Appendix

Learning and Pricing with Models that Do Not Explicitly Incorporate Competition

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Proposition A–1. *Suppose that $F : \mathbb{R}^d \to \mathbb{R}^d$ is a contraction mapping such that $\|F(x) - F(y)\| \leq \lambda \|x - y\|$ where $\lambda \in [0,1)$. Consider a closed convex set $A \subseteq \mathbb{R}^d$ and let π denote projection onto A; i.e., $\pi(x) = \arg\min\{\|x - y\| : y \in A\}$. Let $x^* = (x_1^*, \dots, x_d^*)$ denote the unique fixed point of F and suppose $x^* \in A$. Consider stochastic processes $\{X^k = (X_1^k, \dots, X_d^k) : k = 0, 1, 2, \dots\}$ and $\epsilon^k = \{(\epsilon_1^k, \dots, \epsilon_d^k) : k = 1, 2, \dots\}$ such that*

$$X^k = \pi\left(\left(1 - k^{-1}\right) X^{k-1} + k^{-1}\left(F(X^{k-1}) + \epsilon^k\right)\right) \qquad k = 1, 2, \dots \tag{A–1}$$

Let $\mathcal{F}^k$ denote the σ-algebra generated by $X_0, \epsilon^1, \dots, \epsilon^k$, and suppose $\mathbb{E}[\epsilon_i^k | \mathcal{F}^{k-1}] = 0$ and $\mathbb{E}[(\epsilon_i^k)^2 | \mathcal{F}^{k-1}] < M$ for $i = 1, \dots, d$ for some constant $M < \infty$. Then $X^k \to x^*$ w.p.1.

Proof. For $x \in \mathbb{R}^d$, we will use the notation $F_i(x)$ to denote the i-th component of $F(x)$.

Let $Z^k := \left\|X^k - x^*\right\|^2 = \sum_{i=1}^d (X_i^k - x_i^*)^2$. Then

$$\begin{aligned}
Z^k &= \left\|\pi\left(\left(1 - k^{-1}\right) X^{k-1} + k^{-1}\left(F(X^{k-1}) + \epsilon^k\right)\right) - x^*\right\|^2 \\
&\leq \left\|\left(1 - k^{-1}\right) X^{k-1} + k^{-1}\left(F(X^{k-1}) + \epsilon^k\right) - x^*\right\|^2 &\text{(A–2)} \\
&= \sum_{i=1}^d \left[\left(1 - k^{-1}\right) X_i^{k-1} + k^{-1}\left(F_i(X^{k-1}) + \epsilon_i^k\right) - x_i^*\right]^2 \\
&= \sum_{i=1}^d \left[\left(1 - k^{-1}\right)\left(X_i^{k-1} - x_i^*\right) + k^{-1}\left(F_i(X^{k-1}) - x_i^* + \epsilon_i^k\right)\right]^2 &\text{(A–3)}
\end{aligned}$$

where (A–2) holds because x^* is an element of closed convex set A and π is projection onto A.

Let T_i^k denote the i-th term in the sum (A–3). We have

$$\begin{aligned}
T_i^k = &\left(1 - k^{-1}\right)^2 (X_i^k - x_i^*)^2 + 2(1 - k^{-1})k^{-1}(X_i^{k-1} - x_i^*)(F_i(X^{k-1}) - x_i^*) \\
&+ 2(1 - k^{-1})k^{-1}(X_i^{k-1} - x_i^*)\epsilon_i^k + k^{-2}(F_i(X^{k-1}) - x_i^*)^2 \\
&+ 2k^{-2}(F_i(X^{k-1}) - x_i^*)\epsilon_i^k + k^{-2}(\epsilon_i^k)^2
\end{aligned}$$

Taking conditional expectations, we obtain

$$\begin{aligned}
\mathbb{E}[T_i^k | \mathcal{F}^{k-1}] \leq &(1 - k^{-1})^2 (X_i^k - x_i^*)^2 + 2(1 - k^{-1})k^{-1}(X_i^{k-1} - x_i^*)(F_i(X^{k-1}) - x_i^*) \\
&+ k^{-2}(F_i(X^{k-1}) - x_i^*)^2 + k^{-2}M
\end{aligned}$$

Hence by (A–3), we have

$$\mathbb{E}[Z^k|\mathcal{F}^{k-1}] \leq (1-k^{-1})^2 Z^{k-1} + 2(1-k^{-1})k^{-1}\sum_{i=1}^{d}(X_i^{k-1}-x_i^*)(F_i(X^{k-1})-F_i(x^*)) + k^{-2}\sum_{i=1}^{d}(F_i(X^{k-1})-x_i^*)^2 + k^{-2}dM\,.$$

Next, observe that

$$\sum_{i=1}^{d}(X_i^{k-1}-x_i^*)(F_i(X^{k-1})-F_i(x^*)) \leq \left\|X^{k-1}-x^*\right\|\cdot\left\|F(X^{k-1})-F(x^*)\right\| \leq \lambda\left\|X^{k-1}-x^*\right\|^2 = \lambda Z^{k-1}$$

where the first inequality follows from the Cauchy-Schwarz inequality and the second from the fact that F is a contraction. In addition

$$\sum_{i=1}^{d}(F_i(X^{k-1})-x_i^*)^2 = \sum_{i=1}^{d}(F_i(X^{k-1})-F_i(x^*))^2 = \left\|F(X^{k-1})-F(x^*)\right\|^2 \leq \lambda^2\left\|X^{k-1}-x^*\right\|^2 = \lambda^2 Z^{k-1}$$

where the first equality holds because x^* is the fixed point of F.

So,

$$\mathbb{E}[Z^k|\mathcal{F}^{k-1}] \leq (1-k^{-1})^2 Z^{k-1} + 2(1-k^{-1})k^{-1}\lambda Z^{k-1} + k^{-2}\lambda^2 Z^{k-1} + k^{-2}dM\,.$$

Rearranging, we obtain

$$\begin{aligned}\mathbb{E}[Z^k|\mathcal{F}^{k-1}] &\leq \left\{1-2k^{-1}+k^{-2}+2k^{-1}\lambda-2k^{-2}\lambda+k^{-2}\lambda^2\right\}Z^{k-1}+k^{-2}dM\\ &\leq \left\{1-2k^{-1}(1-\lambda)+k^{-2}(1+\lambda^2)\right\}Z^{k-1}+k^{-2}dM\\ &= (1+B^{k-1})Z^{k-1}+C^{k-1}-D^{k-1}\end{aligned}$$

where

$$B^{k-1} := k^{-2}(1+\lambda^2) \qquad C^{k-1} := k^{-2}dM \qquad D^{k-1} := 2k^{-1}(1-\lambda)Z^{k-1}$$

The sequences $\{B^k\}$, $\{C^k\}$, $\{D^k\}$, and $\{Z^k\}$ are non-negative and $\sum_k B^k < \infty$ and $\sum_k C^k < \infty$. Hence, we conclude from Lemma A–3 below that there exists a finite random variable Z such that $Z^k \to Z$ and $\sum_k D_k < \infty$ w.p.1.

Next we show that $Z = 0$ w.p.1, from which it follows that $X^k \to x^*$ w.p.1. Consider any sample path such that $Z^k \to Z > 0$. Then, $\sum_k k^{-1}Z^{k-1} = \infty$, from which it follows that $\sum_k D^k = \infty$. Hence, $P(Z > 0) = 0$. □

Lemma A–1. *Suppose* $\beta_{i,-i} > 0$ *for* $i = \pm 1$ *and consider the demand functions*

$$d_i(p_i, p_{-i}) \quad = \quad \beta_{i,0} + \beta_{i,i} p_i + \beta_{i,-i} p_{-i} \quad i = \pm 1. \tag{A–4}$$

Let $\theta = \sqrt{\beta_{-1,1}\beta_{1,-1}/(\beta_{-1,-1}\beta_{1,1})}$ *and* $\nu = \sqrt{\beta_{-1,-1}\beta_{1,-1}/(\beta_{-1,1}\beta_{1,1})}$*, and consider also the demand functions*

$$\tilde{d}_i(\tilde{p}_i, \tilde{p}_{-i}) \quad = \quad \tilde{\beta}_{i,0} + \tilde{\beta}_{i,i}\tilde{p}_i + \tilde{\beta}_{i,-i}\tilde{p}_{-i} \quad i = \pm 1 \tag{A–5}$$

where $\tilde{\beta}_{i,0} = \beta_{i,0}$ *and* $\tilde{\beta}_{i,-i} = -\theta\tilde{\beta}_{i,i}$ *for* $i = \pm 1$, $\tilde{\beta}_{-1,-1} = \beta_{-1,-1}$, *and* $\tilde{\beta}_{1,1} = \nu\beta_{1,1}$.

The demand functions (A–4) and (A–5) are equivalent in the sense that $\tilde{d}_i(\tilde{p}_i, \tilde{p}_{-i}) = d_i(p_i, p_{-i})$ *for* $i = \pm 1$ *if* $\tilde{p}_{-1} = p_{-1}$ *and* $\tilde{p}_1 = p_1/\nu$. *Moreover, the Nash equilibrium prices* $(\tilde{p}_{-1}^{\mathrm{N}}, \tilde{p}_1^{\mathrm{N}})$ *and modeling error equilibrium prices* $(\tilde{p}_{-1}^{\infty}, \tilde{p}_1^{\infty})$ *for demand functions (A–5) satisfy*

$$\tilde{p}_{-1}^{\mathrm{N}} \;=\; p_{-1}^{\mathrm{N}} \qquad \tilde{p}_1^{\mathrm{N}} \;=\; p_1^{\mathrm{N}}/\nu \qquad \tilde{p}_{-1}^{\infty} \;=\; p_{-1}^{\infty} \qquad \tilde{p}_1^{\infty} \;=\; p_1^{\infty}/\nu \tag{A–6}$$

where $(p_{-1}^{\mathrm{N}}, p_1^{\mathrm{N}})$ *and* $(p_{-1}^{\infty}, p_1^{\infty})$ *are, respectively, the Nash equilibrium and modeling error equilibrium prices for demand functions (A–4).*

Proof. The equivalence of (A–4) and (A–5) can be verified using simple algebra. Likewise, the relations (A–6) follow after some algebra from (6) and (22). □

Proof of Proposition 2. It follows from (6) that

$$p_i^{\mathrm{N}} \quad = \quad \frac{-\theta\beta_{-i,0}\beta_{i,i} - 2\beta_{i,0}\beta_{-i,-i}}{(4-\theta^2)\beta_{-i,-i}\beta_{i,i}}$$

and thus

$$d_i(p_i^{\mathrm{N}}, p_{-i}^{\mathrm{N}}) \quad = \quad \frac{\theta\beta_{-i,0}\beta_{i,i} + 2\beta_{i,0}\beta_{-i,-i}}{(4-\theta^2)\beta_{-i,-i}}$$

and

$$g_i(p_i^{\mathrm{N}}, p_{-i}^{\mathrm{N}}) \quad = \quad -\frac{(\theta\beta_{-i,0}\beta_{i,i} + 2\beta_{i,0}\beta_{-i,-i})^2}{(4-\theta^2)^2\beta_{-i,-i}^2\beta_{i,i}}\,.$$

Similarly, it follows from (22) that

$$p_i^{\infty} \quad = \quad \frac{-\theta\beta_{-i,0}\beta_{i,i} - \beta_{i,0}\beta_{-i,-i}}{2(1-\theta^2)\beta_{-i,-i}\beta_{i,i}}$$

and thus

$$d_i(p_i^{\infty}, p_{-i}^{\infty}) \quad = \quad \frac{\beta_{i,0}}{2}$$

and

$$g_i(p_i^{\infty}, p_{-i}^{\infty}) \quad = \quad -\frac{\theta\beta_{-i,0}\beta_{i,0}\beta_{i,i} + \beta_{i,0}^2\beta_{-i,-i}}{4(1-\theta^2)\beta_{-i,-i}\beta_{i,i}}\,.$$

Note that

$$
\begin{aligned}
&(\beta_{-1,-1},\beta_{1,1}) \in \Gamma_i^{=} \\
&\quad \Leftrightarrow\; g_i(p_i^{\mathrm{N}}, p_{-i}^{\mathrm{N}}) \;=\; g_i(p_i^{\infty}, p_{-i}^{\infty}) \\
&\quad \Leftrightarrow\; -\frac{(\theta\beta_{-i,0}\beta_{i,i} + 2\beta_{i,0}\beta_{-i,-i})^2}{(4-\theta^2)^2\beta_{-i,-i}^2\beta_{i,i}} \;=\; -\frac{\theta\beta_{-i,0}\beta_{i,0}\beta_{i,i} + \beta_{i,0}^2\beta_{-i,-i}}{4(1-\theta^2)\beta_{-i,-i}\beta_{i,i}} \\
&\quad \Leftrightarrow\; \frac{4(1-\theta^2)(\theta\beta_{-i,0}\beta_{i,i} + 2\beta_{i,0}\beta_{-i,-i})^2 - (4-\theta^2)^2\beta_{-i,-i}(\theta\beta_{-i,0}\beta_{i,0}\beta_{i,i} + \beta_{i,0}^2\beta_{-i,-i})}{4(1-\theta^2)(4-\theta^2)^2\beta_{-i,-i}^2\beta_{i,i}} \;=\; 0 \\
&\quad \Leftrightarrow\; 4(1-\theta^2)(\theta\beta_{-i,0}\beta_{i,i} + 2\beta_{i,0}\beta_{-i,-i})^2 - (4-\theta^2)^2\beta_{-i,-i}(\theta\beta_{-i,0}\beta_{i,0}\beta_{i,i} + \beta_{i,0}^2\beta_{-i,-i}) \;=\; 0 \\
&\quad \Leftrightarrow\; \theta^2\Big[\underbrace{4(1-\theta^2)\beta_{-i,0}^2}_{=:\ a}\,\beta_{i,i}^2 + \underbrace{(-(8+\theta^2)\theta\beta_{-i,0}\beta_{i,0}\beta_{-i,-i})}_{=:\ b}\,\beta_{i,i} + \underbrace{(-(8+\theta^2)\beta_{i,0}^2\beta_{-i,-i}^2)}_{=:\ c}\Big] \;=\; 0 \\
&\quad \Leftrightarrow\; \theta \;=\; 0 \qquad \text{or} \\
&\qquad\quad \beta_{i,i} \;=\; \frac{-b \pm \sqrt{b^2-4ac}}{2a} \;=\; \frac{\beta_{i,0}}{\beta_{-i,0}}\left[\frac{(8+\theta^2)\theta \pm (4-\theta^2)\sqrt{8+\theta^2}}{8(1-\theta^2)}\right]\beta_{-i,-i}
\end{aligned}
$$

Thus, for any $(\beta_{-1,0}, \beta_{1,0}, \theta) \in (0,\infty)^2 \times (0,1)$, the set of $(\beta_{-1,-1}, \beta_{1,1})$-points such that $g_i(p_i^{\mathrm{N}}, p_{-i}^{\mathrm{N}}) = g_i(p_i^{\infty}, p_{-i}^{\infty})$ is given by two lines through the origin. Note that $(8+\theta^2)\theta + (4-\theta^2)\sqrt{8+\theta^2} > 0$, and thus one of the lines has a positive slope, and

$$
\begin{aligned}
\theta \;\in\; (0,1) \;&\Rightarrow\; \sqrt{8+\theta^2}\theta + \theta^2 \;<\; 4 \\
&\Rightarrow\; (8+\theta^2)\theta - (4-\theta^2)\sqrt{8+\theta^2} \;<\; 0
\end{aligned}
$$

and thus the other line has a negative slope. Thus the set of $(\beta_{-1,-1}, \beta_{1,1})$-points in $\mathcal{S}$ such that $g_1(p_1^{\mathrm{N}}, p_{-1}^{\mathrm{N}}) = g_1(p_1^{\infty}, p_{-1}^{\infty})$ is given by one line through the origin $\beta_{1,1} = (\beta_{1,0}/\beta_{-1,0})T(\theta)\beta_{-1,-1}$, where $T(\theta) := [(8+\theta^2)\theta + (4-\theta^2)\sqrt{8+\theta^2}]/[8(1-\theta^2)]$. Similarly, the set of $(\beta_{-1,-1}, \beta_{1,1})$-points in $\mathcal{S}$ such that $g_{-1}(p_{-1}^{\mathrm{N}}, p_1^{\mathrm{N}}) = g_{-1}(p_{-1}^{\infty}, p_1^{\infty})$ is given by one line through the origin $\beta_{-1,-1} = (\beta_{-1,0}/\beta_{1,0})T(\theta)\beta_{1,1}$, that is, $\beta_{1,1} = (\beta_{1,0}/\beta_{-1,0})[1/T(\theta)]\beta_{-1,-1}$. Note that $T(\theta) \geq \sqrt{8+\theta^2}/[2(1-\theta^2)] > \sqrt{2} > 1$ on $(0,1)$. Next we verify that $g_i(p_i^{\mathrm{N}}, p_{-i}^{\mathrm{N}}) < g_i(p_i^{\infty}, p_{-i}^{\infty})$ if $\beta_{i,i} > (\beta_{i,0}/\beta_{-i,0})T(\theta)\beta_{-i,-i}$. Specifically, consider any point $(\beta_{-1,-1}, \beta_{1,1})$ such that $\beta_{i,i} = (\beta_{i,0}/\beta_{-i,0})\beta_{-i,-i}$. Note that $\beta_{i,i} > (\beta_{i,0}/\beta_{-i,0})T(\theta)\beta_{-i,-i}$ since $T(\theta) > 1$ and $\beta_{-i,-i} < 0$.

Note that at such a point,

$$
\begin{aligned}
g_i(p_i^{\rm N}, p_{-i}^{\rm N}) &< g_i(p_i^{\infty}, p_{-i}^{\infty}) \\
\Leftrightarrow\ -\frac{(\theta\beta_{i,0}\beta_{-i,-i} + 2\beta_{i,0}\beta_{-i,-i})^2}{(4-\theta^2)^2\beta_{-i,-i}^2\beta_{i,i}} &< -\frac{\theta\beta_{i,0}^2\beta_{-i,-i} + \beta_{i,0}^2\beta_{-i,-i}}{4(1-\theta^2)\beta_{-i,-i}\beta_{i,i}} \\
\Leftrightarrow\ \frac{(\theta+2)^2}{(4-\theta^2)^2} &< \frac{\theta+1}{4(1-\theta^2)} \\
\Leftrightarrow\ \frac{1}{(2-\theta)^2} &< \frac{1}{4(1-\theta)} \\
\Leftrightarrow\ 4-4\theta &< 4-4\theta+\theta^2 .
\end{aligned}
$$

Thus, taking into account that $T(\theta) > 1$ and $\beta_{-i,-i} < 0$, it follows that

$$
\begin{aligned}
\Gamma^* &= \left\{\beta \in \mathcal{S} \ :\ \beta_{i,i} \geq \frac{\beta_{i,0}}{\beta_{-i,0}} T(\theta)\beta_{-i,-i},\ i = \pm 1\right\} \\
&= \left\{\beta \in \mathcal{S} \ :\ \frac{\beta_{1,0}}{\beta_{-1,0}} T(\theta)\beta_{-1,-1} \leq \beta_{1,1} \leq \frac{\beta_{1,0}}{\beta_{-1,0}} \frac{1}{T(\theta)}\beta_{-1,-1}\right\} \neq \varnothing \\
\Gamma_i^* &= \left\{\beta \in \mathcal{S} \ :\ \beta_{i,i} \geq \frac{\beta_{i,0}}{\beta_{-i,0}} T(\theta)\beta_{-i,-i}, \beta_{-i,-i} \leq \frac{\beta_{-i,0}}{\beta_{i,0}} T(\theta)\beta_{i,i}\right\} \\
&= \left\{\beta \in \mathcal{S} \ :\ \beta_{i,i} \geq \max\left\{\frac{\beta_{i,0}}{\beta_{-i,0}} T(\theta)\beta_{-i,-i},\ \frac{\beta_{i,0}}{\beta_{-i,0}} \frac{1}{T(\theta)}\beta_{-i,-i}\right\}\right\} \\
&= \left\{\beta \in \mathcal{S} \ :\ \beta_{i,i} \geq \frac{\beta_{i,0}}{\beta_{-i,0}} \frac{1}{T(\theta)}\beta_{-i,-i}\right\} \neq \varnothing \\
\Gamma^{\rm N} &= \left\{\beta \in \mathcal{S} \ :\ \beta_{i,i} \leq \frac{\beta_{i,0}}{\beta_{-i,0}} T(\theta)\beta_{-i,-i},\ i = \pm 1\right\} \\
&= \left\{\beta \in \mathcal{S} \ :\ \frac{\beta_{i,0}}{\beta_{-i,0}} \frac{1}{T(\theta)}\beta_{-i,-i} \leq \beta_{i,i} \leq \frac{\beta_{i,0}}{\beta_{-i,0}} T(\theta)\beta_{-i,-i}\right\} = \varnothing .
\end{aligned}
$$

□

Proof of Lemma 1. We use Figure A–1 to illustrate the cases in the proof, and without loss of generality, we assume that $b_{-1} \leq b_1$.

Suppose first that (37) holds. If $x \leq -b := (-b_{-1}, -b_1)$, that is, $x \in \mathcal{R}_1$, then $\left\|\Pi(x) - \bar{\beta}^*\right\|_Q = \left\|x - \bar{\beta}^*\right\|_Q$. If $x_{-1} \leq -b_1$ and $x_1 \geq -b_1$, that is, $x \in \mathcal{R}_2$, then $\Pi(x) = (x_{-1}, -b_1)$. Therefore, $\Pi(x)$ is no further in Euclidean distance than x from both $\bar{\beta}^*$ and $\mathcal{D}$ [that is, $\left\|\Pi(x) - \bar{\beta}^*\right\| \leq \left\|x - \bar{\beta}^*\right\|$ and $\mathfrak{d}(\Pi(x)) \leq \mathfrak{d}(x)$], and hence $\left\|\Pi(x) - \bar{\beta}^*\right\|_Q \leq \left\|x - \bar{\beta}^*\right\|_Q$ by (36). Similarly, if $x_{-1} \geq -b_{-1}$ and $x_1 \leq -b_1$, that is, $x \in \mathcal{R}_3$, then $\Pi(x) = (-b_{-1}, x_1)$, and $\left\|\Pi(x) - \bar{\beta}^*\right\|_Q \leq \left\|x - \bar{\beta}^*\right\|_Q$.

In the remainder of the proof we consider $x \geq (-b_1, -b_1)$, that is, $x \in \mathcal{R}_4 \cup \mathcal{R}_5 \cup \mathcal{R}_6$. If $x \geq (-b_{-1}, -b_{-1})$, that is, $x \in \mathcal{R}_4$, then $(-b_{-1}, -b_{-1})$ is no further in Euclidean distance than x from both $\bar{\beta}^*$ and $\mathcal{D}$, and hence $\left\|(-b_{-1}, -b_{-1}) - \bar{\beta}^*\right\|_Q \leq \left\|x - \bar{\beta}^*\right\|_Q$. Also, $\Pi(x) = \Pi(-b_{-1}, -b_{-1})$,

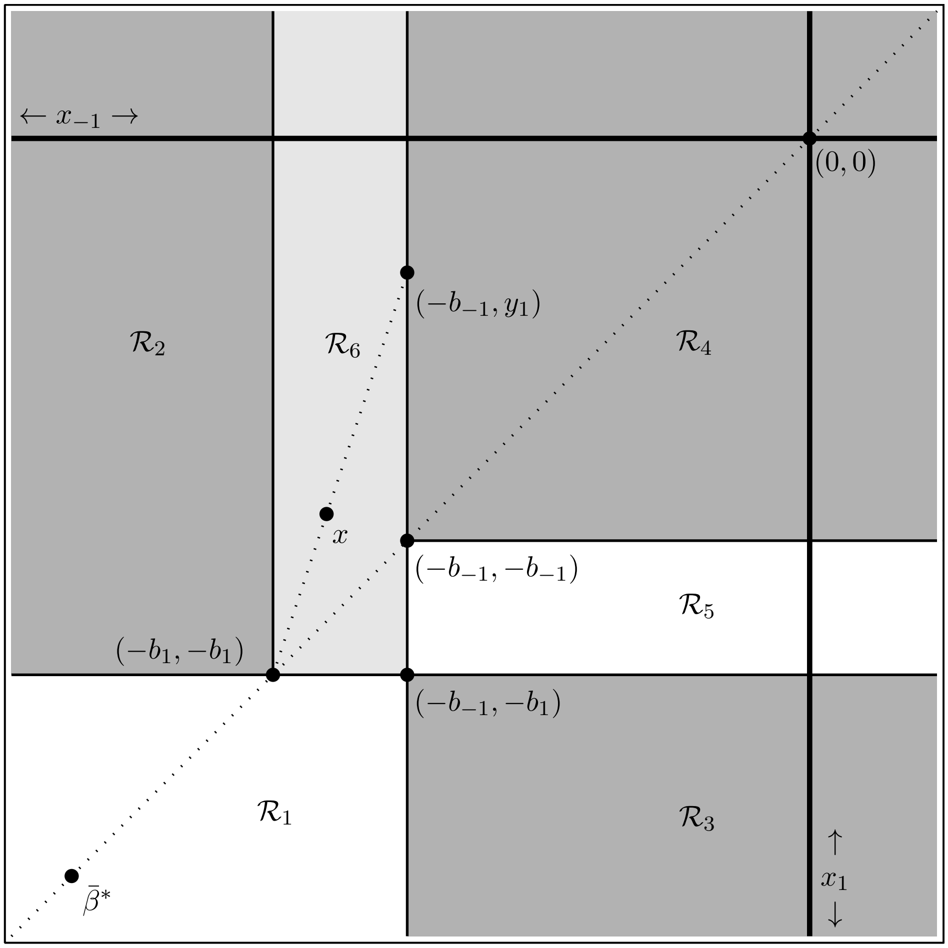

Figure A–1: Regions for x to show that $\left\|\Pi(x) - \bar{\beta}^*\right\|_Q \leq \left\|x - \bar{\beta}^*\right\|_Q$

so $\left\|\Pi(x) - \bar{\beta}^*\right\|_Q = \left\|\Pi(-b_{-1}, -b_{-1}) - \bar{\beta}^*\right\|_Q$. Therefore, (38) will hold for $x \in \mathcal{R}_4$ if (38) holds for $(-b_{-1}, -b_{-1})$.

If $x_{-1} \geq -b_{-1}$ and $-b_1 \leq x_1 < -b_{-1}$, that is, $x \in \mathcal{R}_5$, then $(-b_{-1}, x_1)$ is no further in Euclidean distance than x from both $\bar{\beta}^*$ and $\mathcal{D}$. Thus, $\left\|(-b_{-1}, x_1) - \bar{\beta}^*\right\|_Q \leq \left\|x - \bar{\beta}^*\right\|_Q$. Also, $\Pi(x) = \Pi(-b_{-1}, x_1)$, so $\left\|\Pi(x) - \bar{\beta}^*\right\|_Q = \left\|\Pi(-b_{-1}, x_1) - \bar{\beta}^*\right\|_Q$. Hence, (38) will hold for $x \in \mathcal{R}_5$ if (38) holds for $(-b_{-1}, x_1)$.

Therefore, to show that (38) holds for all $x \in \mathcal{R}_4 \cup \mathcal{R}_5 \cup \mathcal{R}_6$, it is sufficient to show that (38) holds for all points in $\mathcal{R}_6 := \{x \in \mathbb{R}^2 : -b_1 < x_{-1} \leq -b_{-1} \text{ and } x_1 \geq -b_1\}$. We begin by showing that

$$\left\|-b - \bar{\beta}^*\right\|_Q^2 \quad \leq \quad \left\|(-b_{-1}, y_1) - \bar{\beta}^*\right\|_Q^2 \tag{A–7}$$

for all $y_1 \geq -b_1$. If $y_1 = -b_1$ then (A–7) holds with equality. Suppose that $y_1 > -b_1$. It follows from (36) that

$$
\begin{aligned}
&\left\|-b-\bar{\beta}^*\right\|_Q^2 \;\; \leq \;\; \left\|(-b_{-1},y_1)-\bar{\beta}^*\right\|_Q^2 \\
&\iff \quad (1-q)\left\|-b-\bar{\beta}^*\right\|^2+q(b_1-b_{-1})^2 \;\; \leq \;\; (1-q)\left\|(-b_{-1},y_1)-\bar{\beta}^*\right\|^2+q(y_1+b_{-1})^2 \\
&\iff \quad (1-q)\left[(-b_{-1}-\beta^*)^2+(-b_1-\beta^*)^2\right]+q(b_1-b_{-1})^2 \\
&\qquad \leq \;\; (1-q)\left[(-b_{-1}-\beta^*)^2+(y_1-\beta^*)^2\right]+q(y_1+b_{-1})^2 \\
&\iff \quad (1-q)\left[b_1^2+2b_1\beta^*\right]+q\left[b_1^2-2b_{-1}b_1\right] \;\; \leq \;\; (1-q)\left[y_1^2-2y_1\beta^*\right]+q\left[y_1^2+2b_{-1}y_1\right] \\
&\iff \quad b_1^2-b_1\left[-2(1-q)\beta^*+2qb_{-1}\right] \;\; \leq \;\; y_1^2+y_1\left[-2(1-q)\beta^*+2qb_{-1}\right] \\
&\iff \quad (y_1+b_1)\left[(y_1-b_1)-2(1-q)\beta^*+2qb_{-1}\right] \;\; \geq \;\; 0.
\end{aligned}
$$

Since $y_1+b_1>0$, the final inequality above holds if and only if $(y_1-b_1)-2(1-q)\beta^*+2qb_{-1}\geq 0$, i.e., if and only if $2q(\beta^*+b_{-1})\geq b_1-y_1+2\beta^*$. We are assuming that $\beta^*\leq -b_i$, $i=\pm 1$, so it follows that the inequalities above hold if and only if

$$
q \;\; \leq \;\; \frac{b_1-y_1+2\beta^*}{2(\beta^*+b_{-1})}\,. \tag{A–8}
$$

A sufficient condition for (A–8) to hold is that

$$
q \;\; \leq \;\; \inf_{y_1>-b_1}\frac{b_1-y_1+2\beta^*}{2(\beta^*+b_{-1})} \;\; = \;\; \frac{2b_1+2\beta^*}{2(\beta^*+b_{-1})} \;\; = \;\; 1+\frac{b_1-b_{-1}}{\beta^*+b_{-1}}\,, \tag{A–9}
$$

which holds by (37). Therefore, we have shown that (A–7) holds for all $y_1\geq -b_1$.

Consider any $x\in\mathcal{R}_6$ and note that x can be expressed as $x=\lambda(-b_{-1},y_1)+(1-\lambda)(-b_1,-b_1)$ for some $y_1\geq -b_1$ and $\lambda\in[0,1]$; see Figure A–1. Then

$$
\begin{aligned}
&\left\|-b-\bar{\beta}^*\right\|_Q^2-\left\|(-b_{-1},y_1)-\bar{\beta}^*\right\|_Q^2 \\
&\quad = \;\; (1-q)(-b_1-\beta^*)^2+q(b_1-b_{-1})^2-(1-q)(y_1-\beta^*)^2-q(y_1+b_{-1})^2.
\end{aligned} \tag{A–10}
$$

Observe also that $\Pi(x) = \lambda(-b_{-1}, -b_1) + (1-\lambda)(-b_1, -b_1)$. Therefore,

$$\begin{aligned}
&\left\|\Pi(x) - \bar{\beta}^*\right\|_Q^2 - \left\|x - \bar{\beta}^*\right\|_Q^2 \\
&= (1-q)\left([\lambda(-b_{-1}) + (1-\lambda)(-b_1) - \beta^*]^2 + [-b_1 - \beta^*]^2\right) + q\lambda^2(b_1 - b_{-1})^2 \\
&\quad - (1-q)\left([\lambda(-b_{-1}) + (1-\lambda)(-b_1) - \beta^*]^2 + [\lambda(y_1) + (1-\lambda)(-b_1) - \beta^*]^2\right) - q\lambda^2(y_1 + b_{-1})^2 \\
&= (1-q)[\lambda(-b_1) + (1-\lambda)(-b_1) - \beta^*]^2 + q\lambda^2(b_1 - b_{-1})^2 \\
&\quad - (1-q)[\lambda(y_1) + (1-\lambda)(-b_1) - \beta^*]^2 - q\lambda^2(y_1 + b_{-1})^2 \\
&= (1-q)\lambda^2(-b_1 - \beta^*)^2 + q\lambda^2(b_1 - b_{-1})^2 - (1-q)\lambda^2(y_1 - \beta^*)^2 - q\lambda^2(y_1 + b_{-1})^2 \\
&\quad + (1-q)[(1-\lambda)^2(-b_1 - \beta^*)^2 + 2\lambda(1-\lambda)(-b_1 - \beta^*)^2] \\
&\quad - (1-q)[(1-\lambda)^2(-b_1 - \beta^*)^2 + 2\lambda(1-\lambda)(y_1 - \beta^*)(-b_1 - \beta^*)] \\
&= \lambda^2\left(\left\|-b - \bar{\beta}^*\right\|_Q^2 - \left\|(-b_{-1}, y_1) - \bar{\beta}^*\right\|_Q^2\right) + 2(1-q)\lambda(1-\lambda)(-b_1 - \beta^*)(-b_1 - y_1) \\
&\le 2(1-q)\lambda(1-\lambda)(-b_1 - \beta^*)(-b_1 - y_1)\,,
\end{aligned}$$

where the final equality follows from (A–10) and the inequality follows from (A–7). Hence, $\left\|\Pi(x) - \bar{\beta}^*\right\|_Q^2 \le \left\|x - \bar{\beta}^*\right\|_Q^2$ if $(1-q)\lambda(1-\lambda)(-b_1 - \beta^*)(-b_1 - y_1) \le 0$. The preceding inequality does indeed hold because $\beta^* \le -b_1 \le y_1$, $\lambda \in [0,1]$, and $q < 1$. This completes the proof in the case that (37) holds.

Next, suppose that $\beta^* + b_{\min} = 0$. Then, since $\beta^* \le -b_{\max}$, it follows that $\beta^* = -b_{-1} = -b_1$. Therefore, regions $\mathcal{R}_5$ and $\mathcal{R}_6$ are empty. For $x \in \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$, (38) holds by the arguments above. For $x \in \mathcal{R}_4$, $\Pi(x) = \bar{\beta}^*$ and (38) holds trivially. □

Proof of Lemma 2. Let $r = x_1/x_{-1}$. Note that

$$\begin{aligned}
h(x)^T Q y &= \begin{bmatrix} y_{-1} + \beta_d(1/r - 1) \\ y_1 + \beta_d(r-1) \end{bmatrix}^T \begin{bmatrix} 1 & -q \\ -q & 1 \end{bmatrix} \begin{bmatrix} y_{-1} \\ y_1 \end{bmatrix} \\
&= \|y\|_Q^2 + y_{-1}\big[\beta_d(1/r - 1) - q\beta_d(r-1)\big] + y_1\big[-q\beta_d(1/r - 1) + \beta_d(r-1)\big] \\
&= \|y\|_Q^2 + \beta_d(r-1)\big[y_1(1 + q/r) - y_{-1}(q + 1/r)\big]\,. \qquad \text{(A–11)}
\end{aligned}$$

Next, since $r = x_1/x_{-1} = (\beta^* - y_1)/(\beta^* - y_{-1})$, it follows that $y_{-1} = y_1/r + (1 - 1/r)\,\beta^*$ and hence,

$$\begin{aligned}
&\beta_d(r-1)\big[y_1(1+q/r) - y_{-1}(q+1/r)\big] \\
&\qquad = \beta_d(r-1)\big[y_1(1+q/r-q/r-1/r^2) - \beta^*(q+1/r)(1-1/r)\big] \\
&\qquad = \frac{\beta_d(r-1)}{r^2}\big[y_1(r^2-1) - \beta^*(rq+1)(r-1)\big] \\
&\qquad = \frac{\beta_d(r-1)^2}{r^2}\big[y_1(r+1) - \beta^*(rq+1)\big] \\
&\qquad = \frac{\beta_d(r-1)^2(r+1)}{r^2}\left[(y_1-\beta^*) - \beta^*(q-1)\frac{r}{r+1}\right]. \qquad \text{(A–12)}
\end{aligned}$$

Observe that in (A–12) the first term $\beta_d(r-1)^2(r+1)/r^2$ is nonnegative. The second term is always positive, since $y_1 - \beta^* = -x_1 \geq b_1$ and

$$\beta^*(q-1)\frac{r}{r+1} \quad < \quad \beta^*(q-1) \quad \leq \quad b_{\min}.$$

It follows that $\beta_d(r-1)\big[y_1(1+q/r) - y_{-1}(q+1/r)\big] \geq 0$, and hence, it follows from (A–11) that $h^T Q y \geq \|y\|_Q^2$. □

Lemma A–2. *Suppose that $\mathbb{E}[\varepsilon_i^{k+1}|\mathcal{F}^k] = 0$, and there is a constant M such that $\mathbb{E}[(\varepsilon_i^{k+1})^2|\mathcal{F}^k] \leq M$ for each k and each i. Then, there are positive constants A and B such that, for each k and each i,*

$$\mathbb{E}\left[(H^{k+1})^T Q H^{k+1} \,\middle|\, \mathcal{F}^k\right] \quad \leq \quad A + B\left\|\bar{\beta}^* - \widehat{\alpha}^k\right\|_Q^2. \qquad \text{(A–13)}$$

Proof. Note that

$$\begin{aligned}
&\mathbb{E}\left[(H^{k+1})^T Q H^{k+1} \,\middle|\, \mathcal{F}^k\right] \\
&= \mathbb{E}\left[(H_{-1}^{k+1})^2 + (H_1^{k+1})^2 - 2qH_{-1}^{k+1}H_1^{k+1} \,\middle|\, \mathcal{F}^k\right] \\
&= \mathbb{E}\left[\left(h_{-1}^k - 2\frac{\widehat{\alpha}_{-1}^k}{\beta_0}\varepsilon_{-1}^{k+1}\right)^2 + \left(h_1^k - 2\frac{\widehat{\alpha}_1^k}{\beta_0}\varepsilon_1^{k+1}\right)^2 - 2q\left(h_{-1}^k - 2\frac{\widehat{\alpha}_{-1}^k}{\beta_0}\varepsilon_{-1}^{k+1}\right)\left(h_1^k - 2\frac{\widehat{\alpha}_1^k}{\beta_0}\varepsilon_1^{k+1}\right) \,\middle|\, \mathcal{F}^k\right] \\
&= \left(h_{-1}^k\right)^2 + 4\left(\frac{\widehat{\alpha}_{-1}^k}{\beta_0}\right)^2 \mathbb{E}\left[\left(\varepsilon_{-1}^{k+1}\right)^2 \,\middle|\, \mathcal{F}^k\right] - 4h_{-1}^k\frac{\widehat{\alpha}_{-1}^k}{\beta_0}\mathbb{E}\left[\varepsilon_{-1}^{k+1} \,\middle|\, \mathcal{F}^k\right] \\
&\quad + \left(h_1^k\right)^2 + 4\left(\frac{\widehat{\alpha}_1^k}{\beta_0}\right)^2 \mathbb{E}\left[\left(\varepsilon_1^{k+1}\right)^2 \,\middle|\, \mathcal{F}^k\right] - 4h_1^k\frac{\widehat{\alpha}_1^k}{\beta_0}\mathbb{E}\left[\varepsilon_1^{k+1} \,\middle|\, \mathcal{F}^k\right] \\
&\quad - 2qh_{-1}^k h_1^k - 8q\frac{\widehat{\alpha}_{-1}^k}{\beta_0}\frac{\widehat{\alpha}_1^k}{\beta_0}\mathbb{E}\left[\varepsilon_{-1}^{k+1}\varepsilon_1^{k+1} \,\middle|\, \mathcal{F}^k\right] + 4qh_{-1}^k\frac{\widehat{\alpha}_1^k}{\beta_0}\mathbb{E}\left[\varepsilon_1^{k+1} \,\middle|\, \mathcal{F}^k\right] + 4qh_1^k\frac{\widehat{\alpha}_{-1}^k}{\beta_0}\mathbb{E}\left[\varepsilon_{-1}^{k+1} \,\middle|\, \mathcal{F}^k\right] \\
&= \left(h_{-1}^k\right)^2 + 4\left(\frac{\widehat{\alpha}_{-1}^k}{\beta_0}\right)^2 \mathbb{E}\left[\left(\varepsilon_{-1}^{k+1}\right)^2 \,\middle|\, \mathcal{F}^k\right] + \left(h_1^k\right)^2 + 4\left(\frac{\widehat{\alpha}_1^k}{\beta_0}\right)^2 \mathbb{E}\left[\left(\varepsilon_1^{k+1}\right)^2 \,\middle|\, \mathcal{F}^k\right] \\
&\quad - 2qh_{-1}^k h_1^k - 8q\frac{\widehat{\alpha}_{-1}^k}{\beta_0}\frac{\widehat{\alpha}_1^k}{\beta_0}\mathbb{E}\left[\varepsilon_{-1}^{k+1}\varepsilon_1^{k+1} \,\middle|\, \mathcal{F}^k\right]. \qquad \text{(A–14)}
\end{aligned}$$

Also,

$$
\begin{aligned}
\left(h^k_{-1}\right)^2 + \left(h^k_1\right)^2 - 2q h^k_{-1} h^k_1 &= \left\| h^k \right\|^2_Q \\
&= \left\| \begin{bmatrix} (\beta^* - \widehat{\alpha}^k_{-1}) + \beta_d \left(\frac{\widehat{\alpha}^k_{-1}}{\widehat{\alpha}^k_1} - 1 \right) \\ (\beta^* - \widehat{\alpha}^k_1) + \beta_d \left(\frac{\widehat{\alpha}^k_1}{\widehat{\alpha}^k_{-1}} - 1 \right) \end{bmatrix} \right\|^2_Q \\
&= \left\| \begin{bmatrix} \left(1 - \frac{\beta_d}{\widehat{\alpha}^k_1}\right)(\beta^* - \widehat{\alpha}^k_{-1}) \\ \left(1 - \frac{\beta_d}{\widehat{\alpha}^k_{-1}}\right)(\beta^* - \widehat{\alpha}^k_1) \end{bmatrix} + \beta_d \begin{bmatrix} \frac{\beta^*}{\widehat{\alpha}^k_1} - 1 \\ \frac{\beta^*}{\widehat{\alpha}^k_{-1}} - 1 \end{bmatrix} \right\|^2_Q \\
&\leq 2 \left\| \begin{bmatrix} \left(1 - \frac{\beta_d}{\widehat{\alpha}^k_1}\right)(\beta^* - \widehat{\alpha}^k_{-1}) \\ \left(1 - \frac{\beta_d}{\widehat{\alpha}^k_{-1}}\right)(\beta^* - \widehat{\alpha}^k_1) \end{bmatrix} \right\|^2_Q + 2\beta_d^2 \left\| \begin{bmatrix} \frac{\beta^*}{\widehat{\alpha}^k_1} - 1 \\ \frac{\beta^*}{\widehat{\alpha}^k_{-1}} - 1 \end{bmatrix} \right\|^2_Q \qquad \text{(A–15)}
\end{aligned}
$$

where the inequality holds because $\|x + y\|^2_Q \leq 2\|x\|^2_Q + 2\|y\|^2_Q$ (which is a consequence of the equality $\|x + y\|^2_Q + \|x - y\|^2_Q = 2\|x\|^2_Q + 2\|y\|^2_Q$).

Next, we show that there is a finite constant C such that

$$
\left\| \begin{bmatrix} \left(1 - \frac{\beta_d}{\widehat{\alpha}^k_1}\right)(\beta^* - \widehat{\alpha}^k_{-1}) \\ \left(1 - \frac{\beta_d}{\widehat{\alpha}^k_{-1}}\right)(\beta^* - \widehat{\alpha}^k_1) \end{bmatrix} \right\|^2_Q \leq C^2 \left\| \bar{\beta}^* - \widehat{\alpha}^k \right\|^2_Q . \qquad \text{(A–16)}
$$

Recall that $\beta_d \geq 0$ and $\widehat{\alpha}^k_i \leq -b_i < 0$, and thus $1 \leq 1 - \beta_d/\widehat{\alpha}^k_i \leq 1 + \beta_d/b_i$. Therefore, to show (A–16), it suffices to find C such that

$$
\left\| \begin{bmatrix} a_{-1} x_{-1} \\ a_1 x_1 \end{bmatrix} \right\|^2_Q \leq C^2 \|x\|^2_Q \qquad \text{(A–17)}
$$

for all $x \in \mathbb{R}^2$ and all $a = (a_{-1}, a_1) \in \mathcal{S} := [1, 1 + \beta_d/b_1] \times [1, 1 + \beta_d/b_{-1}]$. To this end, consider the function

$$
f(a) := \left\| \begin{bmatrix} a_{-1} x_{-1} \\ a_1 x_1 \end{bmatrix} \right\|^2_Q = a^2_{-1} x^2_{-1} + a^2_1 x^2_1 - 2q a_{-1} a_1 x_{-1} x_1
$$

Note that

$$
\nabla^2 f(a) = \begin{bmatrix} 2x^2_{-1} & -2q x_{-1} x_1 \\ -2q x_{-1} x_1 & 2x^2_1 \end{bmatrix} .
$$

Since $2x^2_{-1} \geq 0$, $2x^2_1 \geq 0$, and $\left|\nabla^2 f(a)\right| = 4(1 - q^2) x^2_{-1} x^2_1 \geq 0$, it follows that $\nabla^2 f(a)$ is positive semidefinite for all a, and thus f is convex. Hence f attains its maximum over $\mathcal{S}$ at an extreme point of $\mathcal{S}$. Therefore, if (A–17) holds for all $x \in \mathbb{R}^2$ and all extreme points of $\mathcal{S}$, then (A–17) holds for all $x \in \mathbb{R}^2$ and all $a \in \mathcal{S}$.

Note that

$$\lim_{C\to\infty} \frac{\sqrt{C^2 - a_{-1}^2}\sqrt{C^2 - a_1^2}}{C^2 - a_{-1}a_1} \quad = \quad 1$$

Recall that $0 \leq q < 1$. Choose $C > \max\{1 + \beta_d/b_1, 1 + \beta_d/b_{-1}\}$ such that

$$q \quad \leq \quad \frac{\sqrt{C^2 - a_{-1}^2}\sqrt{C^2 - a_1^2}}{C^2 - a_{-1}a_1}$$

for all a in the set of four extreme points of $\mathcal{S}$.

Consider $x \in \mathbb{R}^2$ and an extreme point a of $\mathcal{S}$. If $x_{-1}x_1 > 0$ then

$$\begin{aligned}
&C^2 \|x\|_Q^2 - \left\| \begin{bmatrix} a_{-1}x_{-1} \\ a_1 x_1 \end{bmatrix} \right\|_Q^2 \\
&= (C^2 - a_{-1}^2)x_{-1}^2 + (C^2 - a_1^2)x_1^2 - 2q(C^2 - a_{-1}a_1)x_{-1}x_1 \\
&= (C^2 - a_{-1}^2)x_{-1}^2 + (C^2 - a_1^2)x_1^2 - 2\sqrt{C^2 - a_{-1}^2}\sqrt{C^2 - a_1^2}x_{-1}x_1 \\
&\quad + 2\left(1 - q\frac{C^2 - a_{-1}a_1}{\sqrt{C^2 - a_{-1}^2}\sqrt{C^2 - a_1^2}}\right)\sqrt{C^2 - a_{-1}^2}\sqrt{C^2 - a_1^2}x_{-1}x_1 \\
&= \left(\sqrt{C^2 - a_{-1}^2}x_{-1} - \sqrt{C^2 - a_1^2}x_1\right)^2 + 2\left(1 - q\frac{C^2 - a_{-1}a_1}{\sqrt{C^2 - a_{-1}^2}\sqrt{C^2 - a_1^2}}\right)\sqrt{C^2 - a_{-1}^2}\sqrt{C^2 - a_1^2}x_{-1}x_1 \\
&\geq 0
\end{aligned}$$

If $x_{-1}x_1 \leq 0$, that is, $qx_{-1}x_1 \leq 0$, then

$$\begin{aligned}
\left\| \begin{bmatrix} a_{-1}x_{-1} \\ a_1 x_1 \end{bmatrix} \right\|_Q^2 &= a_{-1}^2 x_{-1}^2 + a_1^2 x_1^2 - 2q a_{-1}a_1 x_{-1}x_1 \\
&\leq (1 + \beta_d/b_1)^2 x_{-1}^2 + (1 + \beta_d/b_{-1})^2 x_1^2 - 2q(1 + \beta_d/b_1)(1 + \beta_d/b_{-1})x_{-1}x_1 \\
&\leq C^2 x_{-1}^2 + C^2 x_1^2 - 2qC^2 x_{-1}x_1 \\
&= C^2 \|x\|_Q^2
\end{aligned}$$

Thus, (A–17) holds for all $x \in \mathbb{R}$ and all extreme points of $\mathcal{S}$. Hence, (A–16) holds.

Next, we show that there is a constant K such that

$$\left\| \begin{bmatrix} \frac{\beta^*}{\widehat{\alpha}_1^k} - 1 \\ \frac{\beta^*}{\widehat{\alpha}_{-1}^k} - 1 \end{bmatrix} \right\|_Q^2 \quad \leq \quad K\,. \qquad \text{(A–18)}$$

Recall that $\beta^* < 0$ and $\widehat{\alpha}_i^k \leq -b_i < 0$, and thus $-1 < \beta^*/\widehat{\alpha}_i^k - 1 \leq -\beta^*/b_i - 1$. As before, the function $g(a) := \|a\|_Q^2$ is convex, and thus attains its maximum at an extreme point of $[-1, -\beta^*/b_1 - 1] \times [-1, -\beta^*/b_{-1} - 1]$. Consequently, (A–18) holds with

$$K \quad := \quad \max\left\{g(a) \, : \, a \in [-1, -\beta^*/b_1 - 1] \times [-1, -\beta^*/b_{-1} - 1]\right\}.$$

Therefore, it follows from (A–15), (A–16), and (A–18) that

$$\left(h_{-1}^k\right)^2 + \left(h_1^k\right)^2 - 2q h_{-1}^k h_1^k \quad \leq \quad 2C^2 \left\|\bar{\beta}^* - \widehat{\alpha}^k\right\|_Q^2 + 2\beta_d^2 K. \tag{A–19}$$

Next, observe that $q\widehat{\alpha}_{-1}^k \widehat{\alpha}_1^k \geq 0$, and $\varepsilon_{-1}^{k+1}\varepsilon_1^{k+1} \geq -[(\varepsilon_{-1}^{k+1})^2 + (\varepsilon_1^{k+1})^2]/2$, and thus

$$-q\widehat{\alpha}_{-1}^k \widehat{\alpha}_1^k \mathbb{E}\left[\varepsilon_{-1}^{k+1}\varepsilon_1^{k+1} \,\middle|\, \mathcal{F}^k\right] \quad \leq \quad q\widehat{\alpha}_{-1}^k \widehat{\alpha}_1^k \mathbb{E}\left[(\varepsilon_{-1}^{k+1})^2 + (\varepsilon_1^{k+1})^2]/2 \,\middle|\, \mathcal{F}^k\right] \quad \leq \quad q\widehat{\alpha}_{-1}^k \widehat{\alpha}_1^k M.$$

Hence,

$$\begin{aligned} 4\left(\frac{\widehat{\alpha}_{-1}^k}{\beta_0}\right)^2 \mathbb{E}\left[\left(\varepsilon_{-1}^{k+1}\right)^2 \,\middle|\, \mathcal{F}^k\right] + 4\left(\frac{\widehat{\alpha}_1^k}{\beta_0}\right)^2 \mathbb{E}\left[\left(\varepsilon_1^{k+1}\right)^2 \,\middle|\, \mathcal{F}^k\right] - 8q\frac{\widehat{\alpha}_{-1}^k}{\beta_0}\frac{\widehat{\alpha}_1^k}{\beta_0}\mathbb{E}\left[\varepsilon_{-1}^{k+1}\varepsilon_1^{k+1} \,\middle|\, \mathcal{F}^k\right] \\ \leq \quad \frac{4}{\beta_0^2} M \left[\left(\widehat{\alpha}_{-1}^k\right)^2 + \left(\widehat{\alpha}_1^k\right)^2 + 2q\widehat{\alpha}_{-1}^k \widehat{\alpha}_1^k\right]. \end{aligned} \tag{A–20}$$

Next we show how to choose $\tilde{C} > 1$ so that

$$\tilde{C}^2 \|x\|_Q^2 \quad \geq \quad x_{-1}^2 + x_1^2 + 2q x_{-1} x_1 \tag{A–21}$$

for all $x \in \mathbb{R}_-^2$. Note that

$$\lim_{\tilde{C}\to\infty} \frac{\tilde{C}^2 - 1}{\tilde{C}^2 + 1} \quad = \quad 1$$

Recall that $0 \leq q < 1$. Choose $\tilde{C} > 1$ such that

$$q \quad \leq \quad \frac{\tilde{C}^2 - 1}{\tilde{C}^2 + 1}$$

Then

$$\begin{aligned} & \tilde{C}^2 \|x\|_Q^2 - \left[x_{-1}^2 + x_1^2 + 2q x_{-1} x_1\right] \\ = \quad & (\tilde{C}^2 - 1)x_{-1}^2 + (\tilde{C}^2 - 1)x_1^2 - 2q(\tilde{C}^2 + 1)x_{-1}x_1 \\ = \quad & (\tilde{C}^2 - 1)x_{-1}^2 + (\tilde{C}^2 - 1)x_1^2 - 2\sqrt{\tilde{C}^2 - 1}\sqrt{\tilde{C}^2 - 1}x_{-1}x_1 \\ & + 2\left(1 - q\frac{\tilde{C}^2 + 1}{\tilde{C}^2 - 1}\right)\sqrt{\tilde{C}^2 - 1}\sqrt{\tilde{C}^2 - 1}x_{-1}x_1 \\ = \quad & \left(\sqrt{\tilde{C}^2 - 1}x_{-1} - \sqrt{\tilde{C}^2 - 1}x_1\right)^2 + 2\left(1 - q\frac{\tilde{C}^2 + 1}{\tilde{C}^2 - 1}\right)\sqrt{\tilde{C}^2 - 1}\sqrt{\tilde{C}^2 - 1}x_{-1}x_1 \quad \geq \quad 0. \end{aligned}$$

Thus, it follows from (A–20) and (A–21) that

$$
\begin{aligned}
4\left(\frac{\widehat{\alpha}^k_{-1}}{\beta_0}\right)^2 \mathbb{E}\left[\left(\varepsilon^{k+1}_{-1}\right)^2 \,\middle|\, \mathcal{F}^k\right] + 4\left(\frac{\widehat{\alpha}^k_{1}}{\beta_0}\right)^2 \mathbb{E}\left[\left(\varepsilon^{k+1}_{1}\right)^2 \,\middle|\, \mathcal{F}^k\right] - 8q\frac{\widehat{\alpha}^k_{-1}}{\beta_0}\frac{\widehat{\alpha}^k_{1}}{\beta_0}\mathbb{E}\left[\varepsilon^{k+1}_{-1}\varepsilon^{k+1}_{1} \,\middle|\, \mathcal{F}^k\right]& \\
\leq \frac{4}{\beta_0^2} M\tilde{C}^2 \left\|\widehat{\alpha}^k\right\|_S^2& \\
= \frac{4M\tilde{C}^2}{\beta_0^2} \left\|\widehat{\alpha}^k - \bar{\beta}^* + \bar{\beta}^*\right\|_Q^2& \\
\leq \frac{8M\tilde{C}^2}{\beta_0^2} \left\|\widehat{\alpha}^k - \bar{\beta}^*\right\|_Q^2 + \frac{8M\tilde{C}^2}{\beta_0^2} \left\|\bar{\beta}^*\right\|_Q^2 .& \qquad \text{(A–22)}
\end{aligned}
$$

The result follows from (A–14), (A–19), and (A–22) by choosing

$$
\begin{aligned}
A &:= 2\beta_d^2 K + \frac{8M\tilde{C}^2}{\beta_0^2}\left\|\bar{\beta}^*\right\|_Q^2 \;>\; 0 \\
B &:= 2C^2 + \frac{8M\tilde{C}^2}{\beta_0^2} \;>\; 0.
\end{aligned}
$$

□

In the above developments we have used Lemma A–3, which was established by Robbins and Siegmund (1971).

Lemma A–3. *Consider a sequence of finite, nonnegative random variables B^k, C^k, D^k, Z^k, adapted to the σ-field $\mathcal{F}^k$, that satisfy*

$$
\mathbb{E}\left[Z^{k+1} \,|\, \mathcal{F}^k\right] \;\leq\; (1 + B^k)Z^k + C^k - D^k.
$$

Then, there is a finite random variable Z such that, on the set $\{\sum_k B^k < \infty, \sum_k C^k < \infty\}$, w.p.1,

$$
\lim_{k\to\infty} Z^k \;=\; Z \qquad \text{and} \qquad \sum_k D^k \;<\; \infty.
$$

Proof of Proposition 3. Suppose that each seller i has observed price-quantity pairs $(p_i^0, d_i^1), \ldots, (p_i^k, d_i^{k+1})$ by the end of period $k+1$. First we write p_i^{k+1} as a function of these past observations. To do so, it is useful to define the following quantities:

$$\begin{aligned}
\overline{p}_i^{k+1} &:= \frac{1}{k+1}\sum_{\ell=0}^{k} p_i^\ell &=& \frac{k}{k+1}\overline{p}_i^k + \frac{1}{k+1}p_i^k \\
\overline{d}_i^{k+1} &:= \frac{1}{k+1}\sum_{\ell=1}^{k+1} d_i^\ell &=& \frac{k}{k+1}\overline{d}_i^k + \frac{1}{k+1}d_i^{k+1} \\
\overline{p^2}_i^{k+1} &:= \frac{1}{k+1}\sum_{\ell=0}^{k} (p_i^\ell)^2 &=& \frac{k}{k+1}\overline{p^2}_i^k + \frac{1}{k+1}(p_i^k)^2 \\
\overline{pd}_i^{k+1} &:= \frac{1}{k+1}\sum_{\ell=1}^{k+1} p_i^{\ell-1} d_i^\ell &=& \frac{k}{k+1}\overline{pd}_i^k + \frac{1}{k+1}p_i^k d_i^{k+1} \\
\overline{pp}^{k+1} &:= \frac{1}{k+1}\sum_{\ell=0}^{k} p_i^\ell p_{-i}^\ell, &&
\end{aligned}$$

where p_i^k is given by (41) and d_i^{k+1} is given by (42). It follows from (42) that

$$\begin{aligned}
\overline{d}_i^{k+1} &= \beta_{i,0} + \beta_{i,i}\overline{p}_i^{k+1} + \beta_{i,-i}\overline{p}_{-i}^{k+1} \\
\overline{pd}_i^{k+1} &= \beta_{i,0}\overline{p}_i^{k+1} + \beta_{i,i}\overline{p^2}_i^{k+1} + \beta_{i,-i}\overline{pp}^{k+1}.
\end{aligned}$$

Moreover, the familiar normal equations for linear regression yield

$$\widehat{\alpha}_i^k = \frac{\overline{pd}_i^k - \overline{p}_i^k\overline{d}_i^k}{\overline{p^2}_i^k - \left(\overline{p}_i^k\right)^2} = \beta_{i,i} + \beta_{i,-i}\frac{\overline{pp}^k - \overline{p}_i^k\overline{p}_{-i}^k}{\overline{p^2}_i^k - \left(\overline{p}_i^k\right)^2} \tag{A–23}$$

$$\widehat{\alpha}_{i,0}^k = \overline{d}_i^k - \widehat{\alpha}_i^k\overline{p}_i^k = \beta_{i,0} + \beta_{i,-i}\frac{\overline{p}_{-i}^k\overline{p^2}_i^k - \overline{p}_i^k\overline{pp}^k}{\overline{p^2}_i^k - \left(\overline{p}_i^k\right)^2} \tag{A–24}$$

and then we can use (41) to express p_i^k as a function of $\overline{p}_i^k$, $\overline{p}_{-i}^k$, $\overline{p^2}_i^k$, and $\overline{pp}^k$. It follows from the definition of r_i^k in (43) that

$$r_i^k = \frac{\overline{pp}^k - \overline{p}_i^k\overline{p}_{-i}^k}{\overline{p^2}_i^k - \left(\overline{p}_i^k\right)^2} \tag{A–25}$$

and thus we can write the parameter estimates in (A–23) and (A–24) as

$$\widehat{\alpha}_i^k = \beta_{i,i} + \beta_{i,-i}r_i^k \tag{A–26}$$

$$\widehat{\alpha}_{i,0}^k = \beta_{i,0} + \beta_{i,-i}\left(\overline{p}_{-i}^k - \overline{p}_i^k r_i^k\right). \tag{A–27}$$

Similarly, we can write the prices p_i^k as

$$p_i^k = -\frac{\widehat{\alpha}_{i,0}^k}{2\widehat{\alpha}_i^k} = -\frac{\beta_{i,0} + \beta_{i,-i}\left(\overline{p}_{-i}^k - \overline{p}_i^k r_i^k\right)}{2\left(\beta_{i,i} + \beta_{i,-i}r_i^k\right)}. \tag{A–28}$$

As indicated by the final expression in (43), a useful interpretation of r_i^k is as follows: for a fixed k, let (X_{-1}, X_1) denote a bivariate random vector that takes the values of the k empirically observed price pairs $\{(p_{-1}^0, p_1^0), \ldots, (p_{-1}^{k-1}, p_1^{k-1})\}$ each with probability $1/k$. Then

$$r_i^k = \frac{\mathrm{Cov}^k(X_i, X_{-i})}{\mathrm{Var}^k(X_i)}, \tag{A–29}$$

where the expectations in Cov^k and Var^k are taken with respect to the empirical measure introduced immediately above (A–29). Note that the expression in the denominator of (A–29) is $\mathrm{Var}^k(X_i)$, which is equal to zero if and only if $p_i^0 = \cdots = p_i^{k-1}$. Recall that r_i^k is well defined for all k since $p_i^0 \neq p_i^1$. Also note that

$$r_{-1}^k r_1^k = \frac{\left[\mathrm{Cov}^k(X_{-1}, X_1)\right]^2}{\mathrm{Var}^k(X_{-1})\mathrm{Var}^k(X_1)} = \left[\mathrm{Corr}^k(X_{-1}, X_1)\right]^2$$

and hence

$$0 \leq r_{-1}^k r_1^k \leq 1 \quad \text{for all } k. \tag{A–30}$$

Moreover, it follows that $r_{-1}^k r_1^k = 1$ if and only if all k pairs $(p_{-1}^0, p_1^0), \ldots, (p_{-1}^{k-1}, p_1^{k-1})$ lie on a straight line in $\mathbb{R}^2$. This, of course, is the case at $k = 2$. Also, $r_{-1}^k r_1^k = 0$ if and only if $r_{-1}^k = 0$ and $r_1^k = 0$ — this follows from the fact that $r_i^k = 0 \Leftrightarrow \mathrm{Cov}^k(X_{-1}, X_1) = 0$.

Equation (A–28) relates the prices p_i^k to the empirical mean, variance, and covariance of the previous prices. It follows from (A–28) that

$$r_i^k = \frac{\beta_{i,0} + \beta_{i,-i}\overline{p}_{-i}^k + 2\beta_{i,i}p_i^k}{\beta_{i,-i}\left(\overline{p}_i^k - 2p_i^k\right)}. \tag{A–31}$$

To study the asymptotic behavior of p_i^k, we first characterize potential limits. Suppose that the prices (p_{-1}^k, p_1^k) converge to some limit $(p_{-1}^*, p_1^*) > 0$ as $k \to \infty$. Then it follows from (A–31) that

$$r_i^k \to r_i^* = \frac{\beta_{i,0} + \beta_{i,-i}p_{-i}^* + 2\beta_{i,i}p_i^*}{-\beta_{i,-i}p_i^*} \tag{A–32}$$

for $i = \pm 1$. It follows from (A–30) that

$$0 \leq r_{-1}^* r_1^* \leq 1 \tag{A–33}$$

Moreover, the condition $\widehat{\alpha}_i^k < 0$ for all k and (A–26) imply that

$$r_i^* \leq \frac{-\beta_{i,i}}{\beta_{i,-i}}, \quad i = \pm 1. \tag{A–34}$$

Note that (A–32) can be written as a linear system in (p_{-1}^*, p_1^*):

$$\beta_{i,-i}p_{-i}^* + \left(\beta_{i,-i}r_i^* + 2\beta_{i,i}\right)p_i^* = -\beta_{i,0}\,, \quad i = \pm 1. \tag{A–35}$$

As shown in Lemma A–4 below, if (A–34) holds then the linear system is nonsingular, and yields the following solution for (p^*_{-1}, p^*_1):

$$p^*_i \;=\; p_i(r^*_{-1}, r^*_1), \tag{A–36}$$

where

$$p_i(r_{-1}, r_1) \;:=\; \frac{-2\,\beta_{i,0}\,\beta_{-i,-i} + \beta_{-i,0}\,\beta_{i,-i} - \beta_{i,0}\,\beta_{-i,i}\,r_{-i}}{4\,\beta_{-i,-i}\,\beta_{i,i} + 2\,\beta_{-i,-i}\,\beta_{i,-i}\,r_i + 2\,\beta_{i,i}\,\beta_{-i,i}\,r_{-i} - \beta_{-i,i}\,\beta_{i,-i}\,(1 - r_{-i}\,r_i)}.$$

Let

$$\begin{aligned} \mathcal{R} \;&:=\; \left\{(r_{-1}, r_1) \;:\; 0 \le r_{-1}r_1 \le 1,\; r_i \le \frac{-\beta_{i,i}}{\beta_{i,-i}},\; i = \pm 1\right\} \\ \mathcal{P} \;&:=\; \{(p_{-1}(r_{-1}, r_1), p_1(r_{-1}, r_1)) \;:\; (r_{-1}, r_1) \in \mathcal{R}\} \end{aligned}$$

that is, $\mathcal{R}$ is the set of all pairs (r_{-1}, r_1) that satisfy (A–33) and (A–34), and $\mathcal{P}$ is the set of potential limit prices. It is shown in Lemma A–5 below that $p_i(r_{-1}, r_1) > 0$ for all $(r_{-1}, r_1) \in \mathcal{R}$, and thus $(p_{-1}, p_1) > 0$ for all $(p_{-1}, p_1) \in \mathcal{P}$. Also, the steps for deriving (A–36) from (A–35) can be reversed, and thus it follows that for any $(p_{-1}, p_1) \in \mathcal{P}$ there exists a unique pair $(r_{-1}, r_1) \in \mathcal{R}$, that is, $p(r_{-1}, r_1) := (p_{-1}(r_{-1}, r_1), p_1(r_{-1}, r_1))$ is a bijection from $\mathcal{R}$ to $\mathcal{P}$. Thus, if such (p^*_{-1}, p^*_1) is the limit of (p^k_{-1}, p^k_1), then the unique pair $(r^*_{-1}, r^*_1) = p^{-1}(p^*_{-1}, p^*_1)$ that satisfies (A–35) must be the limit of (r^k_{-1}, r^k_1), and thus must satisfy (A–33) and (A–34). □

Lemma A–4. *If (A–34) holds, then the system (A–35) is nonsingular.*

Proof. Recall (A–35):

$$\beta_{i,-i}p^*_{-i} + (\beta_{i,-i}r^*_i + 2\beta_{i,i})p^*_i \;=\; -\beta_{i,0}\,, \qquad i = \pm 1.$$

Thus, the system (A–35) is nonsingular if and only if the determinant

$$\Delta \;:=\; 4\,\beta_{-1,-1}\,\beta_{1,1} + 2\,\beta_{-1,-1}\,\beta_{1,-1}\,r^*_1 + 2\,\beta_{1,1}\,\beta_{-1,1}\,r^*_{-1} - \beta_{-1,1}\,\beta_{1,-1}\,(1 - r^*_{-1}\,r^*_1) \;\neq\; 0$$

Next we show that if (A–34) holds, then $\Delta > 0$. Note that $r^*_{-i} \le -\beta_{-i,-i}/\beta_{-i,i}$, $\beta_{-i,i} \ge 0$, and $\beta_{-i,-i} < 0$ imply that $\beta_{-i,i}r^*_{-i} + 2\beta_{-i,-i} < 0$. Thus

$$\begin{aligned} \Delta \;&=\; \left(\beta_{-1,1}r^*_{-1} + 2\beta_{-1,-1}\right)\beta_{1,-1}r^*_1 + 2\left(\beta_{-1,1}r^*_{-1} + 2\beta_{-1,-1}\right)\beta_{1,1} - \beta_{-1,1}\beta_{1,-1} \\ &\ge\; -\left(\beta_{-1,1}r^*_{-1} + 2\beta_{-1,-1}\right)\beta_{1,1} + 2\left(\beta_{-1,1}r^*_{-1} + 2\beta_{-1,-1}\right)\beta_{1,1} - \beta_{-1,1}\beta_{1,-1} \\ &=\; \beta_{1,1}\beta_{-1,1}r^*_{-1} + 2\beta_{-1,-1}\beta_{1,1} - \beta_{-1,1}\beta_{1,-1} \\ &\ge\; -\beta_{-1,-1}\beta_{1,1} + 2\beta_{-1,-1}\beta_{1,1} - \beta_{-1,1}\beta_{1,-1} \\ &=\; \beta_{-1,-1}\beta_{1,1} - \beta_{-1,1}\beta_{1,-1} \;>\; 0 \end{aligned}$$

where the first inequality follows from $\beta_{-1,1} r^*_{-1} + 2\beta_{-1,-1} < 0$, $\beta_{1,-1} \geq 0$, and (A–34); the second inequality follows from $\beta_{1,1} < 0$, $\beta_{-1,1} \geq 0$, and (A–34); and the third inequality follows from (3). $\square$

Lemma A–5. $p_i(r_{-1}, r_1) > 0$ *for all* $(r_{-1}, r_1) \in \mathcal{R}$.

Proof. Recall that

$$p_i(r_{-1}, r_1) \quad := \quad \frac{-2\,\beta_{i,0}\,\beta_{-i,-i} + \beta_{-i,0}\,\beta_{i,-i} - \beta_{i,0}\,\beta_{-i,i}\,r_{-i}}{4\,\beta_{-i,-i}\,\beta_{i,i} + 2\,\beta_{-i,-i}\,\beta_{i,-i}\,r_i + 2\,\beta_{i,i}\,\beta_{-i,i}\,r_{-i} - \beta_{-i,i}\,\beta_{i,-i}\,(1 - r_{-i}\,r_i)}.$$

Note that the denominator on the right side is the determinant Δ, and it was shown above in the proof of Lemma A–4 that if $r_i \leq -\beta_{i,i}/\beta_{i,-i}$ for $i = \pm 1$, then $\Delta > 0$. Next, consider the numerator on the right side:

$$\begin{aligned} -2\,\beta_{i,0}\,\beta_{-i,-i} + \beta_{-i,0}\,\beta_{i,-i} - \beta_{i,0}\,\beta_{-i,i}\,r_{-i} \quad &\geq \quad -2\,\beta_{i,0}\,\beta_{-i,-i} + \beta_{-i,0}\,\beta_{i,-i} + \beta_{i,0}\,\beta_{-i,-i} \\ &= \quad -\beta_{i,0}\,\beta_{-i,-i} + \beta_{-i,0}\,\beta_{i,-i} \quad > \quad 0 \end{aligned}$$

where the first inequality follows because $r_i \leq -\beta_{i,i}/\beta_{i,-i}$ for $i = \pm 1$ for $(r_{-1}, r_1) \in \mathcal{R}$ and the second inequality follows from $\beta_{i,0} > 0$, $\beta_{i,i} < 0$, and $\beta_{i,-i} \geq 0$ for $i = \pm 1$. $\square$

Proof of Proposition 4. For $k_0 \geq 2$ we shall use the following conditions:

$$\overline{p}_i^{k_0} \quad = \quad p_i^{k_0} \tag{A–37}$$

and

$$\frac{\overline{p}_{-i}^{k_0} - p_{-i}^{k_0}}{\overline{p}_i^{k_0} - p_i^{k_0}} \quad = \quad r_i^{k_0} \quad \text{and} \quad \overline{p}_i^{k_0} \quad \neq \quad p_i^{k_0}. \tag{A–38}$$

Lemma A–6 below gives necessary and sufficient conditions for the process to be stationary. Using the lemma, we will show that the process can become stationary at most of the potential limit prices $(p^*_{-1}, p^*_1) \in \mathcal{P}$. The only exceptions are the following:

1. The points $(p^*_{-1}, p^*_1) \in \mathcal{P}$ corresponding to (r^*_{-1}, r^*_1) such that $r^*_{-1} r^*_1 = 0$ but either $r^*_{-1} \neq 0$ or $r^*_1 \neq 0$. It follows from the comment after (A–30) that, if the system is stationary at period k_0 with $r_{-1}^{k_0} r_1^{k_0} = 0$, then it must hold that $r^*_{-1} = r_{-1}^{k_0} = r^*_1 = r_1^{k_0} = 0$.

2. The points $(p^*_{-1}, p^*_1) \in \mathcal{P}$ corresponding to (r^*_{-1}, r^*_1) such that $r^*_i = -\beta_{i,i}/\beta_{i,-i}$ for $i = -1$ or $i = 1$, since it follows from (A–26) that in those cases $\widehat{\alpha}_i^{k_0} = \beta_{i,i} + \beta_{i,-i} r_i^{k_0} = \beta_{i,i} + \beta_{i,-i} r^*_i = 0$ and hence the iterative procedure stops [cf. remark after (41)].

The exceptions above could be limit values but not stationary ones. Thus, condition (A–33) is replaced with

$$\text{either} \quad r^*_{-1} \;=\; r^*_1 \;=\; 0 \quad \text{or} \quad 0 \;<\; r^*_{-1} r^*_1 \;\leq\; 1 \tag{A–39}$$

and condition (A–34) is replaced with

$$r^*_i \;<\; \frac{-\beta_{i,i}}{\beta_{i,-i}}, \qquad i = \pm 1. \tag{A–40}$$

We proceed now with the proof. Consider $(p^*_{-1}, p^*_1) \in \mathcal{P}'$. By the definition of $\mathcal{P}'$, there exists a pair $(r^*_{-1}, r^*_1) \in \mathcal{R}'$ such that $p^*_i = p_i(r^*_{-1}, r^*_1)$ for $i = \pm 1$. Suppose initially that $0 < r^*_{-1} r^*_1 < 1$. In what follows, the superscript 3 refers to $k = 3$, i.e. the values calculated from the three initial price pairs p^0, p^1, p^2.

To simplify the notation, define

$$\begin{aligned} w &:= \overline{pp}^3 \qquad & s &:= \overline{p}^3_{-1} \qquad & t &:= \overline{p}^3_1 \\ & & u &:= \overline{p^2}^3_{-1} \qquad & v &:= \overline{p^2}^3_1 \end{aligned}$$

and

$$\begin{aligned} a &:= p^0_{-1} \qquad & c &:= p^1_{-1} \qquad & x &:= p^2_{-1} \\ b &:= p^0_1 \qquad & d &:= p^1_1 \qquad & y &:= p^2_1. \end{aligned}$$

Then the following relations hold:

$$ab + cd + xy = 3w \tag{A–41}$$

$$a + c \;\; + x \;\; = 3s \tag{A–42}$$

$$b + d \;\; + y \;\; = 3t \tag{A–43}$$

$$a^2 + c^2 + x^2 = 3u \tag{A–44}$$

$$b^2 + d^2 + y^2 = 3v. \tag{A–45}$$

Our goal is to determine initial points (i.e., a, b, c, d, x and y) such that the stationary condition $\overline{p}^k_i = p^k_i$ holds for $k = 3$ and also $r^3_i = r^*_i$, $i = \pm 1$ — which then will automatically imply that $p^k_i = p^*_i$. It is easy to see from (A–28) that, if $r^3_i = r^*_i$ holds, then the condition $\overline{p}^3_i = p^3_i$ becomes equivalent to imposing directly that $\overline{p}^3_i = p^*_i$, where p^*_i is given by (A–36). That is, we want the initial points a, b, c, d, x and y to satisfy

$$\frac{w - st}{u - s^2} = r^*_{-1} \tag{A–46}$$

$$\frac{w - st}{v - t^2} = r^*_1 \tag{A–47}$$

$$s = p^*_{-1} \tag{A–48}$$

$$t = p^*_1, \tag{A–49}$$

where w, s, t, u and v are functions of a, b, c, d, x and y (cf. (A–41)-(A–45)). Thus, the above system has four equations and six unknowns, so there are two degrees of freedom. By fixing $y = 0$, we can obtain a, b, c, d as a function of x as follows:

$$\begin{aligned}
b &= \frac{3p^*_1}{2} \pm (x - p^*_{-1} + r^*_1 p^*_1)\sqrt{\frac{3r^*_{-1}}{4r^*_1(1 - r^*_{-1}r^*_1)}} \\
a &= -\frac{1}{2}\left[3\, r^*_{-1}\, x^2 + 3\, r^*_{-1}\, (p^*_{-1})^2 - 6\, r^*_{-1}\, p^*_{-1}\, x + 3\, r^*_1\, (p^*_1)^2\right. \\
&\quad + 6\, r^*_{-1}\, r^*_1\, p^*_1\, p^*_{-1} - 6\, p^*_{-1}\, b\, r^*_{-1}\, r^*_1 - 2\, x\, b + 6\, p^*_{-1}\, b + 2\, x\, b\, r^*_{-1}\, r^*_1 \\
&\quad \left. + 6\, x\, p^*_1 - 12\, p^*_{-1}\, p^*_1\right] / \left[(-1 + r^*_{-1}\, r^*_1)\,(2\, b - 3\, p^*_1)\right] \\
c &= -\frac{1}{2}\left[6\, p^*_{-1}\, b - 6\, p^*_{-1}\, p^*_1 - 6\, p^*_{-1}\, b\, r^*_{-1}\, r^*_1 + 12\, r^*_{-1}\, r^*_1\, p^*_1\, p^*_{-1}\right. \\
&\quad - 2\, x\, b + 2\, x\, b\, r^*_{-1}\, r^*_1 - 6\, r^*_{-1}\, r^*_1\, p^*_1\, x - 3\, r^*_{-1}\, x^2 - 3\, r^*_{-1}\, (p^*_{-1})^2 \\
&\quad \left. + 6\, r^*_{-1}\, p^*_{-1}\, x - 3\, r^*_1\, (p^*_1)^2\right] / \left[(-1 + r^*_{-1}\, r^*_1)\,(2\, b - 3\, p^*_1)\right] \\
d &= 3\, p^*_1 - b.
\end{aligned}$$

It remains to show that it is possible to choose x such that (i) the denominators of the expressions for a and c above do not vanish, and (ii) the denominators of (A–46) and (A–47) do not vanish. To show (i), notice that b is a linear function of x; hence, there is only one value of x for which $2b - 3p^*_1 = 0$. Since $r^*_{-1}r^*_1 < 1$, those denominators are non-zero for all but one value of x. To show (ii), note that, as observed earlier, the denominator of (A–46) vanishes if and only if $p^0_{-1} = p^1_{-1} = p^2_{-1}$ (and similarly for (A–47)). It follows from the above expressions for a and c that

$$c \;=\; a \quad \Leftrightarrow \quad \text{either } x \;=\; p^*_{-1} - p^*_1 r^*_1 \;\text{ or }\; x \;=\; p^*_{-1} - p^*_1/r^*_{-1}$$

so we can always choose x such that $c \neq a$, which guarantees that $u - s^2 > 0$. Finally, by choosing x such that $3p^*_1 - b \neq 0$, we have that $d \neq 0 = y$, which guarantees that $v - t^2 > 0$.

Next, consider the case $r^*_1 = r^*_{-1} = 0$. It is easy to see that the system (A–46)-(A–49) has three equations and six unknowns (provided that the denominators in (A–46) and (A–47) are nonzero), so there are three degrees of freedom. By fixing two of the variables to be zero (say, x and c), we can find values for the remaining variables that ensure that $u - s^2 \neq 0$ and $v - t^2 \neq 0$. One such set of values is

$$\begin{aligned} a &= 3p^*_{-1} \\ b &= p^*_1 \\ d &= 0 \\ y &= 2p^*_1. \end{aligned}$$

Finally, we consider the case $r^*_1 r^*_{-1} = 1$. We will use only two initial data points p^0, p^1, so $w,\ s,\ t,\ u$ and v are re-defined accordingly with 3 replaced by 2. Relations (A–41)-(A–45) become

$$ab + cd = 2w \tag{A–50}$$

$$a + c = 2s \tag{A–51}$$

$$b + d = 2t \tag{A–52}$$

$$a^2 + c^2 = 2u \tag{A–53}$$

$$b^2 + d^2 = 2v. \tag{A–54}$$

As before, we want to impose the conditions $r^2_i = r^*_i$ and $\overline{p}^2_i = p^*_i$, that is, (A–46)-(A–49). However, in this case the system is much simpler, since it follows from (A–50)–(A–54) that

$$r^*_{-1} = \frac{d-b}{c-a}, \qquad r^*_1 = \frac{c-a}{d-b}.$$

The second equation is redundant, since by assumption $r^*_{-1} r^*_1 = 1$. Together with (A–48) and (A–49), we then have three equations and four unknowns. By fixing $b = 0$ it follows that

$$\begin{aligned} a &= p^*_{-1} - r^*_1 p^*_1 \\ c &= p^*_{-1} + r^*_1 p^*_1 \\ d &= 2p^*_1. \end{aligned}$$

We conclude by noticing that condition (A–40) ensures that $\widehat{\alpha}^3_i < 0$, cf. (A–26). Therefore, the same values are repeated all $k = 4, 5, \ldots$, so the system becomes stationary. □

Lemma A–6. *The process is stationary at period k_0 if and only if either (A–37) holds for both $i = \pm 1$ or (A–38) holds for both $i = \pm 1$. Moreover, if the process is stationary with $r^{k_0}_{-1} r^{k_0}_1 < 1$, then (A–37) must necessarily hold for both $i = \pm 1$.*

Proof. Fix $i = \pm 1$. By writing r_i^{k+1} as in (A–25) and using recursive expressions for the averages, we obtain the following equivalences:

$$
\begin{aligned}
& r_i^k \quad = \quad \frac{\overline{pp}^{k+1} - \overline{p}_{-i}^{k+1}\overline{p}_i^{k+1}}{\overline{p^2}_i^{k+1} - \left(\overline{p}_i^{k+1}\right)^2} \quad \left(= \; r_i^{k+1} \right) \\
\Leftrightarrow \quad & \overline{pp}^{k+1} - \overline{p}_{-i}^{k+1}\overline{p}_i^{k+1} \quad = \quad r_i^k \left[\overline{p^2}_i^{k+1} - \left(\overline{p}_i^{k+1}\right)^2 \right] \\
\Leftrightarrow \quad & \frac{k}{k+1}\overline{pp}^k + \frac{1}{k+1}p_{-i}^k p_i^k - \left(\frac{k}{k+1}\overline{p}_{-i}^k + \frac{1}{k+1}p_{-i}^k \right)\left(\frac{k}{k+1}\overline{p}_i^k + \frac{1}{k+1}p_i^k \right) \\
& = \quad r_i^k \left[\frac{k}{k+1}\overline{p^2}_i^k + \frac{1}{k+1}\left(p_i^k\right)^2 - \left(\frac{k}{k+1}\overline{p}_i^k + \frac{1}{k+1}p_i^k \right)^2 \right] \\
\Leftrightarrow \quad & \frac{k}{k+1}\left(\overline{pp}^k - \overline{p}_{-i}^k\overline{p}_i^k\right) + \frac{k}{k+1}\overline{p}_{-i}^k\overline{p}_i^k - \left(\frac{k}{k+1}\right)^2 \overline{p}_{-i}^k\overline{p}_i^k + \frac{1}{k+1}p_{-i}^k p_i^k - \left(\frac{1}{k+1}\right)^2 p_{-i}^k p_i^k \\
& \quad - \frac{k}{(k+1)^2}\overline{p}_{-i}^k p_i^k - \frac{k}{(k+1)^2}\overline{p}_i^k p_{-i}^k \\
& = \quad r_i^k \left[\frac{k}{k+1}\left(\overline{p^2}_i^k - \left(\overline{p}_i^k\right)^2 \right) + \frac{k}{k+1}\left(\overline{p}_i^k\right)^2 - \left(\frac{k}{k+1}\right)^2 \left(\overline{p}_i^k\right)^2 + \frac{1}{k+1}\left(p_i^k\right)^2 - \left(\frac{1}{k+1}\right)^2\left(p_i^k\right)^2 \right. \\
& \quad \left. - \frac{2k}{(k+1)^2}\overline{p}_i^k p_i^k \right] \\
\Leftrightarrow \quad & \frac{k}{(k+1)^2}\overline{p}_{-i}^k\overline{p}_i^k + \frac{k}{(k+1)^2}p_{-i}^k p_i^k - \frac{k}{(k+1)^2}\left(\overline{p}_{-i}^k p_i^k + \overline{p}_i^k p_{-i}^k\right) + \frac{k}{k+1}r_i^k\left(\overline{p^2}_i^k - \left(\overline{p}_i^k\right)^2\right) \\
& = \quad r_i^k \left[\frac{k}{(k+1)^2}\left(\overline{p}_i^k\right)^2 + \frac{k}{(k+1)^2}\left(p_i^k\right)^2 - \frac{2k}{(k+1)^2}\overline{p}_i^k p_i^k \right] + \frac{k}{k+1}r_i^k\left(\overline{p^2}_i^k - \left(\overline{p}_i^k\right)^2\right) \\
\Leftrightarrow \quad & \left(\overline{p}_{-i}^k - p_{-i}^k\right)\left(\overline{p}_i^k - p_i^k\right) \quad = \quad r_i^k\left(\overline{p}_i^k - p_i^k\right)^2 .
\end{aligned}
$$

It follows that the ratio r_i^k defined in (43) satisfies $r_i^{k_0+1} = r_i^{k_0}$ if and only if either (A–37) or (A–38) holds. In that case, it follows from (A–26) that $\widehat{\alpha}_i^{k_0+1} = \widehat{\alpha}_i^{k_0}$.

Next we verify that $\widehat{\alpha}_{i,0}^{k_0+1} = \widehat{\alpha}_{i,0}^{k_0}$ if (A–37) holds for both $i = \pm 1$ or if (A–38) holds. In either case, $r_i^{k_0+1} = r_i^{k_0}$ by the above argument, and hence from (A–27) we have the following equivalences:

$$
\begin{aligned}
& \widehat{\alpha}_{i,0}^{k_0+1} \quad = \quad \widehat{\alpha}_{i,0}^{k_0} \\
\Leftrightarrow \quad & \overline{p}_{-i}^{k_0+1} - \overline{p}_i^{k_0+1}r_i^{k_0+1} \quad = \quad \overline{p}_{-i}^{k_0} - \overline{p}_i^{k_0}r_i^{k_0} \\
\Leftrightarrow \quad & \left(\frac{k_0}{k_0+1}\overline{p}_{-i}^{k_0} + \frac{1}{k_0+1}p_{-i}^{k_0}\right) - \left(\frac{k_0}{k_0+1}\overline{p}_i^{k_0} + \frac{1}{k_0+1}p_i^{k_0}\right)r_i^{k_0} \quad = \quad \overline{p}_{-i}^{k_0} - \overline{p}_i^{k_0}r_i^{k_0} \\
\Leftrightarrow \quad & \frac{1}{k_0+1}\left(\overline{p}_{-i}^{k_0} - p_{-i}^{k_0}\right) \quad = \quad \frac{1}{k_0+1}\left(\overline{p}_i^{k_0} - p_i^{k_0}\right)r_i^{k_0} \\
\Leftrightarrow \quad & \overline{p}_{-i}^{k_0} - p_{-i}^{k_0} \quad = \quad r_i^{k_0}\left(\overline{p}_i^{k_0} - p_i^{k_0}\right) . \qquad \text{(A–55)}
\end{aligned}
$$

The statement (A–55) is true if (A–37) holds for both $i = \pm 1$ or if (A–38) holds.

It follows from the above developments that if (A–37) holds for both $i = \pm 1$ or (A–38) holds for both $i = \pm 1$, then $r_i^{k_0+1} = r_i^{k_0}$, $\widehat{\alpha}_i^{k_0+1} = \widehat{\alpha}_i^{k_0}$, and $\widehat{\alpha}_{i,0}^{k_0+1} = \widehat{\alpha}_{i,0}^{k_0}$ for both $i = \pm 1$. Consequently, $p_i^{k_0+1} = p_i^{k_0}$ for both $i = \pm 1$ by (A–28). It follows that

$$\overline{p}_i^{k_0+1} - p_i^{k_0+1} \quad = \quad \frac{k_0}{k_0+1}\overline{p}_i^{k_0} + \frac{1}{k_0+1}p_i^{k_0} - p_i^{k_0} \quad = \quad \frac{k_0}{k_0+1}\left(\overline{p}_i^{k_0} - p_i^{k_0}\right) \qquad \text{for } i = \pm 1,$$

and thus (A–37) holds for both $i = \pm 1$ or (A–38) holds for both $i = \pm 1$ each with $k_0 + 1$ in place of k_0. This implies $r_i^{k_0+2} = r_i^{k_0+1}$ and $p_i^{k_0+2} = p_i^{k_0+1}$ for $i = \pm 1$. Repeating the argument, it follows that $r_i^{k+1} = r_i^k$ and $p_i^{k+1} = p_i^k$ for $i = \pm 1$ for all $k \geq k_0$ and therefore $r_i^k = r_i^{k_0}$ and $p_i^k = p_i^{k_0}$ for $i = \pm 1$ for all $k \geq k_0$; that is, the process is stationary at period k_0.

Conversely, suppose the process is stationary at period k_0. Then $r_i^{k_0+1} = r_i^{k_0}$ for $i = \pm 1$ and, as seen above, it must hold that $\widehat{\alpha}_i^{k_0+1} = \widehat{\alpha}_i^{k_0}$ for $i = \pm 1$. Together with the fact that $p_i^{k_0+1} = p_i^{k_0}$ for $i = \pm 1$, this implies by (A–28) that $\widehat{\alpha}_i^{k_0+1} = \widehat{\alpha}_i^{k_0}$ for $i = \pm 1$, and hence it follows from (A–55) that $\overline{p}_{-i}^{k_0} - p_{-i}^{k_0} = r_i^{k_0}\left(\overline{p}_i^{k_0} - p_i^{k_0}\right)$ for $i = \pm 1$. This implies that either (A–37) holds for both $i = \pm 1$ or else (A–38) holds for both $i = \pm 1$.

To show the second assertion of the lemma, consider the linear system

$$\begin{aligned} \overline{p}_{-1}^{k_0} - p_{-1}^{k_0} &= r_1^{k_0}\left(\overline{p}_1^{k_0} - p_1^{k_0}\right) \\ \overline{p}_1^{k_0} - p_1^{k_0} &= r_{-1}^{k_0}\left(\overline{p}_{-1}^{k_0} - p_{-1}^{k_0}\right). \end{aligned}$$

It is easy to see that the determinant of this system is $1 - r_1^{k_0} r_{-1}^{k_0}$. It follows that if $r_1^{k_0} r_{-1}^{k_0} < 1$, then the system has a unique solution, given by $\overline{p}_i^{k_0} = p_i^{k_0}$, $i = \pm 1$. □

Lemma A–7. *In the context of the discussion in Section 4.3, necessary conditions for the models of both sellers* 1 *and* −1 *to capture competition are that*

$$r_i^\infty \quad = \quad -\frac{\beta_{-i,i}}{2\beta_{-i,-i}} \qquad \textit{and} \qquad p_i^\infty \quad = \quad \frac{\beta_{-i,0}\beta_{i,-i} - 2\beta_{i,0}\beta_{-i,-i}}{4\beta_{-i,-i}\beta_{i,i} - \beta_{-i,i}\beta_{i,-i}},$$

for $i = \pm 1$.

Proof. Expression (44) for the reaction function implies that the demand of seller i is

$$d_i(p_i, p_{-i}(p_i)) \quad = \quad \beta_{i,0} + \beta_{i,i}p_i + \beta_{i,-i}p_{-i}(p_i) \quad = \quad \left(\beta_{i,0} - \frac{\beta_{-i,0}\beta_{i,-i}}{2\beta_{-i,-i}}\right) + \left(\beta_{i,i} - \frac{\beta_{-i,i}\beta_{i,-i}}{2\beta_{-i,-i}}\right)p_i$$

The monopoly demand model of seller i captures competition for the reaction function (44) if and only if

$$\widehat{\alpha}_{i,0}^\infty + \widehat{\alpha}_i^\infty p_i \quad = \quad \left(\beta_{i,0} - \frac{\beta_{-i,0}\beta_{i,-i}}{2\beta_{-i,-i}}\right) + \left(\beta_{i,i} - \frac{\beta_{-i,i}\beta_{i,-i}}{2\beta_{-i,-i}}\right)p_i \quad \text{for all } p_i,$$

that is, if and only if

$$\widehat{\alpha}_{i,0}^{\infty} = \beta_{i,0} - \frac{\beta_{-i,0}\beta_{i,-i}}{2\beta_{-i,-i}} \tag{A–56}$$

$$\widehat{\alpha}_{i}^{\infty} = \beta_{i,i} - \frac{\beta_{-i,i}\beta_{i,-i}}{2\beta_{-i,-i}}\,. \tag{A–57}$$

It follows from (A–26) and (A–57) that a necessary condition for the model of seller i to capture competition for the reaction function (44) is that

$$r_i^{\infty} = -\frac{\beta_{-i,i}}{2\beta_{-i,-i}}\,. \tag{A–58}$$

Substitution of expression (A–58) for r_i^{∞} into (A–35) gives

$$\beta_{i,-i}p_{-i}^{\infty} + \left(2\beta_{i,i} - \frac{\beta_{-i,i}\beta_{i,-i}}{2\beta_{-i,-i}}\right)p_i^{\infty} = -\beta_{i,0}\,. \tag{A–59}$$

Also, it follows from (A–27), (A–56), and (A–58) that another necessary condition is that

$$2\beta_{-i,-i}p_{-i}^{\infty} + \beta_{-i,i}p_i^{\infty} = -\beta_{-i,0}\,. \tag{A–60}$$

Condition (A–60) holds for $i = \pm 1$ if and only if

$$p_i^{\infty} = \frac{\beta_{-i,0}\beta_{i,-i} - 2\beta_{i,0}\beta_{-i,-i}}{4\beta_{-i,-i}\beta_{i,i} - \beta_{-i,i}\beta_{i,-i}}\,,$$

for $i = \pm 1$. □

References

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