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Stochastic Comparisons in Airline Revenue Management

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Consider two markets of different sizes but similar costs and fare structure. All other things being equal, is an airline's expected revenue larger in the market with larger demand? If not, under what circumstances is it possible to compare expected revenues without carrying out a detailed analysis? In this article, we provide answers to these questions by studying the relationship between the optimal expected revenue and the demand distributions when the latter are comparable according to various stochastic orders. For the two-fare class problem with dependent demand we obtain three results. We show that airlines should prefer lesser positive dependence between fare classes when marginal demand distributions are the same. We also describe particular dependence structures under which stochastically larger marginal demand distributions improve optimal expected revenue. Finally, when the dependence between effective demands in the two fare classes arises due to "sell ups," we show that stochastically larger marginal demand distributions should be preferred. (Sell ups occur when some lower-fare-class customers buy higher-fare tickets upon finding that the former tickets are sold out.) For a problem with an arbitrary number of fare classes and independent demands, we show that stochastically larger demand distributions should be preferred. Numerical examples demonstrating the effect of parameterized demand distributions (with appropriate stochastic ordering) and dependence structures are also presented.

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1. Introduction

Airlines sell seats on any given flight in multiple fare classes. They also use advertising campaigns and promotions (such as an offer to customers to earn more than the usual number of frequent-flyer miles) to increase demand for all fare classes. Assuming that such marketing efforts are successful and that revenue/cost parameters remain invariant, is the airline better off when demand for each fare class is larger? If not, is there a way to compare demand vectors that guarantees greater expected revenues for the airline? In this paper, we use stochastic comparisons to answer such questions. Although motivated by real-life situations, the focus of our inquiry is, for the most part, technical. That is, we do not focus on the precise mechanisms that may give rise to such changes in demand. Note that the ability to compare expected revenues may also be important to an airline that is interested in assessing profitability in two markets of different sizes but similar costs and fare structures.

Consider first a stylized example that helps to establish that a common notion of larger demand is not necessarily superior. More realistic examples will be discussed later in the paper. The airline in this example offers two types of fares for a particular flight with capacity $\kappa = 100$. Class-1 customers pay fare $f_1 = \$800$, and class-2 customers pay $f_2 = \$500$. A random amount of class-2 demand, denoted as D_2, arrives first, followed by a random amount of class-1 demand, denoted as D_1. The airline optimizes its expected revenue by choosing a booking limit b, which represents the maximum number of class-2 seats to sell. Seats not sold to class-2 customers are then available for sale to class-1 customers. Because class-1 demand has not yet materialized when the airline must make class-2 booking decisions, there is a trade-off between selling class-2 tickets or waiting in the hope of selling those seats to higher-paying class-1 customers. We denote by $\rho(b, d_1, d_2)$ the airline's revenue when class-i realized demand is d_i, $i = 1, 2$, and

$b \in \{0, 1, \ldots, \kappa\}$ is the booking limit. Formally,

$$\rho(b, d_1, d_2) = f_2 \min\{b, d_2\} + f_1 \min\{\kappa - \min\{b, d_2\}, d_1\}. \quad (1)$$

An optimal booking limit maximizes the expected revenue $E\rho(b, D_1, D_2)$.

Suppose that in the base case $P(D_1 = 100, D_2 = 0) = P(D_1 = 0, D_2 = 100) = 1/2$. For any integer booking limit $b \in [0, 100]$, the expected revenue is given by $250b + 40{,}000$. It is easy to see that setting $b^\dagger = 100$ gives an optimal booking limit, which yields an optimal expected revenue of $65,000. After a marketing campaign, the new demands are $(\widehat{D}_1, \widehat{D}_2)$, where $P(\widehat{D}_1 = 100, \widehat{D}_2 = 0) = P(\widehat{D}_1 = 0, \widehat{D}_2 = 100) = 1/4$ and $P(\widehat{D}_1 = 100, \widehat{D}_2 = 100) = 1/2$. The marginal distributions of $\widehat{D}_i$ are still two-point distributions with masses at 0 and 100. Observe that $P(\widehat{D}_1 = 100) = P(\widehat{D}_2 = 100) = 0.75$, which means that each fare class is likely to have a larger demand realization after the marketing campaign. (In fact, each $\widehat{D}_i$ as well as the bivariate demand vector is stochastically larger than the corresponding base-case quantity.) Now, for any integer booking limit $b \in [0, 100]$ the expected revenue is given by $-25b + 60{,}000$. The optimal booking limit is $\hat{b}^\dagger = 0$, which gives an optimal expected revenue of $60,000. We see that although larger demand is more likely after marketing efforts, the airline's expected revenue is smaller, even when the booking limit is chosen optimally. An identical reduction in optimal expected revenue will be seen if the airline uses a randomized booking limit.

How can we explain the apparent anomaly in the above example? The remainder of this article contains formal models to answer such questions in more complex settings. However, for this particular example, it is possible to explain the anomaly through intuitive arguments. In the base case, the demand in each fare class is perfectly negatively correlated. Therefore, by choosing the right booking limit, the airline can sell all its seats. In fact, the seats are sold to the higher-paying customers whenever there is nonzero class-1 demand. After marketing efforts, the demand is more positively correlated. Positive correlation makes it more likely that the airline is forced to choose between selling a seat to one type of customer, and selling the same seat to another type, which lowers profits. In closing this discussion of the example, it is worthwhile to point out that in addition to promotional activities, special events such as conventions and tournaments, changes in economic conditions, and "sell ups" can affect demand correlation (see Brumelle et al. 1990). Sell ups occur when some low-fare customers buy the high-fare ticket upon finding that the low-fare tickets are sold out.

The situation described above is an example of a revenue-management problem, wherein a firm (the airline) controls the availability of products (low- and high-fare tickets) comprised of perishable resources (the seats) to maximize revenues. An extensive survey of the revenue-management literature can be found in Talluri and van Ryzin (2004), so here we will confine our remarks to closely related material. Our general approach in this paper will consider revenue management problems in which customer classes arrive in sequential blocks for a single-leg flight—see, for instance, Belobaba (1989), Curry (1990), Wollmer (1992), Brumelle and McGill (1993), and Robinson (1995).

Lautenbacher and Stidham (1999) relate the research mentioned above with queueing control problems using a Markov decision-process framework. Li and Oum (2002) also discuss and unify some of the earlier work. Moving outside the context of independent demands, Brumelle et al. (1990) analyze the effects of stochastic dependence between the low- and high-class demand in a two-class problem. Other related work includes research by Cooper et al. (2006), who describe potential long-term negative effects of using models that do not incorporate customer choice; Pfeifer (1989), who models the diversion of customers from one class to another; Bodily and Weatherford (1995), who analyze both diversion and overbooking; Weatherford (1997), who considers an integrated pricing and inventory-control problem; van Ryzin and McGill (2000), who describe an adaptive procedure to determine inventory controls from past booking data; and Netessine and Shumsky (2005), who study a game-theoretic version of the problem.

The remainder of this paper is organized as follows. Section 2 considers two-class models in which the demands for tickets in each fare class are dependent. We identify stochastic orders of demand distributions that lead to greater expected revenue. We also present a more specialized model in this section that deals with sell ups. Section 3 generalizes the models to an arbitrary number of fare classes; in this more complicated setting we can derive meaningful results only when demands for different fare classes are independent. Section 4 contains examples and insights, and concluding remarks can be found in §5.

2. The Two-Fare-Class Model

We begin this section by formalizing the model underlying the example in §1. Recall that there is a single-flight leg with κ seats, two fare classes, fares f_i ($f_1 \geq f_2$), and integer-valued demands D_i. The joint distribution of the random demand vector $\mathbf{D} = (D_1, D_2)$ is given by the function $H(\cdot, \cdot)$, and the marginal distributions are given by $H_i(\cdot)$; $i = 1, 2$. In addition, the conditional distribution of D_i, given $D_j = d$, for $i \neq j$, is denoted by $G_i(d_i \mid d)$; that is, $G_i(d_i \mid d) = P(D_i \leq d_i \mid D_j = d)$. Lowercase letters will denote mass functions; for instance, $g_i(d_i \mid d) = P(D_i = d_i \mid D_j = d)$, $h_2(d_2) = P(D_2 = d_2)$, and so on. The airline must determine the class-2 booking limit for use during period 2—this limit places an upper bound on the number of seats that can be sold to class-2 customers. Note that the number of remaining seats for sale to the higher class is at least $\kappa - b$; this quantity is sometimes called the protection level for class 1.

Let $\rho(b, d_1, d_2)$, defined in (1), be the airline's revenue when realized demand for classes 1 and 2 are respectively, d_1 and d_2, and b is the class-2 booking limit. The objective is to pick an integer-valued $b \in [0, \kappa]$ that maximizes the expected revenue $E\rho(b, D_1, D_2)$. This maximizer is denoted $b^\dagger$. For references dealing with this model, see §§2.2 and 2.7 of Talluri and van Ryzin (2004). In the remainder of this section we investigate, using stochastic orders, the effect of the joint demand distribution on $E\rho(b^\dagger, D_1, D_2)$.

Before we proceed, we first review some material from the theory of stochastic comparisons (for complete details, see Shaked and Shanthikumar 1994 or Müller and Stoyan 2002). For random variables X and Y with respective distribution functions F_X and F_Y, we say that Y is stochastically larger than X (written $X \leq_{\text{st}} Y$) if $F_Y(u) \leq F_X(u)$ for all u, or equivalently if $Ef(X) \leq Ef(Y)$ for all increasing functions $f(\cdot)$ for which the expectations exist. Note that because X and Y are not necessarily defined on the same probability space, the expectation operators may be different, but this is not indicated by the notation. An important result in the theory of stochastic comparisons (see, e.g., Shaked and Shanthikumar 1994, Theorem 1.A.1, or Müller and Stoyan 2002, Theorem 1.2.4) states that there exist a probability space $\tilde{\Omega}$ endowed with probability measure $\tilde{P}$ and random variables $\tilde{X}$ and $\tilde{Y}$ defined on $\tilde{\Omega}$ so that $\tilde{P}(\{\omega \in \tilde{\Omega}\colon \tilde{X}(\omega) \leq u\}) = F_X(u)$ and $\tilde{P}(\{\omega \in \tilde{\Omega}\colon \tilde{Y}(\omega) \leq u\}) = F_Y(u)$ for all $u \in \mathcal{R}$ and $\tilde{P}(\{\omega \in \tilde{\Omega}\colon \tilde{X}(\omega) \leq \tilde{Y}(\omega)\}) = 1$. Therefore, for the purpose of comparing expectations of functions of X and Y, there is no loss of generality in assuming that X and Y are themselves defined on a common probability space and $X \leq Y$ with probability one. Note, if initially this is not the case, we may construct $\tilde{X}$ and $\tilde{Y}$ as above.

Similarly, given two-dimensional random vectors $\mathbf{V} = (V_1, V_2)$ and $\hat{\mathbf{V}} = (\hat{V}_1, \hat{V}_2)$, we say that $\hat{\mathbf{V}}$ is stochastically larger than $\mathbf{V}$ (written $\mathbf{V} \leq_{\text{st}} \hat{\mathbf{V}}$) when $Ef(\mathbf{V}) \leq Ef(\hat{\mathbf{V}})$ for all increasing functions $f\colon \mathcal{R}^2 \to \mathcal{R}$ for which the expectations exist.

On an intuitive level, the stochastic order ($\leq_{\text{st}}$) compares random variables based upon their size. The example in §1 shows that "larger" (in the sense of $\leq_{\text{st}}$) is not necessarily desirable. It also gives rise to the question—what other property is desirable? The basic approach we take is to ask what happens when we replace $\mathbf{D} = (D_1, D_2)$ by some other random vector $\hat{\mathbf{D}} = (\hat{D}_1, \hat{D}_2)$, where $\mathbf{D}$ and $\hat{\mathbf{D}}$ are comparable with respect to some stochastic order, while keeping other model parameters unchanged. We will indicate quantities related to $\hat{\mathbf{D}}$ by affixing a "hat" ($\hat{}$) to the corresponding notation for $\mathbf{D}$. Viewed in terms of this notation, the example from §1 shows that it is possible to have $\mathbf{D} \leq_{\text{st}} \hat{\mathbf{D}}$ and $E\rho(b^\dagger, D_1, D_2) \geq E\rho(\hat{b}^\dagger, \hat{D}_1, \hat{D}_2)$. Other papers on the application of stochastic orders to operations management problems include Song (1994), Ridder et al. (1998), and Gupta and Cooper (2005), the last of which provides additional references.

We provide three different comparisons. The first two focus on isolating the effect of dependence and demand size on the airline's expected profit. Specifically, we assume that an airline's efforts may either predominantly change the dependence between D_1 and D_2 but not their marginal distributions (see §2.1), or that they may predominantly change the marginal demand distributions while retaining a certain dependence structure (see §2.2). We understand that promotional efforts may change demand in other ways not captured by our analysis. The goal of this article is to provide a sharper understanding of the role of demand dependence and size, without actually linking such stylized changes to specific promotional activities. We also include a third case (see §2.3), in which both marginal distributions of demand and correlation are affected; however, this effect is produced by a particular mechanism whereby some low-fare customers purchase high-fare tickets when the low-fare tickets are sold out.

2.1. The Effect of Dependence

A function $f\colon \mathscr{R}^2 \to \mathscr{R}$ is called *supermodular* if it satisfies $f(x_2, y_2) - f(x_1, y_2) \geq f(x_2, y_1) - f(x_1, y_1)$ for all $x_2 \geq x_1$ and $y_2 \geq y_1$ (see Topkis 1998 for an extensive treatment of supermodularity). If the first inequality above is reversed, that is, $\geq$ is replaced by $\leq$, then f is submodular. If the function f measures the "benefit" associated with the pair (x, y) and f is submodular, then the preceding condition can be interpreted as saying that x and y are substitutes; i.e., the defining inequality states that the incremental benefit from increasing the level of x is decreasing in the level of y.

For two-dimensional random vectors $\mathbf{V} = (V_1, V_2)$ and $\widehat{\mathbf{V}} = (\widehat{V}_1, \widehat{V}_2)$, we say that $\widehat{\mathbf{V}}$ is smaller than $\mathbf{V}$ in the supermodular order (written $\widehat{\mathbf{V}} \leq_{\text{sm}} \mathbf{V}$) if $Ef(\widehat{\mathbf{V}}) \leq Ef(\mathbf{V})$ whenever f is a supermodular function for which the expectations exist. From the definitions above observe that a function f is supermodular if and only if $-f$ is submodular. So, an alternative condition for $\widehat{\mathbf{V}} \leq_{\text{sm}} \mathbf{V}$ is that $E\phi(\widehat{\mathbf{V}}) \geq E\phi(\mathbf{V})$ whenever ϕ is a submodular function for which the expectations exist.

Loosely speaking, $\widehat{\mathbf{V}} \leq_{\text{sm}} \mathbf{V}$ means that there is a stronger positive dependence between V_1 and V_2 than between $\widehat{V}_1$ and $\widehat{V}_2$. Because the function $f(\mathbf{v}) = v_1 v_2$ is supermodular, it follows that if $\widehat{\mathbf{V}} \leq_{\text{sm}} \mathbf{V}$, then $E(\widehat{V}_1 \widehat{V}_2) \leq E(V_1 V_2)$. Comparability according to the supermodular order implies equal marginals; to see this, observe that for any $u \in \mathscr{R}$, the function $f(\mathbf{v}) = \mathrm{I}(v_1 \leq u)$, where $\mathrm{I}(\cdot) = 1$, if the logical statement in its argument is true, and 0 otherwise, is trivially supermodular, and so $P(V_1 \leq u) = E\mathrm{I}(V_1 \leq u) \leq E\mathrm{I}(\widehat{V}_1 \leq u) = P(\widehat{V}_1 \leq u)$. A similar argument shows that $P(V_1 > u) \leq P(\widehat{V}_1 > u)$, and, hence, $P(V_1 \leq u) = P(\widehat{V}_1 \leq u)$ must hold. Combining the above, it follows that $\widehat{\mathbf{V}} \leq_{\text{sm}} \mathbf{V}$ implies $\mathrm{Corr}(\widehat{V}_1, \widehat{V}_2) \leq \mathrm{Corr}(V_1, V_2)$; the converse does not hold in general. Müller and Scarsini (2000) provide a discussion of the use of the supermodular order in comparing random vectors. For additional background, see Nelsen (1999) or Müller and Stoyan (2002). Applications in the revenue-management context are described by Karaesmen and van Ryzin (2004), who use submodularity in their analysis of multiclass overbooking problems; and by Netessine and Shumsky (2005), who employ supermodularity to prove the existence of equilibria in a game-theoretic formulation.

For each b, the function $\rho(b, \cdot, \cdot)$ defined in (1) is submodular in (d_1, d_2). Because the two customer classes are sold seats from the same pool, the fact that d_1 and d_2 act as "substitutes" in terms of their impact on $\rho(b, \cdot, \cdot)$ is apparent on an intuitive level. A formal argument can be constructed by noting that in (1), the function $\rho(b, \cdot, \cdot)$ is expressed as the sum of two functions, each of which is submodular, and that the sum of submodular functions is submodular. To see the submodularity of each term in (1), note first that $f_2 \min\{d_2, b\}$ is trivially submodular in (d_1, d_2) because it is a function just of d_2. The submodularity of $f_1 \min\{\kappa - \min\{b, d_2\}, d_1\}$ can be proved by checking several cases to see that the defining property of submodular functions holds. We are now ready for the first result of this section.

Proposition 1. *Suppose* $\widehat{\mathbf{D}} \leq_{\text{sm}} \mathbf{D}$. *Then,*

$$E\rho(b, D_1, D_2) \leq E\rho(b, \widehat{D}_1, \widehat{D}_2)$$

for any $b \in [0, \kappa]$. *Therefore,*

$$E\rho(b^\dagger, D_1, D_2) \leq E\rho(\hat{b}^\dagger, \widehat{D}_1, \widehat{D}_2).$$

A proof of Proposition 1 follows immediately from the definition of the supermodular order and the observation that the function $\phi_b(x, y) := -\rho(b, x, y)$ is supermodular in (x, y) for each b. Netessine and Rudi (2003) employ a similar argument in the proof of their Proposition 2.

Proposition 1 shows that in addition to the distribution of demand for each fare class, airlines must also consider the degree to which the demands are dependent. When all other things are equal, airlines should prefer the case in which the demands are less positively dependent (in the sense of the supermodular order). We give examples of demand vectors comparable according to the supermodular order in §4.

2.2. The Effect of Marginal Distributions

The example in §1 showed a situation in which stochastically larger demands lead to lower expected revenue. In this section we provide positive results in the form of conditions under which it is indeed the case that $E\rho(b^*, D_1, D_2) \le E\rho(\hat{b}^*, \widehat{D}_1, \widehat{D}_2)$ whenever $D_1 \le_{st} \widehat{D}_1$ and $D_2 \le_{st} \widehat{D}_2$. These conditions involve the dependence structure between class-1 and 2 demands. First, we need to introduce some more notation.

Let $D_1(d_2)$ (respectively $\widehat{D}_1(d_2)$) denote a random variable with distribution function $G_1(\cdot \mid d_2)$ (resp. $\widehat{G}_1(\cdot \mid d_2)$). That is, $P(D_1(d_2) \le x) = P(D_1 \le x \mid D_2 = d_2) = G_1(x \mid d_2)$ and $P(\widehat{D}_1(d_2) \le x) = P(\widehat{D}_1 \le x \mid \widehat{D}_2 = d_2) = \widehat{G}_1(x \mid d_2)$. Below, we consider cases in which conditional class-1 demand distributions are comparable in the usual stochastic order, that is, $D_1(d_2) \le_{st} \widehat{D}_1(d_2)$ for each d_2. Note that $D_1(d_2) \le_{st} \widehat{D}_1(d_2)$ means that for all increasing functions $f\colon \mathcal{R} \to \mathcal{R}$, $E[f(D_1) \mid D_2 = d_2] = Ef(D_1(d_2)) \le Ef(\widehat{D}_1(d_2)) = E[f(\widehat{D}_1) \mid \widehat{D}_2 = d_2]$, or equivalently $G_1(x \mid d_2) \ge \widehat{G}_1(x \mid d_2)$ for all x.

A random variable X_1 is stochastically increasing in another random variable X_2, denoted by $\mathrm{SI}(X_1 \mid X_2)$, if $P(X_1 > y \mid X_2 = x)$ is increasing in x for all y—see Nelsen (1999) or Müller and Stoyan (2002) for background. Others (e.g., Lehmann 1966 and Tong 1980) study SI under the name "positive regression dependence." If $\mathrm{SI}(D_1 \mid D_2)$ holds, it means that $G_1(d_1 \mid d_2)$ is decreasing in d_2. It is also straightforward to check that if $D_2 \le_{st} \widehat{D}_2$, $D_1(d_2) \le_{st} \widehat{D}_1(d_2)$, and $\mathrm{SI}(D_1 \mid D_2)$, then these conditions together imply that $D_1 \le_{st} \widehat{D}_1$. To see this, note that under these assumptions we have

$$G_1(d_1) = \sum_{d_2} G_1(d_1 \mid d_2) h_2(d_2) \ge \sum_{d_2} G_1(d_1 \mid d_2) \hat{h}_2(d_2) \ge \sum_{d_2} \widehat{G}_1(d_1 \mid d_2) \hat{h}_2(d_2) = \widehat{G}_1(d_1). \tag{2}$$

When $P(D_1 > \kappa - b \mid D_2 \ge b)$ is increasing in b, Brumelle et al. (1990) have shown that an optimal b is given by

$$b^* = \max\{b \in [0, \kappa]\colon P(D_1 > \kappa - b \mid D_2 \ge b) \le f_2/f_1\}. \tag{3}$$

A few computations show that if $b \le b'$, then $\mathrm{SI}(D_1 \mid D_2)$ implies $P(D_1 > a \mid D_2 \ge b) \le P(D_1 > a \mid D_2 \ge b')$. Taking $a = \kappa - b$ yields

$$P(D_1 > \kappa - b \mid D_2 \ge b) \le P(D_1 > \kappa - b \mid D_2 \ge b') \le P(D_1 > \kappa - b' \mid D_2 \ge b'). \tag{4}$$

Therefore, expression (4) tells us that if $\mathrm{SI}(D_1 \mid D_2)$ holds, then $P(D_1 > \kappa - b \mid D_2 \ge b)$ is increasing in b, and, hence, b^* in (3) is an optimal booking limit. Below, the notation $X \overset{d}{=} Y$ means that X and Y have the same distribution.

Proposition 2. *Suppose (D_1, D_2) and $(\widehat{D}_1, \widehat{D}_2)$ are two demand vectors.*

1. *If $D_1(d_2) \le_{st} \widehat{D}_1(d_2)$, for all d_2, and $D_2 \overset{d}{=} \widehat{D}_2$, then*

$$\rho(b, D_1, D_2) \le_{st} \rho(b, \widehat{D}_1, \widehat{D}_2) \quad \text{for any booking limit } b \in [0, \kappa]. \tag{5}$$

2. *If $D_1(d_2) \le_{st} \widehat{D}_1(d_2)$ for all d_2, $D_2 \le_{st} \widehat{D}_2$, and $\mathrm{SI}(D_1 \mid D_2)$ holds, then*

$$E\rho(b^*, D_1, D_2) \le E\rho(b^*, \widehat{D}_1, \widehat{D}_2) \le E\rho(\hat{b}^*, \widehat{D}_1, \widehat{D}_2). \tag{6}$$

Proof. To prove Part 1, fix $b \in [0, \kappa]$ and observe from (1) that $\rho(b, d_1, d_2)$ is increasing in d_1 for each fixed value of $d_2 \ge 0$. Then for each increasing function $f\colon \mathcal{R} \to \mathcal{R}$ and each fixed value of d_2, the function $f(\rho(b, d_1, d_2))$ is again increasing in d_1. So, if $D_1(d_2) \le_{st} \widehat{D}_1(d_2)$ for all d_2, we have

$$\sum_{d_1 \ge 0} f(\rho(b, d_1, d_2)) g_1(d_1 \mid d_2) \le \sum_{d_1 \ge 0} f(\rho(b, d_1, d_2)) \hat{g}_1(d_1 \mid d_2) \quad \text{for all } d_2.$$

Since $D_2 \overset{d}{=} \widehat{D}_2$ implies $h_2(\cdot) = \hat{h}_2(\cdot)$, the proof of Part 1 now follows from the definition of the order $\le_{st}$ and (7) below:

$$\begin{aligned} Ef(\rho(b, D_1, D_2)) &= \sum_{d_2 \ge 0} \sum_{d_1 \ge 0} f(\rho(b, d_1, d_2)) g_1(d_1 \mid d_2) h_2(d_2) \\ &\le \sum_{d_2 \ge 0} \sum_{d_1 \ge 0} f(\rho(b, d_1, d_2)) \hat{g}_1(d_1 \mid d_2) \hat{h}_2(d_2) \\ &= Ef(\rho(b, \widehat{D}_1, \widehat{D}_2)). \end{aligned} \tag{7}$$

Turning to Part 2, define

$$\phi(d_2) = \sum_{d_1 \ge 0} \rho(b^*, d_1, d_2) g_1(d_1 \mid d_2) = E\rho(b^*, D_1(d_2), d_2)$$

and

$$\hat{\phi}(d_2) = E\rho(b^*, \widehat{D}_1(d_2), d_2),$$

and observe that

$$E\rho(b^*, D_1, D_2) = \sum_{d_2 \ge 0} \phi(d_2) h_2(d_2).$$

Moreover, since $\phi(d_2) \le \hat{\phi}(d_2)$ for all d_2, we have

$$E\rho(\hat{b}^*, \widehat{D}_1, \widehat{D}_2) \ge E\rho(b^*, \widehat{D}_1, \widehat{D}_2) = \sum_{d_2 \ge 0} \hat{\phi}(d_2)\hat{h}_2(d_2) \ge \sum_{d_2 \ge 0} \phi(d_2)\hat{h}_2(d_2).$$

Therefore, if we can show that $\phi(d_2)$ is an increasing function of d_2, the proof will be complete by the definition of $D_2 \le_{\text{st}} \widehat{D}_2$. To this end, we will demonstrate that $\phi(d_2) - \phi(d_2 - 1) \ge 0$ for all d_2.

For fixed d_2, the $\text{SI}(D_1 \mid D_2)$ assumption implies that $D_1(d_2 - 1) \le_{\text{st}} D_1(d_2)$. In what follows, we assume without loss of generality that $D_1(d_2 - 1)$ and $D_1(d_2)$ are constructed on a common probability space so that

$$D_1(d_2 - 1) \le D_1(d_2) \quad \text{with probability one.} \tag{8}$$

With $D_1(d_2 - 1)$ and $D_1(d_2)$ so-constructed we have

$$\phi(d_2) - \phi(d_2 - 1) = E\big[\rho(b^*, D_1(d_2), d_2) - \rho(b^*, D_1(d_2 - 1), d_2 - 1)\big]. \tag{9}$$

We now verify that the expression in (9) is nonnegative by checking two different cases.

Case 1 ($d_2 \le b^*$). Using (1), expression (9) becomes

$$\begin{aligned}
\phi(d_2) - \phi(d_2 - 1) &= f_2 + f_1 E\big[\min\{\kappa - d_2, D_1(d_2)\} - \min\{\kappa - d_2 + 1, D_1(d_2 - 1)\}\big] \\
&\ge f_2 + f_1 E\big[\min\{\kappa - d_2, D_1(d_2)\} - \min\{\kappa - d_2 + 1, D_1(d_2)\}\big] \qquad (10) \\
&= f_2 - f_1 E[\mathrm{I}(D_1(d_2) > \kappa - d_2)] \\
&= f_2 - f_1 P(D_1 > \kappa - d_2 \mid D_2 = d_2) \\
&\ge f_2 - f_1 P(D_1 > \kappa - d_2 \mid D_2 \ge d_2) \qquad (11) \\
&\ge 0. \qquad (12)
\end{aligned}$$

In the above, (10) follows from (8), and (11) follows because $\text{SI}(D_1 \mid D_2)$ implies $P(D_1 > a \mid D_2 \ge b) \ge P(D_1 > a \mid D_2 = b)$ for any $a, b \ge 0$. Finally, (12) follows from $d_2 \le b^*$, the definition (3) of b^*, and because $P(D_1 > \kappa - \cdot \mid D_2 \ge \cdot)$ is increasing.

Case 2 ($d_2 \ge b^* + 1$). Again using (1), expression (9) gives us

$$\phi(d_2) - \phi(d_2 - 1) = f_1 E\big[\min\{\kappa - b^*, D_1(d_2)\} - \min\{\kappa - b^*, D_1(d_2 - 1)\}\big] \ge 0,$$

where the final inequality follows from (8). This completes the proof. □

On an intuitive level, in both Parts 1 and 2 of the proposition (not to be confused with the two cases in the proof), the particular dependence structure allows an airline to realize the benefits of greater demand, while protecting it from any adverse effects. In Part 1, the high-fare class has stochastically greater demand for each realization of the demand for the low-fare class. Such a change is beneficial to the airline for any booking limit because it will sell at least as many low-fare tickets and more high-fare tickets. Similarly, in Part 2, due to the SI property, the greater demand for the low-fare class is associated with larger high-fare demand. Therefore, greater low-fare demand is beneficial. In Part 2, note that using a booking limit of either b^* or $\hat{b}^*$ when demand is given by $(\widehat{D}_1, \widehat{D}_2)$ will give higher expected revenue than when demand is (D_1, D_2)—regardless of what booking limit is in effect for demand (D_1, D_2).

Observe that Part 2 of the proposition describes situations in which stochastically larger marginal distributions are desirable (because (2) shows that under the assumptions in Part 2, we have $D_1 \le_{\text{st}} \widehat{D}_1$ in addition to $D_2 \le_{\text{st}} \widehat{D}_2$). Hence, apparent anomalies like that in the introduction cannot occur when the dependence between low- and high-fare demand is as described in Part 2. Note also that Proposition 2 does not contradict the example in the introduction. In the example, the class-2 demand distribution changes so Part 1 does not apply, and $\text{SI}(D_1 \mid D_2)$ does not hold, so Part 2 does not apply either.

Proposition 2 applies to situations in which class-1 and 2 demands are independent. In particular, if we assume independence of the components of $\mathbf{D}$ and independence of the components of $\widehat{\mathbf{D}}$, then Part 1 says that stochastically greater class-1 demand leads to stochastically larger revenues for any fixed booking limit. Similarly, Part 2 says that stochastically greater class-2 demand yields higher expected revenue when the original optimal booking limit (corresponding to $\mathbf{D}$) is used. These observations are also true when there are more than two classes of independent demand, as we will show in §3.

2.3. A Model with Sell Ups

In this section, we explore the effect of stochastically ordered demand distributions in the sell-up model from Brumelle et al. (1990, §2.3). In this setting, if a class-2 customer is unable to purchase a low-fare ticket because the low-fare booking limit has already been reached, then there is a fixed probability that the customer in question will "sell up" and purchase a class-1 ticket rather than go away empty-handed. Formally, suppose $\{u_i\}$ is a $\{0, 1\}$-valued sequence, where $u_i = 1$ if the ith class-2 customer is willing to sell up if she is denied a class-2 ticket, and $u_i = 0$ otherwise. The revenue function is given by

$$\begin{aligned}&\rho(b, d_1, d_2, \{u_i\})\\ &= f_2 \min\{b, d_2\} + f_1 \min\left\{\kappa - \min\{b, d_2\},\ d_1 + \sum_{i=b+1}^{d_2} u_i\right\}.\end{aligned} \tag{13}$$

(We define an empty sum to be zero). Let $\{U_i\}$ be an i.i.d. sequence of Bernoulli random variables, each with parameter $\gamma \in (0, 1)$, where u_i in (13) is the realized value of U_i. As in Brumelle et al. (1990), D_1, D_2, and $\{U_i\}$ are assumed independent throughout this subsection. The objective is to select b to maximize $E\rho(b, D_1, D_2, \{U_i\})$. For each b, revenue (13) is increasing in d_1 and u_i. Therefore, it follows immediately that stochastically increasing class-1 demand or increasing the sell-up probability γ (while maintaining the independence assumptions) will increase expected revenue for each fixed b, and, hence, will also increase optimal expected revenue.

Brumelle et al. (1990) show that an optimal booking limit is given by

$$b^\ddagger = \max\left\{b \in [0, \kappa]\colon P\left(D_1 + \sum_{i=b+1}^{D_2} U_i > \kappa - b\right) < \frac{f_2 - \gamma f_1}{(1-\gamma) f_1}\right\}.$$

The following result shows that an analog of Part 2 of Proposition 2 holds in the sell-up setting. As before, we append a "hat" on quantities associated with the problem with larger demand.

Proposition 3. *Suppose that $D_1 \le_{\mathrm{st}} \widehat{D}_1$, $D_2 \le_{\mathrm{st}} \widehat{D}_2$, demands are independent, and $\gamma \le \hat{\gamma}$. Then,*

$$\begin{aligned}E\rho(b^\ddagger, D_1, D_2, \{U_i\}) &\le E\rho(b^\ddagger, \widehat{D}_1, \widehat{D}_2, \{\widehat{U}_i\})\\ &\le E\rho(\hat{b}^\ddagger, \widehat{D}_1, \widehat{D}_2, \{\widehat{U}_i\}).\end{aligned}$$

Proof. The second inequality is obvious, so it remains only to show the first. In view of the comment regarding monotonicity of ρ after (13), we need only to focus on the case in which $D_1 \overset{d}{=} \widehat{D}_1$ and $\gamma = \hat{\gamma}$. Proceeding as in the proof of Part 2 of Proposition 2, it suffices to verify that

$$E\rho(b^\ddagger, D_1, d_2, \{U_i\}) \ge E\rho(b^\ddagger, \widehat{D}_1, d_2 - 1, \{\widehat{U}_i\}) \tag{14}$$

for each $d_2 \ge 1$. Again, without loss of generality, we construct all random variables on a common probability space so that, with probability one, $D_1 = \widehat{D}_1$ and $\{U_i\} = \{\widehat{U}_i\}$. With such a construction, to prove (14), it suffices to demonstrate that

$$E[\rho(b^\ddagger, D_1, d_2, \{U_i\}) - \rho(b^\ddagger, D_1, d_2 - 1, \{U_i\})] \ge 0. \tag{15}$$

We again consider two cases. If $d_2 \ge b^\ddagger + 1$, then the quantity inside the expectation on the left side of (15) is nonnegative with probability one, and, hence, (15) holds. Next, suppose that $d_2 \le b^\ddagger$. In this case, it follows that $D_1 + \sum_{i=b^\ddagger+1}^{d_2} U_i = D_1$, and therefore

$$\begin{aligned}&E[\rho(b^\ddagger, D_1, d_2, \{U_i\}) - \rho(b^\ddagger, D_1, d_2 - 1, \{U_i\})]\\ &\quad = f_2 - f_1 E[\mathrm{I}(D_1 > \kappa - d_2)] \qquad (16)\\ &\quad = f_2 - f_1 P(D_1 > \kappa - d_2). \qquad (17)\end{aligned}$$

Because $d_2 \le b^\ddagger$, the independence of D_1 and D_2, and the definition of $b^\ddagger$, we have that

$$\begin{aligned}P(D_1 > \kappa - d_2) &\le P(D_1 > \kappa - b^\ddagger)\\ &= P(D_1 > \kappa - b^\ddagger \mid D_2 \ge b^\ddagger)\\ &\le P\left(D_1 + \sum_{i=b^\ddagger+1}^{D_2} U_i > \kappa - b^\ddagger \mid D_2 \ge b^\ddagger\right)\\ &< \frac{f_2 - f_1\gamma}{f_1(1-\gamma)}.\end{aligned}$$

Hence, (17) is bounded below by $\gamma(f_1 - f_2)/(1-\gamma)$, which is positive. This completes the proof. □

3. The Multiple-Fare-Class Model

Consider a single flight leg with κ seats and $n \geq 2$ demand classes. In this model, demand classes are again analogous to time periods, because a different demand class arrives in each period. Throughout this section, we assume that demands in the n classes are mutually independent, leading to a Markov decision process (MDP) formulation. Many other papers that address the multiple-fare-class model have also assumed the demands to be independent; see Talluri and van Ryzin (2004) for references. At the end of this section, we revisit the independence assumption.

The MDP framework is similar to that described in Lautenbacher and Stidham (1999, §3). (Although our formulation is slightly different, it can be shown that both models give rise to the same value function and there is a simple correspondence between optimal policies). As we move closer to the time of departure, the time index decreases, so the first time period is $t = n$, and the last is $t = 1$. During time-period t, a random amount of class-t demand D_t arrives. Each accepted class-t customer pays fare f_t. Note that we do not need to assume $f_n < \cdots < f_2 < f_1$. At the beginning of time-period t, the airline must decide how many seats to make available based on the number of seats sold by that time. Let $v_t(s)$ be the value function when the state (seats sold) is s just prior to time t. That is, $v_t(s)$ represents the maximum expected revenue obtainable in periods $t, \ldots, 1$, given state s at time t. The overall-optimal expected revenue is $v_n(0)$. It is well known that $v_t(\cdot)$ is determined by the optimality equations;

$$v_t(s) = \max_{x=0,\ldots,\kappa-s} E[f_t \min\{x, D_t\} + v_{t-1}(s + \min\{x, D_t\})] \qquad s = 0, 1, \ldots, \kappa - 1;\ t = 1, \ldots, n, \quad (18)$$

with boundary conditions $v_0(s) = 0$ and $v_t(\kappa) = v_{t-1}(\kappa)$. Furthermore, a policy that employs, for each s and t, a maximizer of (18) is optimal and gives expected revenue $v_n(0)$. For the $n = 2$ case with $f_1 > f_2$, we have $v_2(0) = E\rho(b^*, D_1, D_2)$, in which D_1 and D_2 are independent and b^* is given by (3).

Lautenbacher and Stidham (1999, Theorem 5; see also Robinson 1995) shows that

$$x_t^*(s) = (b_t^* - s)^+ \quad (19)$$

is an optimal action for state s at time t, in which $b_t^* = \min\{i \geq 0\colon f_t < v_{t-1}(i) - v_{t-1}(i+1)\}$, with the convention that $v_t(\kappa + 1) = -\infty$ for all t. That is, $x_t^*(s)$ in (19) maximizes the right side of (18), and the policy that prescribes the actions specified by (19) yields the maximum expected revenue $v_n(0)$. Lautenbacher and Stidham (1999) also show that for each t, the function $v_t(\cdot)$ is decreasing and concave, so

$$s < b_t^* \iff f_t + v_{t-1}(s+1) - v_{t-1}(s) \geq 0. \quad (20)$$

With independent demands and $n = 2$, Proposition 2 shows that airlines realize greater optimal expected revenue when demand is stochastically larger. In the sequel, we will show that when $n \geq 2$, greater demand according to the *increasing concave* order results in greater optimal expected revenue. For random variables X and Y, we say that $X \leq_{\text{icv}} Y$ if $Ef(X) \leq Ef(Y)$ for all increasing, concave functions f for which the expectations exist; see Müller and Stoyan (2002) or Shaked and Shanthikumar (1994).

Proposition 4. *Suppose that $D_t \leq_{\text{icv}} \widehat{D}_t$ for $t = 1, \ldots, n$ and that $\{D_t\}$ and $\{\widehat{D}_t\}$ are both independent sequences. Let $\hat{v}_t(\cdot)$ be the value function for the MDP with demand sequence $\{\widehat{D}_t\}$, and let $\tilde{v}_t(\cdot)$ be the expected revenue from using actions* (19) *for demand sequence $\{\widehat{D}_t\}$. That is, $\tilde{v}_0(s) = 0$ and*

$$\tilde{v}_t(s) = E[f_t \min\{x_t^*(s), \widehat{D}_t\} + \tilde{v}_{t-1}(s + \min\{x_t^*(s), \widehat{D}_t\})] \qquad t = 1, \ldots, n. \quad (21)$$

Then $v_t(s) \leq \tilde{v}_t(s) \leq \hat{v}_t(s)$ for all $s = 0, \ldots, \kappa$ and $t = 1, \ldots, n$.

Proof. It is immediate that $\tilde{v}_t(s) \leq \hat{v}_t(s)$. Hence, it remains only to show that $v_t(s) \leq \tilde{v}_t(s)$. The proof is by induction on t. For $t = 1$, we have $v_1(s) = E[f_1 \min\{x_t^*(s), D_1\}]$ and $\tilde{v}_1(s) = E[f_1 \min\{x_t^*(s), \widehat{D}_1\}]$. Because $\psi_s(d) = f_1 \min\{x_t^*(s), d\}$ is increasing and concave in d, for each fixed s, it follows that $v_1(s) \leq \tilde{v}_1(s)$ for all s, and so the result holds for $t = 1$.

For general $t > 1$, suppose that $v_{t-1}(s) \leq \tilde{v}_{t-1}(s)$. Note that $v_t(s) = E[V_t^s(D_t)]$, in which we define

$$V_t^s(d) = f_t \min\{x_t^*(s), d\} + v_{t-1}(s + \min\{x_t^*(s), d\}).$$

Observe that

$$V_t^s(d+1) - V_t^s(d) = \begin{cases} f_t + v_{t-1}(s+d+1) - v_{t-1}(s+d) & \text{if } d < x_t^*(s) \\ 0 & \text{if } d \geq x_t^*(s). \end{cases}$$

When $0 \le d < x_t^*(s) := (b_t^* - s)^+$, it follows that $s + d < b_t^*$, so in view of (20), we have that $f_t + v_{t-1}(s+d+1) - v_{t-1}(s+d) \ge 0$. Consequently, $V_t^s(\cdot)$ is an increasing function of its argument. Moreover, $V_t^s(\cdot)$ is concave because $v_{t-1}(\cdot)$ is concave (see Lautenbacher and Stidham 1999l, Theorem 5). So, $V_t^s(\cdot)$ is an increasing and concave function on the domain $\{0, 1, 2, \ldots\}$.

We now have that

$$v_t(s) = E[V_t^s(D_t)] \le E[V_t^s(\widehat{D}_t)] \tag{22}$$

$$= E[f_t \min\{x_t^*(s), \widehat{D}_t\} + v_{t-1}(s + \min\{x_t^*(s), \widehat{D}_t\})] \tag{23}$$

$$\le E[f_t \min\{x_t^*(s), \widehat{D}_t\} + \tilde{v}_{t-1}(s + \min\{x_t^*(s), \widehat{D}_t\})]$$

$$= \tilde{v}_t(s). \tag{24}$$

The inequality in (22) holds because $V_t^s(d)$ is increasing and concave in d and $D_t \le_{\text{icv}} \widehat{D}_t$. Equation (23) comes from the definition of $V_t^s(d)$, and the inequality in (24) follows from the inductive hypothesis. This completes the proof. □

It is easy to see that if $X \le_{\text{st}} Y$, then $X \le_{\text{icv}} Y$. Therefore, an immediate corollary to the proposition is the fact that if $D_t \le_{\text{st}} \widehat{D}_t$ for $t = 1, \ldots, n$ and $\{\widehat{D}_t\}$ are independent, then $\hat{v}_t(s) \ge v_t(s)$. Just as the usual stochastic order ($\le_{\text{st}}$) compares distributions based on "size," the convex order ($\le_{\text{cx}}$) compares distributions based on "variability." The convex order is defined as follows: if $Ef(Y) \le Ef(X)$ for all convex f, then $Y \le_{\text{cx}} X$. It can be proved that if $Y \le_{\text{cx}} X$, then $EY = EX$ and $\text{Var}(Y) \le \text{Var}(X)$; see Müller and Stoyan (2002) or Shaked and Shanthikumar (1994). If $Y \le_{\text{cx}} X$, it is straightforward to check that $X \le_{\text{icv}} Y$. Hence, the proposition also implies that if $\widehat{D}_t \le_{\text{cx}} D_t$ for $t = 1, \ldots, n$ and $\{\widehat{D}_t\}$ are independent, then $\hat{v}_t(s) \ge v_t(s)$. In summary, Proposition 4 implies that when demands are independent, it is desirable to have larger demand, and it is also desirable to have less-variable demand.

Proposition 4 has also been used in Zhang and Cooper (2005) to help obtain easy-to-compute bounds on the multidimensional value function of an MDP in which customers choose among multiple "parallel" flights between a common origin and destination. These bounds help motivate a procedure that yields heuristics for a model not amenable to exact solution.

Returning to the assumption of independent demands, our efforts to obtain results for $n > 2$ dependent demand classes have proved unsuccessful. Note that the proof of Proposition 4 relies on the optimality of the policy that uses actions given by (19). Without the independence assumption, such simple characterizations of optimal policies do not presently exist; in fact, without independence, the problem is not an MDP. An alternative approach would be to simply restrict attention to booking-limit policies, and to try to show that for *any* booking-limit policy, the revenue function is submodular in demand. This parallels the approach taken in Proposition 1. Unfortunately, there are examples in which $n = 3$ and the revenue function is not submodular. In addition, our attempts to develop a different modeling framework to allow meaningful comparisons based on some measure of dependence have not yet proven fruitful.

In summary, it is an open question how to extend the results for dependent demands to more than two classes. A somewhat related issue is how to apply the techniques of this article when demand is an endogenous parameter determined by consumer preferences and airlines' actions. The emergence of the Internet as a distribution channel has made the booking process more transparent to many customers, which has served to elevate the importance of models that do not rely on the notion of exogenous demand for fare classes. The material in §2.3 is a first step toward a stochastic-comparison analysis of endogenous-demand models; however, much more work is needed.

4. Examples and Insights

To demonstrate the effects of dependence we first identify classes of demand distributions comparable in the supermodular order. The comparison of copulas provides one method for finding such classes. A k-dimensional copula is a distribution function on the k-dimensional unit cube with uniform-$[0, 1]$ marginal distributions. Copulas are particularly useful for relating multivariate distributions and their marginals; for instance, Sklar's theorem states that any k-dimensional distribution can be expressed as the composition of a copula with the distribution's marginals. Nelsen (1999) provides comprehensive coverage of copulas. Below, we collect facts from this reference most pertinent to our analysis.

Copulas and the supermodular order can be linked through the notion of concordance. Given two-dimensional random vectors $\mathbf{V}$ and $\widehat{\mathbf{V}}$, we say $\mathbf{V}$ is more concordant than $\widehat{\mathbf{V}}$ (written $\widehat{\mathbf{V}} \leq_c \mathbf{V}$) if $P(V_1 \leq v_1, V_2 \leq v_2) \geq P(\widehat{V}_1 \leq v_1, \widehat{V}_2 \leq v_2)$ and $P(V_1 > v_1, V_2 > v_2) \geq P(\widehat{V}_1 > v_1, \widehat{V}_2 > v_2)$ for all v_1, v_2. In the bivariate case, it is known (see Müller and Scarsini 2000, p. 110) that $\widehat{\mathbf{V}} \leq_c \mathbf{V}$ is equivalent to $\widehat{\mathbf{V}} \leq_{sm} \mathbf{V}$. (For higher dimensions, this equivalence does not hold; however, we limit our discussion to the two-dimensional setting.) Greater concordance occurs when V_1 and V_2 are more likely to be simultaneously large, or simultaneously small. Intuitively, that also means that V_1 and V_2 are more positively dependent. As described in Müller and Stoyan (2002, Theorem 3.8.2), the concordance order is also related to the lower and upper orthant orders.

Suppose now that demand vectors $\mathbf{D}$ and $\widehat{\mathbf{D}}$ have respective distributions H and $\widehat{H}$ satisfying

$$H(d_1, d_2) = C(H_1(d_1), H_2(d_2)) \tag{25}$$

$$\widehat{H}(d_1, d_2) = \widehat{C}(H_1(d_1), H_2(d_2)), \tag{26}$$

where H_1 and H_2 are one-dimensional distributions and C and $\widehat{C}$ are each two-dimensional copulas. Note that this is consistent with our earlier definitions of H_1 and H_2, because, e.g., $P(D_1 \leq d_1) = H(d_1, \infty) = C(H_1(d_1), 1) = H_1(d_1)$, where the final equality follows because the copula C has uniform marginals.

If $\widehat{C}(u_1, u_2) \leq C(u_1, u_2)$ for all $u_1, u_2 \in [0, 1]$, then it follows immediately that

$$\begin{aligned}\widehat{H}(d_1, d_2) &= P(\widehat{D}_1 \leq d_1, \widehat{D}_2 \leq d_2) \leq P(D_1 \leq d_1, D_2 \leq d_2)\\ &= H(d_1, d_2) \quad \text{for all } d_1,\ d_2.\end{aligned}$$

In addition, it is easily seen that

$$P(\widehat{D}_1 > d_1, \widehat{D}_2 > d_2) \leq P(D_1 > d_1, D_2 > d_2).$$

Therefore, $\widehat{\mathbf{D}} \leq_c \mathbf{D}$, and, in turn, $\widehat{\mathbf{D}} \leq_{sm} \mathbf{D}$. For details, see Nelsen (1999).

So, by way of (25)–(26), random vectors can be compared using the supermodular order when $\widehat{C}(u_1, u_2) \leq C(u_1, u_2)$. A similar approach, taken below, involves considering a family of copulas $\{C^\theta\}$, parameterized by θ, so that $\theta_1 \leq \theta_2$ implies $C^{\theta_1}(u_1, u_2) \leq C^{\theta_2}(u_1, u_2)$. Nelsen (1999) provides examples of many such families. Corbett and Rajaram (2002) describe how these ideas can be applied to problems involving aggregation of uncertainty.

Returning to our specific revenue-management problem, suppose that $\{\mathbf{D}^\theta\colon \theta \in \Theta\}$ is a parameterized family of bivariate demand vectors so that

$$\mathbf{D}^{\theta_1} \leq_{sm} \mathbf{D}^{\theta_2} \quad \text{if } \theta_1 \leq \theta_2, \tag{27}$$

where Θ is a set of real-valued parameters. For each $\theta \in \Theta$, suppose that $\mathbf{D}^\theta$ has joint distribution function H^θ and joint mass function h^θ.

For a fixed θ, we can recast the optimization problem in §2 as follows:

$$\max_{b \in [0, \kappa]} r(\theta, b) \tag{28}$$

where

$$r(\theta, b) := \sum_{d_1, d_2} \rho(b, d_1, d_2) h^\theta(d_1, d_2) = E\rho(b, D_1^\theta, D_2^\theta). \tag{29}$$

Let $b^\dagger(\theta) = \min\{b\colon\ r(\theta, b) \geq r(\theta, b')$ for all $b' = 0, 1, \ldots, \kappa\}$ and $r^\dagger(\theta) = r(\theta, b^\dagger(\theta))$. In terms of the above notation, Proposition 1 says that if $\theta_1 \leq \theta_2$ and (27) holds, then $r(\theta_2, b) \leq r(\theta_1, b)$ for all b, and consequently $r^\dagger(\theta_2) \leq r^\dagger(\theta_1)$.

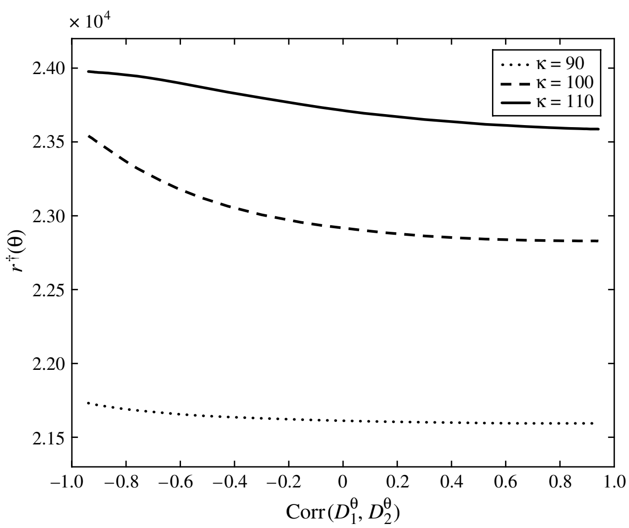
In the examples depicted in Figures 1 and 2, we consider a family of problems parameterized by θ, which indicates that for the θth problem the demand is $\mathbf{D}^\theta$. In particular, we assume that class-1 demand is Poisson with mean 20, and class-2 demand is Poisson with mean 80. So, $H_1(\cdot)$ (respectively $H_2(\cdot)$) is the Poisson distribution function with mean 20 (resp. 80). It remains still to specify the dependence structure.

For $\theta \neq 0$ consider the two-dimensional copula

$$C^\theta(u_1, u_2) = -\frac{1}{\theta} \ln\left(1 + \frac{(e^{-\theta u_1} - 1)(e^{-\theta u_2} - 1)}{e^{-\theta} - 1}\right). \tag{30}$$

Such copulas are described in Nelsen (1999, Chapter 4)—see Equation (4.2.5). For our purposes, the family (30) has several useful properties. Specifically, if $\theta_1 \leq \theta_2$, then $C^{\theta_1}(u_1, u_2) \leq C^{\theta_2}(u_1, u_2)$ for all u_1, u_2; so, we can obtain different levels of dependence between components through the choice of θ. In particular, independence can be obtained as $\theta \to 0$, perfect negative correlation can be obtained as $\theta \to -\infty$, and perfect positive correlation follows when $\theta \to +\infty$ (see Nelsen 1999 for details and references). We

Figure 1 Optimal Expected Revenue as a Function of Correlation



assume that

$$H^\theta(d_1, d_2) = P(D_1^\theta \le d_1, D_2^\theta \le d_2)$$
$$= C^\theta(H_1(d_1), H_2(d_2)), \qquad (31)$$

akin to (25). Because (30) is not defined for $\theta = 0$, we take $H^0(d_1, d_2) = H_1(d_1) \times H_2(d_2)$ to obtain the case of independent demands. As detailed above, this construction gives us demand vectors that satisfy (27).

Because $\mathbf{D}^{\theta_1} \le_{\text{sm}} \mathbf{D}^{\theta_2}$ for $\theta_1 \le \theta_2$, it follows that $\text{Corr}(D_1^\theta, D_2^\theta)$ is increasing in θ (see comment in §2.1). Figure 1 shows the optimal expected revenue as a function of $\text{Corr}(D_1^\theta, D_2^\theta)$ for $f_1 = 600$ and $f_2 = 150$ and different values of capacity. As shown in Proposition 1, optimal expected revenue is decreasing as correlation (as measured by $\le_{\text{sm}}$) increases. For a particular value of θ, the value reported for expected revenue on Figure 1 is the result of simulating 25,000 draws from the distribution H^θ, for each draw evaluating $\rho(b, D_1^\theta, D_2^\theta)$ for each $b \in 0, \ldots, \kappa$, and maximizing the resulting empirical average revenue functions over b. (All other figures are also based on similar Monte Carlo estimates.)

Figure 2 Optimal Booking Limit as a Function of Correlation

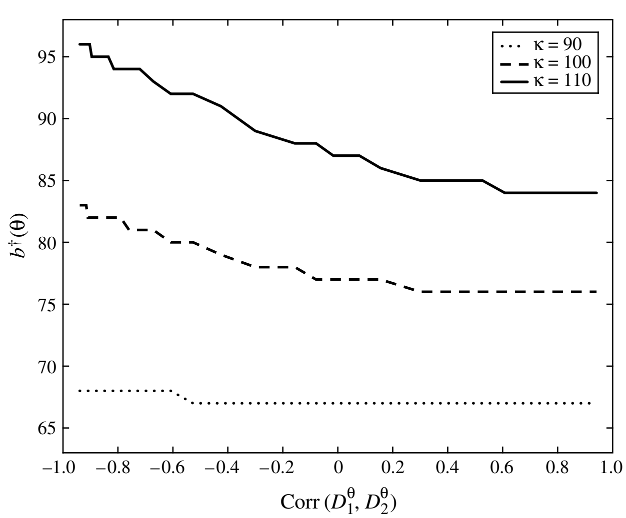


Figure 2 shows that as the dependence increases, optimal booking limits decrease. As the following result demonstrates, this is a particular example of a general monotonicity property. Note that the result does *not* depend upon the construction (30) and (31), and requires only that the problems be parameterized according to demand-vector comparisons with respect to the supermodular order as in (27).

Proposition 5. *If* $\theta_1 \le \theta_2$, *then* $b^\dagger(\theta_1) \ge b^\dagger(\theta_2)$.

Proof. By Topkis (1998, Theorem 2.8.3(a)), it suffices to verify that $r(\theta, b)$ is submodular in (θ, b). That is, we need to check that

$$r(\theta_1, b+1) - r(\theta_1, b) \ge r(\theta_2, b+1) - r(\theta_2, b)$$
$$\text{for all} \quad b \in [0, \kappa - 1] \text{ and } \theta_1 \le \theta_2. \qquad (32)$$

To show (32), observe that for any θ, we have

$$\begin{aligned} r(\theta, b+1) - r(\theta, b) &= E\rho(b+1, D_1^\theta, D_2^\theta) - E\rho(b, D_1^\theta, D_2^\theta) \\ &= E[\rho(b+1, D_1^\theta, D_2^\theta) - \rho(b, D_1^\theta, D_2^\theta)] \\ &= E[f_2 \mathrm{I}(D_2^\theta > b) - f_1 \mathrm{I}(D_2^\theta > b, D_1^\theta > \kappa - b - 1)] \\ &= f_2 P(D_2^\theta > b) - f_1 P(D_2^\theta > b, D_1^\theta > \kappa - b - 1). \qquad (33) \end{aligned}$$

Because $\mathbf{D}^{\theta_1} \le_{\text{sm}} \mathbf{D}^{\theta_2}$, the marginal distributions of $D_2^{\theta_1}$ and $D_2^{\theta_2}$ are identical and furthermore,

$$P(D_1^{\theta_1} > d_1, D_2^{\theta_1} > d_2) \le P(D_1^{\theta_2} > d_1, D_2^{\theta_2} > d_2) \quad \text{for all } (d_1, d_2).$$

Therefore, from (33), we have

$$\begin{aligned} &[r(\theta_1, b+1) - r(\theta_1, b)] - [r(\theta_2, b+1) - r(\theta_2, b)] \\ &\quad = f_1[P(D_2^{\theta_2} > b, D_1^{\theta_2} > \kappa - b - 1) \\ &\qquad - P(D_2^{\theta_1} > b, D_1^{\theta_1} > \kappa - b - 1)] \ge 0. \end{aligned}$$

This completes the proof. □

Brumelle et al. (1990) note that optimal booking limits decrease as correlation increases for two-class problems with bivariate normal demand (see Brumelle et al., p. 187). Müller and Stoyan (2002, Theorem 3.13.5) tells us that for two bivariate normal vectors with the same marginal distributions, the ordering of the vectors' correlations is equivalent to ordering the vectors according to the supermodular order. Therefore, Proposition 5 is in agreement with the statement made by Brumelle et al. (1990).

Next, we demonstrate the impact of changing D_2 as described in Proposition 2. Two sets of examples are constructed. In the first set, we assume that when D_2 is made stochastically larger, $D_1(d_2) \stackrel{d}{=} \widehat{D}_1(d_2)$ for all d_2, and D_1 is stochastically increasing in D_2. Under these conditions, Figure 3 shows that the optimal expected revenue increases with D_2. In contrast, when $\mathrm{SI}(D_1 \mid D_2)$ does not hold, Figure 4 shows that the optimal expected revenue is not monotone in D_2.

In both Figures 3 and 4, D_2 is Poisson distributed. Its mean, ED_2, is varied from 0 to 120 to generate different problem instances with stochastically varying D_2. (For Poisson distributions, the ordering of the means is equivalent to the $\leq_{\mathrm{st}}$ order.) In all instances, the flight leg has capacity $\kappa = 100$ and the two types of customers pay fares $f_2 = 150$ and $f_1 = 600$, respectively. The demand D_1 has a discrete uniform distribution that depends on d_2. Its range is fixed at 40.

Figure 3 **Expected Revenue as a Function of $E[D_2]$ when D_1 Is Stochastically Increasing in D_2**

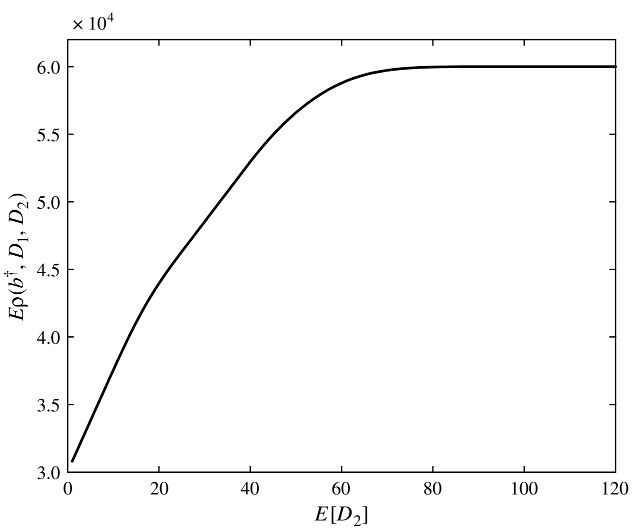
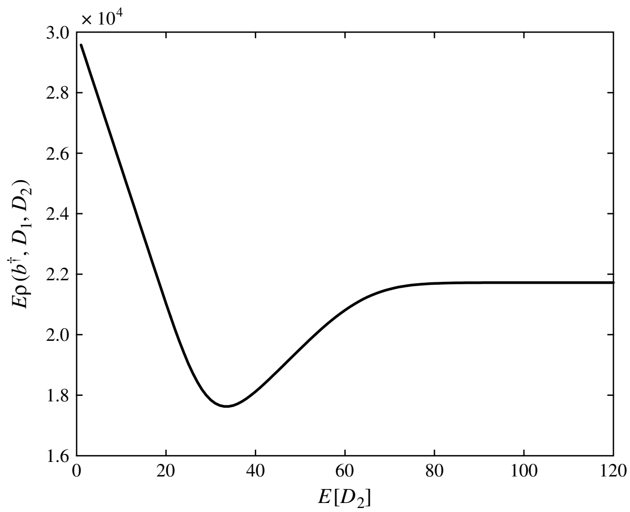


Figure 4 **Expected Revenue as a Function of $E[D_2]$ when D_1 Is Stochastically Decreasing in D_2**



For Figure 3 it is assumed that $ED_1 = 50 + d_2$, which immediately implies that D_1 is uniform in the range $(30 + d_2, 70 + d_2)$ for each d_2. Similarly, for Figure 4, $ED_1 = 50 - \min\{30, d_2\}$. Because the range of D_1 is fixed at 40, in this instance D_1 is uniform over $(30 - \min\{30, d_2\}, 70 - \min\{30, d_2\})$.

For each instance, we first generate a realization of D_2 and use it to generate a dependent realization of $D_1(d_2)$. A Monte Carlo simulation, using 25,000 draws of each pair of demands, is used to estimate $E\rho(b, D_1, D_2)$ for each b, and the optimal b is determined by enumeration. Observe that in Figure 3 the optimal expected revenue function is increasing in ED_2 (or stochastically larger D_2) as predicted by Proposition 2. This happens because the unchanged $D_1(d_2)$ and larger D_2 also implies stochastically larger D_1 on account of their positive dependence. For the problems depicted in Figure 3, demand correlation is positive and increasing in $E[D_2]$. The conditions outlined in Proposition 2 are violated in Figure 4 with the result that the optimal expected revenue first decreases with ED_2 and eventually starts to rise as ED_2 becomes larger. The reason for this pattern is that at first, larger D_2 comes at the cost of smaller D_1, which generates smaller overall expected revenue since $f_2 < f_1$. Eventually, a very large D_2 implies that $D_1 \to \mathrm{Uniform}[0, 40]$; i.e., D_1 becomes virtually independent of D_2. Further increases in D_2 therefore increase optimal expected revenue due to the increased demand for type-2 bookings. In the setting of Figure 4, demand correlations (data not shown) are negative; initially they decrease, reaching a minimum of approximately -0.35 around $E[D_2] = 20$, before increasing to almost 0 once $E[D_2]$ exceeds 45.

5. Concluding Remarks

Significant strides have occurred recently in modeling airline revenue-management problems. However, the effect on an airline's optimal expected revenue of having a different demand distribution for one or more fare-classes, when the latter are possibly correlated, has not been studied. The need to compare different demand distributions can arise either when an airline operates (or wishes to operate) in two markets of different sizes that are otherwise similar or when promotional efforts/exogenous factors simultaneously alter the demand distribution for more than one fare class. In this article, we have used stochastic comparisons to identify situations in which one joint distribution of demand is preferred over another. Our results caution managers against assuming that a larger demand distribution is always beneficial. The results also underscore the importance of considering how demands in different fare classes are interdependent and the importance of using optimal booking policies. When promotional or exogenous factors (such as scheduling a convention or a change in national and international economic conditions) increase demand stochastically, our analysis shows that an improvement in expected revenue is guaranteed in each of the following three scenarios:

1. Demands in different fare classes are independent.
2. There are two fare classes, and their demands depend on each other according to a particular structure (see Proposition 2).
3. There are two fare classes, exogenous demands are independent, and some class-2 customers attempt to purchase high-fare tickets upon finding that low-fare tickets are sold out.

If the stochastically larger distributions arise from a shift in demand induced by promotional efforts or external factors, airline managers may find it difficult to ascertain the exact timing and magnitude of the change, which would be required to compute the postchange optimal booking limits. Our analysis reveals that managers need not be concerned about being worse off so long as (a) one of the above scenarios applies, and (b) they were using an optimal booking limit prior to the shift. The fact that booking limits need not be updated to have an improvement in expected revenue (though proper updating will improve revenues even more) is satisfying from a practical viewpoint.

Many possibilities exist for future work to build on ideas in this article. For instance, as revenue management models continue to evolve to include endogenous demand and consumer-choice behavior, it will be important to understand the effects of changes in underlying sources of randomness. Stochastic comparisons may prove useful for this endeavor. It is also of potential interest to study empirically how various marketing efforts affect such sources of randomness.

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