}$ forms a Markov chain with state space $S = \{x \in \mathbb{Z}_+^n : \sum_{i=1}^n x_i = m\}$ where $\mathbb{Z}_+$ denotes the nonnegative integers. Note that S has $\binom{m+n-1}{n-1}$ elements (see, e.g., Chapter 1, Proposition 6.2 in Ross 1998). For example, when $n = 20$ and $m = 100$ (the values assumed in the examples of §6), the cardinality of S is approximately 4.9×10^{21}; this shows why direct calculation may be impossible and why easily computable bounds are important.

For $1 \leqslant i \leqslant n$, we define the n-dimensional vector $m^{(i)} = (m_1^{(i)}, \ldots, m_n^{(i)})$ given by $m_i^{(i)} = m$ and $m_j^{(i)} = 0$ for $j \neq i$. These states, in which all inventory is the same age, will play an important role later.

A major quantity of interest is the number of outdates that occur at time $t \geqslant 1$, defined by

$$Z_t \equiv (X_{t-1}^n - D_t)^+. \tag{3}$$

We assume throughout that $P(D_1 \geqslant m/n) > 0$. Under this assumption, it can be shown that X has a single absorbing, aperiodic, positive-recurrent class and possibly some transient states. Thus, there exists a unique stationary distribution π, satisfying

$$\pi = \pi \mathbb{P}, \tag{4}$$

where $\mathbb{P}$ is the transition matrix of the Markov chain X. We will denote by X_∞ the random variable for which

$$P(X_\infty = x) = \lim_{t \to \infty} P(X_t = x) = \pi(x).$$

Furthermore, it follows from (3) that there exists a random variable Z_∞, independent of X_0, such that

$$P(Z_\infty = z) = \lim_{t \to \infty} P(Z_t = z) = P((X_\infty^n - D)^+ = z).$$

We also define the cumulative number of outdates up to and including time t by

$$W_t \equiv \sum_{s=1}^{t} Z_s. \tag{5}$$

In the subsequent section, we will study the long-run number of outdates per time period,

$$\mu \equiv \lim_{t \to \infty} t^{-1} W_t.$$

In light of the previous discussion, the above limit holds with probability one. Furthermore, $\mu = \lim_{t\to\infty} E[t^{-1} W_t]$, and $\mu = EZ_\infty$. By standard properties of Markov chains, it follows that if X_0 has distribution π, then X is stationary; consequently $EZ_t = \mu$, and Z_t has the same distribution as Z_∞ for all $t \geqslant 1$. Later we will provide computable bounds for μ, because direct computation is not viable in view of the size of S.

In addition to the outdates, there are a number of other performance measures of interest, such as orders, sales, lost sales, and the number of items held in inventory. The structure of a fixed-critical number policy makes most of these easy to compute once there is an expression for the number of outdates; we shall address this issue in more detail in §8.

We end this section by introducing a transformation of the state vector that will allow us to derive a number of important sample-path relations for the outdate process $\{Z_t\}$. Consider the n-dimensional process $\{Y_t\}$, where the quantity Y_t^i represents the total inventory of age i or greater when the state of the system is X_t; formally,

$$Y_t^i \equiv \sum_{j=i}^{n} X_t^j, \qquad i = 1, \ldots, n. \tag{6}$$

The state space of Y is $S' = \{y \in \mathbb{Z}_+^n : 0 \leqslant y_n \leqslant y_{n-1} \leqslant \cdots \leqslant y_1 = m\}$. In an abuse of notation, we say that $Y_t = m^{(i)}$ when $Y_t^j = m$ for $j \leqslant i$ and $Y_t^j = 0$ for $j > i$ (i.e., when $X_t = m^{(i)}$). Because X_t converges in distribution to X_∞, we can also talk about the limiting random variable Y_∞, defined in the obvious way. In the remainder of the paper we will switch freely between X and Y, using whichever state representation is more convenient.

LEMMA 1. *The process Y satisfies the recursion*

$$Y_t^1 = m, \tag{7}$$

$$Y_t^i = \left[Y_{t-1}^{i-1} - \max\{D_t, Y_{t-1}^n\} \right]^+, \qquad i = 2, \ldots, n. \tag{8}$$

We omit the proof, which requires the rather tedious verification of several cases. Recalling our assumptions on $\{D_t\}$, it can be seen from (7) and (8) that Y is a Markov chain. The relation (8) is Equation (7), in Cohen (1976).

Using induction (on j) and the fact that if $a, b \in \mathbb{R}$ and $b \geqslant 0$ then $[a^+ - b]^+ = [a - b]^+$, we can now obtain from (7) and (8) the following key result, which is needed in many proofs.

LEMMA 2. *The process Y satisfies the relation*

$$Y_t^k = \left[Y_{t-j}^{k-j} - \sum_{s=t-j+1}^{t} \max\{D_s, Y_{s-1}^n\} \right]^+,$$
$$1 \leqslant j \leqslant \min\{k-1, t\}, \quad 2 \leqslant k \leqslant n. \tag{9}$$

The relationships in Lemma 2 have a simple interpretation. The amount of inventory at time t of age k or greater is the amount of inventory at time $t-j$ of age $k-j$ or greater, minus what has left the system since time $t-j$. Inventory leaves the system in two possible ways: It is taken by demand or it outdates; therefore, the amount that leaves the system at time s is the maximum of D_s and Y_{s-1}^n. When $j=1$, this argument also yields an intuitive explanation for Lemma 1.

2. THE OUTDATE PROCESS

We begin by describing a fundamental sample-path relation satisfied by the outdate process Z. In the interest of brevity, we present the following without proof.

PROPOSITION 1. *The outdate process Z satisfies*

$$Z_t = \left[Y_{t-k}^{n-k+1} - \sum_{s=t-k+1}^{t} D_s - \sum_{s=t-k+1}^{t-1} Z_s \right]^+, \quad k = 1, \ldots, \min\{t, n\}. \tag{10}$$

Observe that (10) shows that Z is not a Markov chain. A variation of (10) appears in Cohen (1976) for the special case in which $k=n$. In such a case $Y_{t-n}^1 = m$ when $t-n \geqslant 0$. Proposition 1 has the following interpretation: The amount of outdates in period t is exactly the amount of inventory that was age at least $n-k+1$ at time $t-k$ and that has not left the system by time t by either outdating or being purchased by some demand. Note that inventory younger than $n-k+1$ at time $t-k$ cannot outdate in or before period t, since it will be younger than n at the end of period $t-1$ if it is still in the system. Adding W_{t-1} to both sides of (10) and letting $k = \min\{t, n\}$ yields

$$W_t = \begin{cases} \max\{W_{t-1}, Y_0^{n-t+1} - \sum_{s=1}^{t} D_s\} \\ \quad \text{if } 1 \leqslant t \leqslant n-1, \\ \max\{W_{t-1}, m + W_{t-n} - \sum_{s=t-n+1}^{t} D_s\} \\ \quad \text{if } t \geqslant n. \end{cases} \tag{11}$$

Next, we will restate a result originally appearing in Chazan and Gal (1977), which is a simple consequence of (11). Hereafter, statements about random variables such as $A \geqslant B$ should be taken to mean that the relation holds for every possible realization of the demand sequence $\{D_t\}$. Such statements involving vectors are taken to mean that the relation holds componentwise. Intuitively, Corollary 1 states that older initial stocks of inventory lead to larger cumulative amounts of outdates up to t, for every time t.

COROLLARY 1 (CHAZAN AND GAL). *Suppose W is the cumulative outdating process when the initial condition is Y_0, and $\overline{W}$ is the cumulative outdating process when the initial condition is $\overline{Y}_0$. If $Y_0 \leqslant \overline{Y}_0$, then $W_t \leqslant \overline{W}_t$ for $t \geqslant 0$.*

The next result describes the pathwise behavior of the outdate process when demand in an n-period interval is large relative to the fixed critical number m. In particular, if the demand in each period always exceeds the ratio m/n, then a typical unit of inventory ordered in period t will always see at least m units of demand before it becomes old enough to perish in period $t+n$. Consequently, after allowing a sufficient amount of time to overcome initial conditions, the number of outdates each period will be zero. The following can be proved by induction on t.

PROPOSITION 2. *Suppose that $P(D_1 < m/n) = 0$, and let Y_0 have an arbitrary distribution on S'. Then*

$$Y_t^j \leqslant \left[m - (j-1)\frac{m}{n} \right]^+, \quad 1 \leqslant j \leqslant \min\{n, t+1\}, \quad t \geqslant 0, \tag{12}$$

and $Z_t = 0$ for $t \geqslant n$.

This result can be viewed as the analog of the result of Chazan and Gal (1977), which gives an exact expression for the expected number of outdates in an n-period span for the special case when $P(D_1 \leqslant m/n) = 1$. In addition, the argument used in the proof of the above can, after minor modifications, be used to demonstrate rigorously that $P(D_1 < m/n) < 1$ is a sufficient condition for the existence of a single aperiodic, positive-recurrent class for the Markov chain X. In the case when $P(D_1 < m/n) = 1$, the Markov chain may be reducible or periodic. When $P(D_1 \leqslant m/n) = 1$, Chazan and Gal show that

$$\mu = m/n - ED_1. \tag{13}$$

Observe that when demand is deterministically equal to m/n (this is the "borderline" case covered by both the result of Chazan and Gal and Proposition 2), this expression gives $\mu = 0$, which agrees with Proposition 2. In fact, neither of these degenerate cases is particularly realistic. Any distribution with support on the entire nonnegative integers falls into neither of the degenerate categories. Furthermore, even when the possible values of demand are limited, a system operator would likely choose m so that we would have positive probability of demand values both above and below m/n. Of course, the choice of m depends on what sorts of goals and costs are involved for the operator. For instance, a cost function that imposes a severe penalty on outdates will lead to a relatively low choice of m. Likewise, a cost function that places high costs on unsatisfied demand will cause the system operator to choose a large value for m.

We conclude this section with a simple bound on the distribution and mean of Z_t. The following statement includes the important special case when $t = \infty$.

PROPOSITION 3. *For $n \leqslant t \leqslant \infty$,*

$$P(Z_t > z) \leqslant G^{*n}(m-1-z), \quad z = 0, \ldots, m-1. \tag{14}$$

Hence,

$$\mu \leqslant \beta^{\dagger} \equiv \sum_{z=0}^{m-1} G^{*n}(m-1-z) = E\left(\left[m - \sum_{s=1}^{n} D_s \right]^+ \right). \tag{15}$$

PROOF. For $t \geqslant n$, Proposition 1 implies

$$Z_t = \left[Y^1_{t-n} - \sum_{s=t-n+1}^{t} D_s - \sum_{s=t-n+1}^{t-1} Z_s\right]^+$$
$$\leqslant \left[m - \sum_{s=t-n+1}^{t} D_s\right]^+. \quad (16)$$

Then (14) holds for finite t, and it holds for $t = \infty$ by letting $t \to \infty$. Since Z_∞ is a nonnegative, integer-valued random variable, $\mu = EZ_\infty = \sum_{z=0}^{\infty} P(Z_\infty > z)$, and so (15) follows from (14). □

We note that a similar result has been obtained, although not specifically pertaining to the limiting distribution of outdates under a fixed-critical number policy, by Nahmias (1976) (see Lemma 2.1 on p. 1,004 and the discussion surrounding it). In §5, we will present refinements of both (14) and (15). Observe that the pathwise inequality (16) is, in general, tight in the sense that there are sample paths for which there is equality. Specifically, if $X^1_{t-n} = m$, then it follows from (1) and (2) that $\sum_{s=t-n+1}^{t-1} Z_s = 0$, giving exactly the right-hand side of (16).

3. THE OLDEST AND YOUNGEST INVENTORY

In this section we derive bounds on the steady-state distribution of the age of the oldest and youngest inventory in the system. These results will be used in subsequent sections in the development of bounds on both the distribution and the expected value of the outdates. First we introduce two functions that will be used throughout the development of the bounds, $\phi : S' \to \{1, \ldots, n\}$, and $\psi : S' \to \{1, \ldots, n\}$ are given by

$$\phi(y) = \max\{i : y_i = m\},$$
$$\psi(y) = \max\{i : y_i > 0\}.$$

Intuitively, $\phi(y)$ represents the age of the youngest inventory in the system when the state is y. Likewise, $\psi(y)$ represents the age of the oldest inventory in the system when the state is y. To simplify future notation, we define

$$\gamma_0 = 0,$$
$$\gamma_i = \overline{G}^{*i}(m-1), \quad i = 1, \ldots, n-1,$$
$$\gamma_n = 1.$$

PROPOSITION 4. *The distributions of the oldest and youngest inventory in the system satisfy*

$$P(\psi(Y_\infty) \leqslant i) \geqslant \gamma_i, \quad 1 \leqslant i \leqslant n-1. \quad (17)$$
$$P(\phi(Y_\infty) \geqslant i) \geqslant G(0)^{i-1}\gamma_{n-i+1}, \quad 2 \leqslant i \leqslant n. \quad (18)$$

The proof of Proposition 4 can be found in the appendix. Statement (17) has the simple interpretation that if, at time t, the cumulative demand over the last i (with $i \leqslant n-1$) periods has been m or more, then there is no inventory left that was in the system at time $t-i$. The bound provided in (18) can be understood as follows. Suppose we have fixed a time t, and suppose that there was no inventory in the system of age greater than or equal to $n-i+2$ at time $t-i+1$. Then, there could have been no outdates in periods $t-i+2, \ldots, t$. If, in addition, demands were zero in periods $t-i+2, \ldots, t$, then no new inventory would have been ordered in periods $t-i+2, \ldots, t$; hence, there can be no inventory with age less than i in the system at time t. It is evident from the proof that (17) and (18) hold for Y_t when t is larger than n. However, we will not require this level of generality for our later results.

We will now briefly discuss a pathwise relationship between perishable and nonperishable inventory systems relevant to the proof of the proposition. Consider the process Y on S', which satisfies the recursion

$$\widetilde{Y}^1_t = m, \quad (19)$$
$$\widetilde{Y}^i_t = \left[\widetilde{Y}^{i-1}_{t-1} - D_t\right]^+, \quad i = 2, \ldots, n. \quad (20)$$

The process $\widetilde{Y}$ is identical to Y except that in $\widetilde{Y}$, inventory that reaches the end of its lifetime does not perish. Rather, it continues to remain at age n until it is consumed by some demand. Equivalently, we could allow inventories to "continue aging" past age n; however, then the process $\widetilde{Y}$ would not be defined on S'. Note that this alternative description of the nonperishable system does not change the recursions (19) and (20) for the range $i = 1, \ldots, n$; it simply requires us to expand the dimension of the state space to get a complete picture of the system. In either case, the quantity $\widetilde{Y}^i_t$ represents the amount of inventory of age greater than or equal to i at time t. For the purposes of a comparison to the perishable system, this is the only range in which we are interested. Nandakumar and Morton (1993) have used the idea of comparing a perishable system with a nonperishable system to obtain bounds on optimal order quantities within a Markov decision process framework.

A consequence of Lemma 1 and expressions (19) and (20) is the following, which relates the paths of the processes Y and $\widetilde{Y}$. Lemma 3 states that, if at some time, the nonperishable system has older inventories than does the perishable system, then the nonperishable system will continue to have older inventories at all subsequent times.

LEMMA 3. *Suppose* $t \geqslant 0$. *If* $Y_t \leqslant \widetilde{Y}_t$, *then* $Y_{t+s} \leqslant \widetilde{Y}_{t+s}$ *for all* $s \geqslant 0$.

Note that Lemma 3 would not be valid for X and $\widetilde{X}$, a process corresponding to a nonperishable system (and defined in the obvious way). We cannot, in general, conclude that $\widetilde{X}$ will dominate X componentwise, regardless of the initial states. Combining (19) and (20) and Lemma 3, we see that inequality (33) in the appendix arises by comparing, in a pathwise sense, the process Y and its analog $\widetilde{Y}$ in which no inventory outdates. Specifically, suppose we take $\widetilde{Y}_{t-i} = Y_{t-i}$; then,

$$Y^{i+1}_t \leqslant \widetilde{Y}^{i+1}_t = \left[\widetilde{Y}^1_{t-i} - \sum_{s=t-i+1}^{t} D_s\right]^+ = \left[m - \sum_{s=t-i+1}^{t} D_s\right]^+.$$

4. BOUNDS ON THE EXPECTED NUMBER OF OUTDATES

In this section we derive families of upper and lower bounds on the average outdates, μ. The basic method used in the derivation of the upper bounds will be to employ the previous section's bound, for each possible age, on the steady-state probability that the oldest item in the system is younger than (or equal to) the age in question. We then consider the process started from steady state, and we condition on the age of the oldest inventory being, say, i. This allows us to obtain a pathwise upper bound on the cumulative outdate process by considering another outdate process started from the state with all m units of inventory of age exactly i. The outdates for this comparison process can be expressed by a simple formula.

A similar method is used in the derivation of the lower bounds. For each i, we use the bound on the steady-state probability that the age of the youngest item will be greater than or equal to i. Starting the process from steady state, we condition on the age of the youngest item in the system and obtain a lower bound by comparing the cumulative outdates in a pathwise sense with those generated by the process that starts with all inventory of age exactly i. As we will see in §6, a simpler variant of this method of pathwise comparison is also the key behind previously published lower bounds on μ.

We are now ready to present the main results of this section. Note that in the following theorem, the bounds include only terms involving convolutions of the demand distribution (and expectations with respect to those convolutions). Furthermore, the highest convolution needed for computation of the bounds is the n-fold convolution. Observe also that the following theorem contains a family of upper and lower bounds.

THEOREM 1. *For $k = 1, \ldots, n$, the long-run number of outdates per period μ satisfies*

$$\mu \leqslant \beta_k^* \equiv \frac{1}{k}\left[\sum_{i=n-k+1}^{n} (\gamma_i - \gamma_{i-1}) E\left(\left[m - \sum_{t=1}^{n-i+1} D_t\right]\right)\right], \quad (21)$$

$$\mu \geqslant \alpha_k^* \equiv \frac{1}{k}\left[\sum_{i=n-k+1}^{n} \left(G(0)^{i-1}\gamma_{n-i+1} - G(0)^i \gamma_{n-i}\right) \times E\left(\left[m - \sum_{t=1}^{n-i+1} D_t\right]^+\right)\right]. \quad (22)$$

PROOF. We prove (21) first. Fix $k \in \{1, \ldots, n\}$, and suppose that Y_0 has the stationary distribution (equivalently, X_0 has distribution π). If the initial state Y_0 is $m^{(i)}$, then we can use (11) to show that

$$W_k = \begin{cases} 0 & \text{if } k < n-i+1 \\ \left[m - \sum_{u=1}^{n-i+1} D_u\right]^+ & \text{if } k \geqslant n-i+1. \end{cases}$$

If $\psi(Y_0) = i$, then $Y_0 \leqslant m^{(i)}$. Therefore, by Corollary 1 and the fact that the demand sequence is independent of the initial state, we have

$$E(W_k | \psi(Y_0) = i) \leqslant 1_{\{k \geqslant n-i+1\}} E\left(\left[m - \sum_{u=1}^{n-i+1} D_u\right]^+\right).$$

So,

$$E(W_k) = \sum_{i=1}^{n} P(\psi(Y_0) = i) E(W_k | \psi(Y_0) = i) \leqslant \sum_{i=1}^{n} P(\psi(Y_0) = i) 1_{\{k \geqslant n-i+1\}} \times E\left(\left[m - \sum_{u=1}^{n-i+1} D_u\right]^+\right). \quad (23)$$

By (17) we see that $\sum_{j=1}^{i} P(\psi(Y_0) = i) \geqslant \gamma_i$ for $i = 1, \ldots, n-1$, because the system is stationary. We can now apply Lemma 4, Part (a) from the appendix to (23) with

$$c_i = 1_{\{k \geqslant n-i+1\}} E\left(\left[m - \sum_{u=1}^{n-i+1} D_u\right]^+\right)$$

$$p_i = P\big(\psi(Y_0) = i\big)$$

$$q_i = \gamma_i$$

to get

$$E(W_k) \leqslant \sum_{i=1}^{n} \left(\gamma_i - \gamma_{i-1}\right) 1_{\{k \geqslant n-i+1\}} E\left(\left[m - \sum_{u=1}^{n-i+1} D_u\right]^+\right) = \sum_{i=n-k+1}^{n} (\gamma_i - \gamma_{i-1}) E\left(\left[m - \sum_{u=1}^{n-i+1} D_u\right]^+\right).$$

The result (21) follows after division by k, since $E(W_k) = k\mu$ when the system is stationary.

Using similar methods, we will now prove (22). Fix k and assume the system is stationary. If $\phi(Y_0) = i$ then $Y_0 \geqslant m^{(i)}$, and hence

$$E(W_k | \phi(Y_0) = i) \geqslant 1_{\{k \geqslant n-i+1\}} E\left(\left[m - \sum_{u=1}^{n-i+1} D_u\right]^+\right).$$

So,

$$E(W_k) = \sum_{i=1}^{n} P(\phi(Y_0) = i) E(W_k | \phi(Y_0) = i) \geqslant \sum_{i=1}^{n} P(\phi(Y_0) = i) 1_{\{k \geqslant n-i+1\}} \times E\left(\left[m - \sum_{u=1}^{n-i+1} D_u\right]^+\right). \quad (24)$$

We now apply Lemma 4 Part (b) to (24) with

$$c_i = 1_{\{k \geqslant n-i+1\}} E\left(\left[m - \sum_{u=1}^{n-i+1} D_u\right]^+\right)$$

$$p_i = P\big(\phi(Y_0) = i\big)$$

$$r_i = G(0)^{i-1} \gamma_{n-i+1}$$

to get

$$E(W_k) \geqslant \sum_{i=1}^{n} \big(G(0)^{i-1}\gamma_{n-i+1} - G(0)^i\gamma_{n-i}\big)1_{\{k\geqslant n-i+1\}} \times E\left(\left[m - \sum_{u=1}^{n-i+1} D_u\right]^+\right)$$
$$= \sum_{i=n-k+1}^{n} \big(G(0)^{i-1}\gamma_{n-i+1} - G(0)^i\gamma_{n-i}\big) \times E\left(\left[m - \sum_{u=1}^{n-i+1} D_u\right]^+\right).$$

Dividing by k completes the proof, because $E(W_k) = k\mu$ since the system is stationary. □

5. REFINED BOUNDS ON THE OUTDATE DISTRIBUTION

In the previous section, we were able to obtain bounds on the long-run average number of outdates per period by conditioning on the ages of the oldest and youngest items in inventory. The bounds on the stationary probabilities of these quantities, embodied by Proposition 4, can also be used to provide a refinement of the bound on the distribution function of Z_∞ (14). This refinement also yields an improvement on the earlier bound on the expectation of Z_∞ given by (15). For $k = n$, we see immediately that (25) improves (14), because $1-\gamma_0 - G(0)\gamma_{n-1} = 1 - G(0)\gamma_{n-1} \leqslant 1$. Likewise, it follows that $\beta_n^\ddagger \leqslant \beta^\dagger$.

THEOREM 2. *For* $k = 1, \ldots, n$,

$$P(Z_\infty > z) \leqslant \big(1-\gamma_{n-k} - G(0)^{n-k+1}\gamma_{k-1}\big)G^{*k}(m-1-z), \quad z = 0, \ldots, m-1. \tag{25}$$

Hence,

$$\mu \leqslant \beta_k^\ddagger \equiv (1-\gamma_{n-k} - G(0)^{n-k+1}\gamma_{k-1}) \times E\left(\left[m - \sum_{t=1}^{k} D_t\right]^+\right). \tag{26}$$

The proof can be found in the appendix. It is also possible to obtain bounds on higher moments of Z_∞ by using (25). For instance, it is straightforward to verify that $E(A^2) = \sum_{a=0}^{\infty}(2a+1)P(A > a)$ when A is a nonnegative, integer-valued random variable. Applying this fact to Z_∞, and then using (25), we obtain

$$E\big(Z_\infty^2\big) \leqslant \delta_k \equiv \big(1-\gamma_{n-k} - G(0)^{n-k+1}\gamma_{k-1}\big) \times \sum_{z=0}^{m-1}(2z+1)G^{*k}(m-1-z). \tag{27}$$

We can now combine (27) with lower bounds on μ from Theorem 1 to obtain an upper bound on the variance of Z_∞.

COROLLARY 2. *Let* $\delta^* = \min\{\delta_1, \ldots, \delta_n\}$ *and* $\alpha^* = \max\{\alpha_1^*, \ldots, \alpha_n^*\}$. *Then*

$$\mathrm{Var}(Z_\infty) \leqslant \delta^* - (\alpha^*)^2. \tag{28}$$

The results of this section may potentially be useful when evaluating fixed-critical number policies within the context of a constrained optimization problem. Previous work in perishable inventory theory has largely focused on choosing ordering policies so as to minimize expected costs. In light of this, a direction for future work will be to study the effect of including probability or variance constraints. The inclusion of such constraints may be more important for the cargo problem, where outdates may be the cause of customer dissatisfaction, which is typically difficult to measure in terms of a cost function.

6. COMPARISON OF THE BOUNDS

In this section, we compare the new bounds with those existing in the literature. Chazan and Gal (1977) obtained the result that $\mu \geqslant \alpha$, where $\alpha = n^{-1}m - E\min\{n^{-1}\sum_{t=1}^{n} D_t, n^{-1}m\}$. After some routine algebra, this lower bound can be written as

$$\alpha = n^{-1}E\left[m - \sum_{t=1}^{n} D_t\right]^+. \tag{29}$$

To see that Theorem 1 provides a tighter lower bound, note that

$$\alpha_n^* - \alpha = \frac{1}{n}\left[\sum_{i=1}^{n}\big(G(0)^{i-1}\gamma_{n-i+1} - G(0)^i\gamma_{n-i}\big) \times \left(E\left(\left[m - \sum_{t=1}^{n-i+1} D_t\right]^+\right) - E\left(\left[m - \sum_{t=1}^{n} D_t\right]^+\right)\right)\right].$$

This quantity is nonnegative because $\gamma_{n-i+1} \geqslant G(0)\gamma_{n-i}$ and $[m - \sum_{t=1}^{n-i+1} D_t]^+ \geqslant [m - \sum_{t=1}^{n} D_t]^+$ for $i = 1, \ldots, n$. The difference will be strictly positive if $G(0) \neq 0$ and there is an i for which $\gamma_{n-i+1} > G(0)\gamma_{n-i}$ and $E([m - \sum_{t=1}^{n-i+1} D_t]^+) > E([m - \sum_{t=1}^{n} D_t]^+)$. These conditions are met by any distribution that has positive support on $\{0, 1, \ldots, m\}$. We also note that the method of proof in Theorem 1 yields and interpretation of the lower bound of Chazan and Gal. We start the process according to the stationary distribution as in the proof and then note that $y \geqslant m^{(1)}$ for any initial state y. Thus, the cumulative number of outdates up to time n when the process starts in $m^{(1)}$ provides a pathwise lower bound for the cumulative outdates up to time n for any initial state. Of course, the cumulative outdates up to time n when $Y_0 = m^{(1)}$ are exactly $[m - \sum_{t=1}^{n} D_t]^+$. Taking expectations then yields the result. The proof presented by Chazan and Gal is based on a similar idea.

In addition, Chazan and Gal (1977) show that $\mu \leqslant \beta$, where $\beta = n^{-1}m - E\min\{D_1, n^{-1}m\}$. Rearranging, we see that

$$\beta = n^{-1}E[m - nD_1]^+.$$

Table 1. Numerical comparison of bounds.

Distribution	Mean	α^*	β^*	α	β	α-PI	β-PI
All-or-None(100,0.975)	2.5	3.32288	4.44482	3.01344	4.87500	4.88	4.62
All-or-None(100,0.950)	5.0	2.43499	3.49524	1.79243	4.75000	15.20	15.22
All-or-None(100,0.925)	7.5	1.79836	2.54986	1.05149	4.62500	26.21	28.92
Geometric	2.5	2.50028	4.87487	2.50026	2.96484	0.00	−24.36
Geometric	5.0	0.49668	4.38329	0.48650	2.00939	1.03	−37.13
Geometric	7.5	0.04059	0.67080	0.03752	1.51119	3.92	38.51
Poisson	2.5	2.50000	4.87500	2.50000	2.56195	0.00	−31.10
Poisson	5.0	0.19962	3.97758	0.19931	0.87734	0.08	−63.86
Poisson	7.5	0.00000	0.00002	0.00000	0.21672	0.35	100.00
Uniform(0,5)	2.5	2.50000	4.87500	2.50000	2.50000	0.00	−32.20
Uniform(0,10)	5.0	0.28766	4.54020	0.28270	1.36364	0.87	−53.81
Uniform(0,15)	7.5	0.00257	0.04359	0.00232	0.93750	5.10	91.11

There are cases for which $\beta_k^* < \beta$ for each k. It can be proven that one such example occurs when the demand distribution is given by

$$P(D=d) = \begin{cases} p & \text{if } d=0, \\ 1-p & \text{if } d=m, \\ 0 & \text{otherwise,} \end{cases} \tag{30}$$

where $0 < p < 1$. This situation will arise if demand is nearly deterministic in the sense that with a very high probability it is m, but in some very rare cases it fails to materialize. In such cases it is reasonable that a system operator would choose a critical level of m, since this would exactly meet the demand nearly all of the time. Alternatively, if p is large, then demand is usually zero, except in some relatively rare cases, when it is m. If there are significant lost-sales costs, the operator may again choose a critical level of m. Note also that this case is easy to analyze, because the only possible states for the process X are $m^{(i)}, i=1,\ldots,n$; this allows explicit computation of the stationary distribution. Note that slight perturbations of G can yield the same qualitative all-or-nothing behavior but will destroy the simple structure that allows us to write down π explicitly. Observe that it is not surprising that the upper bounds from Theorem 1 do well in this case, because the bounds are derived from comparisons with initial states $m^{(i)}$. For this demand distribution, these are the only possible states for X.

There are many examples for which $\beta_k^* > \beta$ for all k. In light of this, it is best to compute all of the upper bounds and use the smallest of them. In cases where demand is "frequently" smaller than m/n, the bound β is quite good. An analysis of Chazan and Gal's clever derivation of β shows why this is true. The argument involves truncating each demand at m/n (i.e., replacing each D_t by $\min\{m/n, D_t\}$), and then using the exact formula for μ given by (13). This exact formula gives an upper bound on the mean number of outdates for the original system, because cumulative outdating is pathwise monotone nonincreasing in demand. When demand is not above m/n too often (and not too far above m/n when it is above m/n), then this truncation does not entail much of a loss. Thus, the expression (13) is almost the average number of outdates for the untruncated demand stream. Conversely, when demand is "frequently" larger than m/n, the truncation causes a significant change in the demand stream. In these cases, (13) may not give a good approximation to the untruncated system.

Table 1 gives a numerical comparison of the bounds obtained above and those provided in Chazan and Gal (1977) for a number of different distributions with $m=100$ and $n=20$. We will use the best choices of new upper and lower bounds in the comparison. Define

$$\beta^* = \min\{\beta_1^*, \ldots, \beta_n^*, \beta_1^\ddagger, \ldots, \beta_n^\ddagger\},$$
$$\alpha^* = \max\{\alpha_1^*, \ldots, \alpha_n^*\}.$$

For each distribution, we have computed the bounds for three separate mean demand values: 2.5, 5.0, and 7.5. These were chosen to reflect cases when demand is less than, equal to, and greater than m/n. Observe that the numerical results support the intuitive explanation given above of what conditions lead to β^* being better than β, and what conditions lead to β being better than β^*. All-or-None(a, p) refers to the distribution where $P(D=0)=p$ and $P(D=a)=1-p$.

We will use α-percentage improvement (α-PI) and β-percentage improvement (β-PI) as a measurement of the relative amount of improvement in the new bounds. Define

$$\alpha-\text{PI} = 100 \times \left(\frac{\alpha^*-\alpha}{\alpha^*+\alpha}\right),$$
$$\beta-\text{PI} = 100 \times \left(\frac{\beta-\beta^*}{\beta+\beta^*}\right).$$

Note each percentage improvement is a number between -100 and 100. Positive values mean that the new bound is tighter than the bounds previously available in the literature. Larger positive numbers reflect larger relative improvements. Similarly, negative values indicate that the previously available bounds are tighter than the new bounds. For distributions that yield very few outdates, these measures may not be particularly meaningful, because the absolute difference in the bounds will be very small.

7. CHOOSING THE CRITICAL NUMBER

We next describe how to use the results of the previous sections to choose the critical number m, and compare these choices of m with other choices based on different expressions for the number of outdates. In addition, we compare the resulting policies to finite-horizon-expected-cost optimal policies, i.e., policies that minimize $v \equiv E[\sum_{t=1}^{\tau} C_t]$, where C_t is the cost incurred in decision epoch t and τ is the end of the planning horizon. We do not consider problems with discounting or infinite time horizons here.

When making a choice of m, it will be more convenient to attack the equivalent problem of minimizing v/τ. We consider choices made using two different approximations for the expected number of outdates per period: One approximation will be based on $\alpha^*(m)$ and $\beta^*(m)$ (the parenthetical m indicates the dependence upon m), the other method uses Chazan and Gal's bounds $\alpha(m)$ and $\beta(m)$. The primary conclusions of the computational study described below are that (1) in most cases both sets of bounds lead to the same choice of critical level, and (2) the critical number policy yields an expected cost very close to that of the optimal policy. However, there are cases in which the different bounds yield extremely different answers. The study adds to the mounting body of computational evidence that fixed-critical number ordering policies perform quite well for a wide variety of demand distributions. Other published studies, cited earlier, have also presented similar results under a variety of assumptions.

We denote the quantity ordered in decision epoch t by Q_t, the amount of lost sales in epoch t by L_t, the number of units held through epoch t by H_t, and the outdates during epoch t by Z_t. There are linear ordering, outdating, lost sales, and holding costs, given by (respectively) c_q, c_z, c_l, and c_h. The cost incurred at "decision epoch" $t \in \{1, \ldots, \tau\}$ is given by

$$C_t \equiv c_q Q_t + c_z Z_t + c_l L_t + c_h H_t. \tag{31}$$

Note that, for reasons to be made evident shortly, we are making a distinction between a time period and decision epoch.

For critical-number policies, we have that $Q_t = X_{t-1}^1 = \text{Sales} + \text{Outdates} = (D_{t-1} - L_{t-1}) + Z_{t-1}$, $L_t = (D_t - m)^+$, and $H_t = m - \text{Sales}$. We have chosen these definitions of Q_t and D_t for the critical-number policy so as to make the recursions introduced in §2 and §3 consistent with the sequence of events in other Markov decision process models; the first thing to occur in a "decision epoch" is the ordering/immediate arrival of new inventory; then demand is realized yielding a realization of lost sales, whatever inventory remains is charged a holding cost, then items outdate. The method of assessing outdating costs is identical to that of Fries (1975).

To get a tractable optimization problem, we substitute the expressions of the previous paragraph into (31), and replace the Zs by $Z(m)$ and the Ds by a generic D. Taking expectations yields

$$EC(m) = c_h m + (c_q - c_h)ED + (c_l - c_q + c_h)E(D-m)^+ + (c_q + c_z)EZ(m). \tag{32}$$

We consider two methods of choosing m. Method 1 replaces $EZ(M)$ by the midpoint of the new bounds, $[\alpha^*(m) + \beta^*(m)]/2$. Method 2 replaces $EZ(m)$ by the midpoint of Chazan and Gal's bounds, $[\alpha(m) + \beta(m)]/2$. In each case, we need conduct only a one-dimensional search to pick the critical number m that minimizes $EC(m)$. Note that we are not considering discounted costs, so we are able to obtain directly a fairly simple cost function. Nahmias (1977) proposes a different method (using the midpoint of the Chazan and Gal bounds as a proxy for the expected outdating) that enables one to address discounted-cost problems under the additional complicating assumption of random lifetimes. His method is also applied by Nandakumar and Morton (1993) to the deterministic-lifetime, infinite-horizon, discounted-cost case.

We considered problems with horizon $\tau = 1000$, lifetime $n = 3$, and no salvage value at $t = \tau + 1$. Under these assumptions, we carried out the above optimization procedures for a number of different choices of the cost parameters. We show results for three different demand distributions, each with mean 5.0: Poisson, geometric, and all-or-none(10,0.5) (i.e., $P(D=10) = P(D=0) = 0.5$). For each of the three different distributions, we used $c_h = 0$, $c_q = 1.5$, and allowed c_l to take values in $\{2, 3, 6, 8\}$ and c_z to take values in $\{1, 2, 3, 6, 8\}$, thereby yielding 20 different combinations of cost parameters. In addition, we studied a wide array of other combinations; however, we do not report on those here, because the results were not substantively different.

Tables 2–4 summarize the results. In a given cell, the first line shows the choice of m made by Method 1 and its corresponding average cost. Likewise, the second line in each cell shows the choice of m made by Method 2 and its corresponding average cost. Each displayed cost is the mean (over 10,000 simulations) average cost (over $\tau = 1{,}000$ epochs) yielded by simulation; thus the total number of decision epochs simulated for each case was 10^7. In addition, the third row of each cell shows the choice of m that minimized the mean simulated total cost. This m was determined by simulating 10,000 demand sequences $\{D_t\}_{t=1,\ldots\tau}$ and evaluating the resulting total costs yielded by different values for the critical number. The displayed value of m on the third line is the one that minimized the mean of the simulated total cost (equivalently, the one that minimized mean of the simulated average cost over $t = 1, \ldots \tau$). Also in the third row is the minimum mean simulated average cost, corresponding to the minimizing m. All simulations started with an empty system. Finally, each cell also shows the expected cost per unit time of an optimal policy when the system starts empty; this expectation was obtained by recursively computing the expected total cost using standard Markov decision process techniques and then dividing by τ.

Table 2. Critical numbers and costs for all-or-none demand.

c_l	c_z = 1		2		3		6		8	
2	10	9.29	0	10.00	0	10.00	0	10.00	0	10.00
	0	10.00	0	10.00	0	10.00	0	10.00	0	10.00
	10	9.29	0	10.00	0	10.00	0	10.00	0	10.00
		9.29		10.00		10.00		10.00		10.00
3	10	9.29	10	10.00	10	10.72	10	12.86	10	14.28
	10	9.29	10	10.00	10	10.72	0	14.99	0	14.99
	10	9.29	10	10.00	10	10.72	10	12.86	10	14.28
		9.29		10.00		10.72		12.86		14.28
6	10	9.29	10	10.00	10	10.72	10	12.86	10	14.28
	10	9.29	10	10.00	10	10.72	10	12.86	10	14.28
	10	9.29	10	10.00	10	10.72	10	12.86	10	14.28
		9.29		10.00		10.72		12.86		14.28
8	10	9.29	10	10.00	10	10.72	10	12.86	10	14.28
	10	9.29	10	10.00	10	10.72	10	12.86	10	14.28
	10	9.29	10	10.00	10	10.72	10	12.86	10	14.28
		9.29		10.00		10.72		12.86		14.28

Examining Tables 2–4, we see that in most cases both Methods 1 and 2 picked the m that minimized costs (among critical-number policies) in the simulations. In addition, these choices of m yielded costs that were generally less than 1% higher than those of an optimal policy. These results are consistent with those in Nahmias (1976, 1977) and Nandakumar and Morton (1993).

The most striking example of the difference between Method 1 and Method 2 occurs in the three cases of the all-or-none distribution, where Method 1 chooses $m = 10$ and Method 2 chooses $m = 0$. In the most extreme case this phenomenon causes a cost increase of about 16% when using Method 2. The reason that Method 2 picks $m = 0$ can be traced back to the discussion of the computation of $\beta(m)$ in the previous section. The fact that the only possible positive values of D are "truncated" by $\beta(m)$ causes Method 2 to overestimate the slope of the expected outdating cost when viewed as a function of m.

Table 3. Critical numbers and costs for geometric demand.

c_l	c_z = 1		2		3		6		8	
2	5	8.55	4	8.99	4	9.07	3	9.25	3	9.34
	5	8.85	4	8.99	3	9.13	3	9.25	2	9.38
	5	8.85	4	8.99	4	9.07	3	9.25	3	9.34
		8.83		8.97		9.06		9.24		9.33
3	7	10.37	7	10.67	6	10.98	5	11.56	5	11.83
	9	10.33	7	10.67	6	10.98	5	11.56	4	11.89
	8	10.30	7	10.67	6	10.98	5	11.56	5	11.83
		10.24		10.62		10.90		11.49		11.79
6	11	12.66	9	13.78	9	14.33	7	16.09	7	16.71
	12	12.59	11	13.51	10	14.25	9	15.97	8	16.72
	12	12.59	11	13.51	10	14.25	8	15.88	7	16.71
		12.47		13.37		14.09		15.71		16.48
8	12	13.71	11	14.86	10	15.87	8	18.21	8	19.04
	14	13.56	12	14.74	12	15.76	9	17.91	9	19.00
	14	13.56	12	14.74	11	15.71	9	17.91	9	19.00
		13.40		14.56		15.51		17.65		18.72

Table 4. Critical numbers and costs for Poisson demand.

c_l	c_z = 1		2		3		6		8	
2	8	7.62	8	7.64	8	7.67	7	7.70	7	7.72
	8	7.62	7	7.66	7	7.67	6	7.78	6	7.78
	8	7.62	8	7.64	8	7.67	7	7.70	7	7.72
		7.62		7.64		7.66		7.70		7.72
3	9	7.71	9	7.75	8	7.79	8	7.86	8	7.90
	9	7.71	9	7.75	8	7.79	8	7.86	7	7.98
	9	7.71	9	7.75	8	7.79	8	7.86	8	7.90
		7.70		7.75		7.78		7.85		7.90
6	10	7.83	9	7.91	9	7.96	9	8.11	9	8.20
	10	7.83	9	7.91	9	7.96	9	8.11	9	8.20
	10	7.83	9	7.91	9	7.96	9	8.11	9	8.20
		7.83		7.90		7.95		8.10		8.20
8	10	7.88	10	7.97	9	8.07	9	8.21	9	8.31
	10	7.88	10	7.97	9	8.07	9	8.21	9	8.31
	10	7.88	10	7.97	10	8.06	9	8.21	9	8.31
		7.87		7.96		8.04		8.20		8.30

Table 2 shows identical costs for the optimal policy and the best choice of m. This is because the optimal decision rule for the all-or-none distribution was a critical-number ($m = 10$) rule in every decision epoch except for (in some cases) $t = \tau$ and $t = \tau - 1$. In these epochs close to the end of the horizon, it was sometimes optimal to order nothing, even when having total inventory below 10; however, the difference in total cost between the optimal policy and the critical-number $m = 10$ policy was too small to appear in the table.

Inspection of Table 4 shows the critical-number costs tracking very close to the optimal numbers. In fact, the optimal policy for the Poisson case prescribed the same order quantity as did the critical-number policy in most states and decision epochs. However, the optimal policies were, in general, not critical-number policies. Table 3 shows that for geometric demand, the critical-number policies did not do as well as they did for Poisson demand.

As mentioned in the previous section, the Chazan and Gal upper bound will be tighter in cases when demand is relatively low. However, it is not altogether clear what causes one approximation method to be better, in terms of the resulting choice of m, for any particular problem. By virtue of the fact that the midpoint of the upper and lower bound was used as an approximation, one could potentially arrive at the best possible critical-number policy by finding upper and lower bounds that are always equidistant from the true expected number of outdates. If such bounds were obtained, then the midpoint of the bounds would yield an exact expression for the expected number of outdates, even if each individual bound were far from the true value.

Nahmias (1977, 1978) used the Chazan and Gal bounds to help derive operating policies for perishable inventory systems with, respectively, random product lifetimes and fixed order (setup) costs. In the 1977 paper, he found that policies arising from approximations utilizing the Chazan and Gal bounds tended to perform better than those that came from methods first introduced in Nahmias (1976). It is also possible to use the bounds $\alpha^*(m)$ and $\beta^*(m)$ to obtain approximate cost functions for the purpose of choosing operating policies for the more-complicated systems. This could be accomplished by using the methods employed in the 1977 and 1978 studies but with the new bounds in place of the Chazan and Gal bounds.

8. CONCLUSIONS

In this paper, we have studied a perishable inventory system that uses a fixed-critical number policy, whereby orders are placed each period so that total on-hand inventory, regardless of age, is returned to a single fixed number each period. The building blocks of the study are a number of sample-path recursions that relate the dynamics of the outdating process to those of a transformed state vector. From these, we obtained families of upper and lower bounds on the expected number of outdates per unit time. In addition, we derived bounds on the steady-state distribution of the number of outdates per unit time; these bounds, in turn, led to bounds on higher moments of the number of outdates. Other products of the analysis included comparison results with nonperishable inventory systems, as well as bounds on the distribution of the age of the youngest and oldest inventory in the system.

One of the new lower bounds on the expected number of outdates was shown to be tighter than those previously published, whereas the upper bounds were sometimes better and sometimes worse than those already available. Numerical results suggested that the new upper bounds were better in cases where demand was "often" greater than m/n, whereas the previously available upper bounds (Chazan and Gal 1977) were better in cases where demand was "often" less than m/n. The Chazan and Gal bound performed better in the lower demand situations, because it is based on a comparison to an exact expression for the number of outdates per unit time in a system in which demand is always less than m/n.

We conducted a numerical study in which we compared, for a variety of demand distributions and cost parameters, different choices of critical-number policies and optimal policies. We considered choices of the critical number based on separate cost estimates that employed either the new bounds or the old bounds. Neither approximation method appeared to be better across the board. However, in all cases, the performance of the critical-number policies was nearly as good as that of an optimal policy, thereby supporting the assertion that, in the absence of significant fixed-charge order costs, critical-number policies provide a simple and effective means for managing inventories of a perishable product.

APPENDIX

PROOF (OF PROPOSITION 4). We prove (17) first. Fix $i \in \{1, \dots, n-1\}$ and suppose $t \geqslant i$. From Lemma 2 with $k = i+1$ and $j = i$, together with (7), we see that

$$Y_t^{i+1} \leqslant \left[m - \sum_{s=t-i+1}^{t} D_s \right]^+. \tag{33}$$

Hence, $P(Y_t^{i+1} = 0) \geqslant P([m - \sum_{s=t-i+1}^{t} D_s]^+ = 0) = \gamma_i$. Noting that $\{\psi(y_t) \leqslant i\} = \{Y_t^{i+1} = 0\}$, we get

$$P(\psi(Y_t) \leqslant i) \geqslant \gamma_i.$$

Letting t go to infinity completes the proof of (17).

We will demonstrate that (18) holds when t is large enough. The proposition will then follow by letting t go to ∞. Suppose that $t \geqslant i-1$, then

$$P(Y_{t-i+1}^{n-i+2} = 0; D_s = 0 \text{ all } s = t-i+2, \dots, t)$$
$$= P\big(Y_{t-i+1}^{n-i+2} = 0\big) \prod_{s=t-i+2}^{t} P(D_s = 0) \tag{34}$$
$$= P\big(\psi(Y_{t-i+1}) \leqslant n-i+1\big) G(0)^{i-1} \tag{35}$$
$$\geqslant \gamma_{n-i+1} G(0)^{i-1}. \tag{36}$$

Observe that $\{\phi(Y_t) \geqslant i\} = \{Y_t^i = m\}$ for $i = 2, \dots, n$. Thus, it remains only to be shown that for $t \geqslant i-1$,

$$\{Y_t^i = m\} = \{Y_{t-i+1}^{n-i+2} = 0; D_s = 0 \text{ all } s = t-i+2, \dots, t\}. \tag{37}$$

We see that

$$\big\{Y_{t-i+1}^{n-i+2} = 0\big\} \subseteq \{Y_s^n = 0 \text{ all } s = t-i+1, \dots, t-1\},$$

because, by Lemma 2, for s such that $t-i+1 \leqslant s \leqslant t-1$,

$$Y_s^n = \left[Y_{t-i+1}^{n+t-s-i+1} - \sum_{u=t-i+2}^{s} \max\{D_u, Y_{u-1}^n\} \right]^+$$
$$\leqslant Y_{t-i+1}^{n+t-s-i+1}$$
$$\leqslant Y_{t-i+1}^{n-i+2}.$$

Therefore,

$$\big\{Y_{t-i+1}^{n-i+2} = 0; D_s = 0 \text{ all } s = t-i+2, \dots, t\big\} = A, \tag{38}$$

where we define

$$A \equiv \big\{Y_{t-i+1}^{n-i+2} = 0; Y_s^n = 0 \text{ all } s = t-i+1, \dots, t-1;$$
$$D_s = 0 \text{ all } s = t-i+2, \dots, t\big\}.$$

Lemma 2 with $k = i$ and $j = i-1$ also implies

$$\big\{Y_t^i = m\big\} = \left\{ \left[Y_{t-i+1}^1 - \sum_{s=t-i+2}^{t} \max\{D_s, Y_{s-1}^n\} \right]^+ = m \right\}. \tag{39}$$

Consequently, $A \subseteq \{Y_t^i = m\}$. To prove the opposite inclusion, note that Lemma 2 with $k = n$, $j = i-2$, and t replaced by $t-1$ yields

$$Y_{t-1}^n = \left[Y_{t-i+1}^{n-i+2} - \sum_{s=t-i+2}^{t-1} \max\{D_s, Y_{s-1}^n\} \right]^+. \tag{40}$$

By (39) it follows that $Y_t^i = m$ if and only if both $D_s = 0$ and $Y_{s-1}^n = 0$ for $s = t-i+2, \dots, t$, which in turn, by (40) implies $Y_{t-i+1}^{n-i+2} = 0$. So, $A \supseteq \{Y_t^i = m\}$, completing the proof. □

Proof (of Theorem 2). Fix $i \in \{1, \ldots, n\}$, $z \in \{0, \ldots, m-1\}$, and suppose that X_0 has the steady-state distribution π. Then

$$P(Z_{n-i+1} > z) = P(Z_{n-i+1} > z | X_0^i = 0) P(X_0^i = 0) + P(Z_{n-i+1} > z | X_0^i \neq 0) P(X_0^i \neq 0).$$

Applying (1) and (2), we see that if $X_0^i = 0$, then $X_j^{i+j} = 0$ for $j = 1, \ldots, n-i$. In particular, $X_{n-i}^n = 0$, so $Z_{n-i+1} = 0$. Therefore, we have

$$\begin{aligned} &P(Z_{n-i+1} > z) \\ &\quad = P(Z_{n-i+1} > z | X_0^i \neq 0) P(X_0^i \neq 0) \\ &\quad = P(Z_{n-i+1} > z | X_0^i \neq 0)(1 - P(X_0^i = 0)) \\ &\quad \leqslant P(Z_{n-i+1} > z | X_0^i \neq 0)(1 - \gamma_{i-1} - G(0)^i \gamma_{n-i}). \end{aligned} \tag{41}$$

To derive this inequality, observe that $\{X_0^i = 0\} \supseteq \{\psi(Y_0) \leqslant i-1\} \cup \{\phi(Y_0) \geqslant i+1\}$. Hence, $P(X_0^i = 0) \geqslant P(\{\psi(Y_0) \leqslant i-1\} \cup \{\phi(Y_0) \geqslant i+1\})$. Furthermore, $\{\psi(Y_0) \leqslant i-1\} \cap \{\phi(Y_0) \geqslant i+1\} = \varnothing$, so $P(\{\psi(Y_0) \leqslant i-1\} \cup \{\phi(Y_0) \geqslant i+1\}) = P(\psi(Y_0) \leqslant i-1) + P(\phi(Y_0) \geqslant i+1)$. Proposition 4 now gives (41).

Proposition 1 yields

$$\begin{aligned} &P(Z_{n-i+1} > z | X_0^i \neq 0) \\ &\quad = P\left(\left[Y_0^i - \sum_{t=1}^{n-i+1} D_t - \sum_{t=1}^{n-i} Z_t\right]^+ > z | X_0^i \neq 0\right) \\ &\quad \leqslant P\left(\left[m - \sum_{t=1}^{n-i+1} D_t\right]^+ > z | X_0^i \neq 0\right) \\ &\quad = P\left(\left[m - \sum_{t=1}^{n-i+1} D_t\right]^+ > z\right). \end{aligned}$$

The last equality follows from the fact that $D_1, D_2, \ldots$ are independent of X_0. Thus, we have

$$P(Z_{n-i+1} > z) \leqslant P\left(\left[m - \sum_{t=1}^{n-i+1} D_t\right]^+ > z\right) \times (1 - \gamma_{i-1} - G(0)^i \gamma_{n-i}). \tag{42}$$

We now obtain (25) by letting $k = n-i+1$ and noting that if X_0 has distribution π, then Z_k has the same distribution as Z_∞ for $k > 0$. Finally, (26) follows from (25) just as (15) followed from (14). □

The following technical inequalities, which can be proved by induction on n, are needed in the proof of Theorem 1. Below, we use the convention that $\sum_{i=2}^{1} x_i = 0$.

Lemma 4. *Suppose* $n \geqslant 2$, $c = (c_1, \ldots, c_n)$, $p = (p_1, \ldots, p_n)$, $q = (q_1, \ldots, q_{n-1})$, $r = (r_2, \ldots, r_n)$, *where* $c_1 \leqslant c_2 \leqslant \cdots \leqslant c_n$.

(a) *If* $q_i \leqslant \sum_{j=1}^{i} p_j$ *for* $i = 1, \ldots, n-1$, *then*

$$\sum_{i=1}^{n} p_i c_i \leqslant q_1 c_1 + \sum_{i=2}^{n-1} (q_i - q_{i-1}) c_i + \left(\left(\sum_{j=1}^{n} p_j\right) - q_{n-1}\right) c_n.$$

(b) *If* $r_i \leqslant \sum_{j=i}^{n} p_j$ *for* $i = 2, \ldots, n$, *then*

$$\sum_{i=1}^{n} p_i c_i \geqslant \left(\left(\sum_{j=1}^{n} p_j\right) - r_2\right) c_1 + \sum_{i=2}^{n-1} (r_i - r_{i+1}) c_i + r_n c_n.$$

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