ONLINE APPENDIX

Optimal Dynamic Pricing with Patient Customers

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A More on the Literature

Here we provide a table that summarizes the main assumptions and results in papers cited in Section 2 that establish optimality of cyclic pricing policies. The table is intended to provide a brief summary, and not to provide a complete explanation of all results in the cited papers or our paper. The demand in all the papers in the table is deterministic.

Table 4: Assumptions-Results Matrix.

	Customer behavior	Result
Our paper	Patient customers that buy as soon as price falls below valuation. Willing to wait up to k periods. Arbitrary valuation distribution.	Decreasing cyclic policy is optimal. Cycle length is at most $k + m - 1$ where m = cardinality of price set.
Ahn et al.	Patient customers that buy as soon as price falls below valuation. Willing to wait up to 1 period. Uniform valuation distribution.	Two-period decreasing cyclic policy is optimal.
Besbes and Lobel	Strategic customers that buy at lowest price within time window (patience level). Heterogeneous levels of patience. Arbitrary joint distribution over valuations and patience levels.	Cyclic policy is optimal. Cycle length is at most twice the greatest patience level. Cycles need not consist of decreasing prices.
Conlisk et al.	Strategic customers that time purchases to maximize own discounted surplus. Infinite patience levels (willing to wait indefinitely), two possible valuations.	Decreasing cyclic policy is optimal.

B Proof of Lemma 1 and Propositions 1 and 2

Lemma B.1 *For any policy $p \in \mathcal{P}^{\infty}$, the length of any component of p is at most κ; that is $r(i+1) - r(i) \leq \kappa$ for all $i = 0, 1, 2, \ldots$.*

Proof. The proof is essentially identical to that of Lemma 1 of Besbes and Lobel (2015). Consider a component $(p_{r(i)}, \ldots, p_{r(i+1)-1})$ of a policy p, and suppose for a contradiction that the length of the component is greater than κ; i.e, $r(i+1) - r(i) > \kappa$.

Let $t_0 = r(i)$. There exists $t_1 \in \{t_0+1, \ldots, t_0+k\}$ such that $p_{t_1} < p_{t_0}$. Otherwise $t_0+1 = r(i)+1$ would be an I-regeneration point of p in which case $r(i+1) = t_0+1 = r(i)+1$, which would contradict our supposition that $r(i+1) - r(i) > \kappa$. Note that $t_1 \leq t_0 + k$.

Likewise, there exists $t_2 \in \{t_1+1, \ldots, t_1+k\}$ such that $p_{t_2} < p_{t_1}$. Otherwise $t_1 + 1$ would be an I-regeneration point of p in which case $r(i+1) = t_1 + 1 \leq t_0 + k + 1 = r(i) + k + 1$, which would contradict our supposition that $r(i+1) - r(i) > \kappa$. So there must exist a t_2 as claimed. Note that $t_2 \leq t_1 + k \leq t_0 + 2k$.

Continuing in this fashion for $j = 1, \ldots, m$, we see that there exist times $\{t_j\}$ with $t_j \in \{t_{j-1}+1, \ldots, t_{j-1}+k\}$ such that $p_{t_j} < p_{t_{j-1}}$ and $t_j \leq t_{j-1} + k \leq t_0 + jk$.

We have now established that $p_{t_0} > p_{t_1} > \cdots > p_{t_m}$, which is not possible because the cardinality of the price set $\mathcal{P}$ is m. Hence, the supposition that $r(i+1) - r(i) > \kappa$ cannot hold, and thus $r(i+1) - r(i) \leq \kappa$. □

Lemma B.2 *Consider $p \in \mathcal{P}^{\infty}$ and suppose $C_i(p) = q \in \mathcal{P}^L$ for some $i \geq 0$. Then q has no F-regeneration points. That is, if $q \in \mathcal{P}^L$ is a component of policy $p \in \mathcal{P}^{\infty}$, then q has no F-regeneration points.*

Proof. Suppose $q = C_i(p) = (p_{r(i)}, \ldots, p_{r(i+1)-1})$ for some i so that $L(q) = r(i+1) - r(i)$ and $q_j = p_{r(i)+j-1}$ for $j = 1, \ldots, r(i+1) - r(i)$. If $L(q) = 1$ the lemma is true. So we need only consider the case with $L(q) \geq 2$. Suppose for a contradiction that q has an F-regeneration point at some time $t \in \{2, \ldots, r(i+1) - r(i)\}$. Then $q_{t-1} \leq \min\{q_t, \ldots, q_{\min\{t+k-1, r(i+1)-r(i)\}}\}$, and so

$$p_{r(i)+t-2} \leq \min\{p_{r(i)+t-1}, \ldots, p_{\min\{r(i)+t+k-2, r(i+1)-1\}}\}$$

If $r(i)+t+k-2 \leq r(i+1)-1$, then $p_{r(i)+t-2} \leq \min\{p_{r(i)+t-1}, \ldots, p_{r(i)+t+k-2}\}$, hence $r(i)+t-1 \in \{r(i)+1, \ldots, r(i+1)-1\}$ is an I-regeneration point of p, which contradicts the fact that q is a component of p. If $r(i)+t+k-2 > r(i+1)-1$, then $p_{r(i)+t-2} \leq \min\{p_{r(i)+t-1}, \ldots, p_{r(i+1)-1}\}$. Moreover, $p_{r(i+1)-1} \leq \min\{p_{r(i+1)}, \ldots, p_{r(i+1)+k-1}\}$ because $r(i+1)$ is an I-regeneration point of

p. The preceding two inequalities imply that $r(i)+t-1$ is an I-regeneration point of p, which is again a contradiction. $\square$

Proof of Lemma 1. Fix $L \in \mathbb{N}$, $q = (q_1, \dots, q_L) \in \mathcal{P}^L$, and $p = (q, q, q, \dots) \in \mathcal{P}^\infty$.

Part 1a: We will show if q has no F-regeneration points, then $C_i(p) = q$ for $i = 0, 1, 2, \dots$.

Suppose q has no F-regeneration points. Then for each $t = 2, 3, \dots, L$, there exists some $s \in \{t, \dots, \min\{t+k-1, L\}\}$ with $q_{t-1} > q_s$. Now consider $p = (q, q, q, \dots)$. For each $t = 2, \dots, L$ and $i = 0, 1, 2, \dots$, the time $u = t + iL$ is such that $p_{u-1} = p_{t+iL-1} = q_{t-1} > q_s = p_{s+iL}$ and

$$s + iL \in \{t+iL, \dots, \min\{L, t+k-1\} + iL\} \subset \{t+iL, \dots, t+iL+k-1\} = \{u, \dots, u+k-1\}$$

So $p_{u-1} > p_{s+iL}$ and $s + iL \in \{u, \dots, u+k-1\}$. Hence, u is not an I-regeneration point of p, and consequently the only possible I-regeneration points of p are $\{1+L, 1+2L, 1+3L, \dots\}$.

Next observe that $q_L = \min\{q_1, \dots, q_L\}$. (Otherwise, there would exist $n \in \{1, 2, \dots, L-1\}$ such that $q_n = \min\{q_1, \dots, q_L\}$ in which case time $n+1$ would be an F-regeneration point of q, contradicting the original assumption that q has no F-regeneration points.) Therefore, for each $i = 0, 1, 2, \dots$, we have $p_{L+iL} = q_L \le \min\{p_{1+L+iL}, \dots, p_{k+L+iL}\}$. Consequently $1 + L + iL$ for $i = 0, 1, 2, \dots$ are I-regeneration points of p. This establishes that $C_i(p) = q$ for $i = 0, 1, 2, \dots$.

Part 1b: If $C_i(p) = q$ for $i = 0, 1, 2, \dots$, then q has no F-regeneration points by Lemma B.2. This completes the proof of part 1 of the lemma.

Part 2: Suppose q has no F-regeneration points. By part 1, we have $C_i(p) = q$. Let $n_T = \lfloor T/L \rfloor$. Thus

$$H(p) = \liminf_{T\to\infty} \frac{1}{T}\Big\{ \sum_{j=1}^{n_T} \sum_{t=(j-1)L+1}^{jL} \rho_t(p) + \sum_{t=n_T L+1}^{T} \rho_t(p) \Big\} = \liminf_{T\to\infty} \frac{1}{T}\Big\{ n_T L v(q) + \sum_{t=n_T L+1}^{T} \rho_t(p) \Big\} = v(q)$$

This completes the proof of part 2 of the lemma.

Part 3: Suppose q has no F-regeneration points. It is clear that $p = (q, q, q, \dots)$ is cyclic. To show the cycle length is $L = L(q)$, it suffices to show there does not exist a sequence $\widehat{q} = (\widehat{q}_1, \dots, \widehat{q}_l) \in \mathcal{P}^l$ and finite integer $n \ge 2$ such that $q = (\widehat{q}, \widehat{q}, \dots, \widehat{q})$ and $L = nl$.

Suppose for a contradiction that there exists such a sequence $\widehat{q}$. Suppose $\widehat{q}_i = \min\{\widehat{q}_1, \dots, \widehat{q}_l\}$. Thus $q_i = \widehat{q}_i = \min\{q_1, \dots, q_L\}$. Therefore, we have $q_i \le \min\{q_{i+1}, q_{i+2}, \dots, q_L\}$, which implies that period $i+1$ is an F-regeneration point, contradicting the fact that q has no F-regeneration points. This completes the proof of part 3 of the lemma. $\square$

Proof of Proposition 1. For policy $p \in \mathcal{P}^\infty$, define $N_T = \max\{i : r(i) \leq T\}$. Then

$$\begin{aligned}
H(p) &= \liminf_{T\to\infty} \frac{1}{T}\Big\{\sum_{j=1}^{N_T}\sum_{t=r(j-1)}^{r(j)-1}\rho_t(p) + \sum_{t=r(N_T)}^{T}\rho_t(p)\Big\} \\
&= \liminf_{T\to\infty}\Big\{\sum_{j=1}^{N_T}\frac{r(j)-r(j-1)}{T}v(C_{j-1}(p)) + \frac{\sum_{t=r(N_T)}^{T}\rho_t(p)}{T}\Big\} \\
&\leq \limsup_{T\to\infty}\max\{v(C_0(p)),\dots,v(C_{N_T-1}(p))\} + \limsup_{T\to\infty}\frac{\sum_{t=r(N_T)}^{T}\rho_t(p)}{T} \\
&= \limsup_{T\to\infty}\max\{v(C_0(p)),\dots,v(C_{N_T-1}(p))\} \qquad (10) \\
&= \sup_i v(C_i(p)).
\end{aligned}$$

The equality (10) is justified as follows. By Lemma B.1, we know the length of each component of p is at most κ, thus $\sum_{t=r(N_T)}^{T}\rho_t(p)$ is the sum of at most κ terms, each of which is bounded. Consequently, $\limsup_{T\to\infty}\frac{1}{T}\sum_{t=r(N_T)}^{T}\rho_t(p) = 0$ and hence the equality holds.

Since the length of each component of p is at most κ and $\mathcal{P}$ is finite, there are finitely many distinct components. Consider $q = C_{i^*}(p)$ where $i^* \in \arg\max_i v(C_i(p))$. By Lemma B.2, q has no F-regeneration points. By Lemma 1, the policy $p' = (q, q, q, \dots)$ built by repeating the component q in p with the largest average revenue is such that

$$H(p') = v(q) = \sup_i v(C_i(p)) \qquad (11)$$

and $C_j(p') = q$ for $j = 0, 1, 2, \dots$.

Combining (10) and (11), we have that for any $p \in \mathcal{P}^\infty$, there exists p' for which $H(p) \leq H(p')$, where p' is a cyclic policy with cycle length at most κ, and $C_j(p') = q$ for $j = 0, 1, 2, \dots$. Therefore, we can rewrite the problem (2) as

$$\begin{aligned}
&\sup_p H(p) \qquad (12)\\
s.t.\quad & p = (q, q, q, \dots) \text{ and } C_i(p) = q, \text{ for } i = 0, 1, 2, \dots, \\
& \text{where } q = (q_1, q_2, \dots, q_L) \text{ and } L \leq \kappa
\end{aligned}$$

The length of q is at most κ, so it follows that the set of q above is finite. Therefore, there exists a cyclic policy $p^* = (q^*, q^*, q^*, \dots)$ with $L(q^*) \leq \kappa$ that achieves the supremum in (2). Moreover, the cycle length of p^* is $L(q^*)$ by part 3 of Lemma 1. We have also established that $C_i(p^*) = q^*$ for $i = 0, 1, 2, \dots$. □

Proof of Proposition 2. Let $Z = \max_p H(p)$ in (4) and $Z' = \max_q v(q)$ in (5).

Consider $q = (q_1, \ldots, q_L)$ with $L \leq \kappa$ and $p = (q, q, q, \ldots)$ with $C_i(p) = q$ for every i. Since $C_i(p) = q$, by Lemma 1, q has no F-regeneration points and $H(p) = v(q)$. Thus, $Z \leq Z'$. Next consider $q = (q_1, \ldots, q_L)$ with $L \leq \kappa$ such that q has no F-regeneration points. Then for $p = (q, q, q, \ldots)$, by Lemma 1, we have $C_i(p) = q$, and $H(p) = v(q)$. Thus, $Z' \leq Z$. Therefore, $Z = Z'$, which establishes the equivalence of problems (4) and (5). $\square$

C Proof of Proposition 3

We begin by introducing a new definition. With the aid of Lemma C.1 below, the definition allows us to understand the structure of policies in the set $B = \cup_{L=1}^{\kappa} \cup_{n=1}^{L-1} B^n(L)$, i.e., the structure of those policies in the feasible set of (5) that are not decreasing. As noted at the start of Section 5.2, a key step in our argument is showing that any policy in B can be improved by re-arranging prices between each pair of successive strong markups into a decreasing sequence. After such a re-arrangement, we arrive at a policy in the set E, which is made up of the sets $\{E^n(L)\}$.

For $n \geq 1$, define $C^n(L)$ as the set of sequences $q \in \mathcal{P}^L$ with the property that $q = (s_0, s_1, \ldots, s_n)$ where $s_i = (q_{\tau(i)}, \ldots, q_{\tau(i+1)-1})$ for some times $\{\tau(i) : i = 0, \ldots, n+1\}$ such that $1 = \tau(0) < \tau(1) < \cdots < \tau(n+1) = L + 1$ and

1. s_0 is decreasing ($q_1 \geq q_2 \geq \cdots \geq q_{\tau(1)-1}$)
2. $q_{\tau(i)-1} < q_{\tau(i)}$ for $i = 1, \ldots, n$
3. $\{t \in \{\tau(i) + 1, \ldots, \tau(i+1) - 1\} : q_t < q_{\tau(i)-1}\} \neq \emptyset$ for $i = 1, \ldots, n$
4. $\min\{t \in \{\tau(i)+1, \ldots, \tau(i+1)-1\} : q_t < q_{\tau(i)-1}\} \in \{\tau(i)+1, \ldots, \tau(i)+k-1\}$ for $i = 1, \ldots, n$.
5. $q_{\tau(i)+m(i)+1} \geq q_{\tau(i)+m(i)+2} \geq \cdots \geq q_{\tau(i+1)-1}$ where

$$m(i) = \min\left\{t \in \{\tau(i) + 1, \ldots, \tau(i+1) - 1\} : q_t < q_{\tau(i)-1}\right\} - \tau(i) - 1 \text{ for } i = 1, \ldots, n.$$

It is apparent that if $q \in E^n(L)$, then $q \in C^n(L)$ with $\{\tau(i)\} = \{t(i)\}$. However, the converse is not true. The strings $s_1, \ldots, s_n$ must each be decreasing for a policy in $E^n(L)$, but they need not be decreasing for a policy in $C^n(L)$. Policies that are in $C^n(L)$ may have markups that are not strong markups. This is not the case for policies in $E^n(L)$. The purpose of introducing the sets $\{C^n(L)\}$ is to make the connection between B and E.

For $q \in C^n(L)$, time $\tau(i) + m(i) + 1$ is the first time in $\{\tau(i)+1, \ldots, \tau(i+1)-1\}$ that the price goes below $q_{\tau(i)-1}$. Immediately after time $\tau(i) - 1$, there are $m(i) + 1$ consecutive prices greater than or equal to $q_{\tau(i)-1}$. The length of sequence s_i is $\tau(i+1)-\tau(i)$. There is at least one price lower than $q_{\tau(i)-1}$ in s_i by condition 3; therefore, we have $\tau(i+1)-\tau(i)-[m(i)+1] \geq 1$, thus $m(i) \leq \tau(i+1)-\tau(i)-2$. By condition 4, $\tau(i)+m(i)+1 \leq \tau(i)+k-1$, thus $m(i) \leq k-2$. Hence we have $m(i) \in \{0, \ldots, \min\{k-2, \tau(i+1)-\tau(i)-2\}\}$.

Observe also that for $q \in C^n(L)$, we have

$$\min\{q_{\tau(i)}, q_{\tau(i)+1}, \ldots, q_{\tau(i)+m(i)}\} \geq q_{\tau(i)-1} > q_{\tau(i)+m(i)+1} \geq q_{\tau(i)+m(i)+2} \geq \cdots \geq q_{\tau(i+1)-1}. \quad (13)$$

Lemma C.1 *Consider $q = (q_1, q_2, \ldots, q_L)$. If $q \in B^n(L)$, then $q \in C^n(L)$.*

Proof. Consider $q \in B^n(L)$. By the definition of $B^n(L)$, we know that q has no F-regeneration points and that q has exactly n strong markups: $2 \leq e(1) < e(2) < \cdots < e(n) \leq L$ such that $q_{e(i)} > q_{e(i)-1} = \min\{q_1, \ldots, q_{e(i)-1}\} < q_{e(i-1)-1}$ for $i = 2, \ldots, n$. Let $e(0) = 1$, $e(n+1) = L+1$, then $1 = e(0) < e(1) < \cdots < e(n+1) = L+1$. Thus sequence q can be written in the form of $q = (s_0, s_1, \ldots, s_n)$ where $s_i = (q_{e(i)}, \ldots, q_{e(i+1)-1})$ for $i = 0, \ldots, n$.

Next we will show that conditions 1–5 in the definition of $C^n(L)$ are satisfied by taking $\tau(i) = e(i)$ for $i = 0, \ldots, n+1$.

Consider sequence s_0. We have $e(1) \geq 2$. Also we have $q_1 \geq q_2 \geq \cdots \geq q_{e(1)-1}$, otherwise there would be a strong markup at some time $t < e(1)$, which would contradict the definition of $e(1)$ as the time of the first strong markup of q. Therefore, condition 1 holds.

Next consider sequence $s_i = (q_{e(i)}, q_{e(i)+1}, \ldots, q_{e(i+1)-1})$ for $i = 1, \ldots, n$. Observe that $q_{e(i)-1} < q_{e(i)}$, so condition 2 holds. In sequence s_i, there must exist at least one price strictly lower than $q_{e(i)-1}$. (For $i = 1, \ldots, n-1$, otherwise $e(i+1)$ would not be a strong markup of q. For $i = n$, otherwise $e(i)$ would be an F-regeneration point.) This establishes condition 3. Suppose $e(i) + w(i) + 1$ is the first time that the price is strictly lower than $q_{e(i)-1}$, then we have

$$\min\{q_{e(i)}, \ldots, q_{e(i)+w(i)}\} \geq q_{e(i)-1},$$

which means that beginning at time $e(i)$, there are $w(i)+1$ consecutive prices greater than or equal to $q_{e(i)-1}$. We must have that $w(i)+1 \leq k-1$, because otherwise $e(i)$ would be an F-regeneration point (and we know q has no F-regeneration points). Hence, condition 4 holds.

Finally, if there are multiple prices strictly lower than $q_{e(i)-1}$ in s_i, we must have $q_{e(i)+w(i)+1} \geq q_{e(i)+w(i)+2} \geq \cdots \geq q_{e(i+1)-1}$, otherwise there would be a strong markup at some time $t \in \{e(i) + w(i)+2, \ldots, e(i+1)-1\}$, which would contradict the definition of $e(i+1)$ as the time of the $(i+1)$th strong markup of q. Thus condition 5 holds.

By taking $\tau(i) = e(i)$ and $m(i) = w(i)$, we see that $q \in C^n(L)$. □

Lemma C.2 *For any policy $y \in \mathcal{P}^L$ with $L \leq k+1$, consider the policy $\widetilde{y}$ constructed by rearranging the prices in y from largest to smallest. Then $V(y) \leq V(\widetilde{y})$ and $v(y) \leq v(\widetilde{y})$. Therefore, for an L-period problem with $L \leq k+1$, there exists an optimal policy that is decreasing.*

Proof. Consider $\widetilde{y} = (\widetilde{y}_1, \ldots, \widetilde{y}_L) = (y_{i(1)}, \ldots, y_{i(L)})$ obtained by re-arranging the prices in $y = (y_1, \ldots, y_L)$ from the largest to smallest. Hence, $i(t)$ denotes the time at which price $\widetilde{y}_t$ (the tth largest price in y) appears in y. If a price (say p) appears more than once in y (say at times $i_1 < i_2 < \cdots < i_n$) then for some t we have $\widetilde{y}_t = \widetilde{y}_{t+1} = \cdots = \widetilde{y}_{t+n-1} = p$ and we take $i(t) = i_1, i(t+1) = i_2, \ldots, i(t+n-1) = i_n$.

For each $i = 1, \ldots, L$, define

$$\varphi_i^L(y) = y_i[1 - G(y_i)] + \alpha \sum_{j=i+1}^{\min\{L,i+k\}} y_j[F(\min\{y_i, \ldots, y_{j-1}\}) - F(y_j)]^+ . \tag{14}$$

The quantity $\varphi_i^L(y)$ is the revenue obtained from customers who initially arrive in period i under policy y. Hereafter, we drop the superscript. When $L \leq k+1$, the formula (14) reduces to

$$\varphi_i(y) = y_i[1 - G(y_i)] + \alpha \sum_{j=i+1}^{L} y_j[F(\min\{y_i, \ldots, y_{j-1}\}) - F(y_j)]^+ . \tag{15}$$

Therefore,

$$\begin{aligned}
\sum_{i=1}^{L} \varphi_i(y) &= \sum_{i=1}^{L} \left\{ y_i[1 - G(y_i)] + \alpha \sum_{j=i+1}^{L} y_j[F(\min\{y_i, \ldots, y_{j-1}\}) - F(y_j)]^+ \right\} \\
&= \sum_{j=1}^{L} y_j[1 - G(y_j)] + \alpha \sum_{j=1}^{L} \sum_{i=1}^{j-1} y_j[F(\min\{y_i, \ldots, y_{j-1}\}) - F(y_j)]^+ \\
&= \sum_{j=1}^{L} \rho_j(y) = V(y) .
\end{aligned}$$

Consequently, $V(y) = \sum_{i=1}^{L} \varphi_i(y) = \sum_{t=1}^{L} \varphi_{i(t)}(y)$. We also have $V(\widetilde{y}) = \sum_{t=1}^{L} \varphi_t(\widetilde{y})$ where

$$\begin{aligned}
\varphi_t(\widetilde{y}) &= \widetilde{y}_t[1 - G(\widetilde{y}_t)] + \alpha \sum_{s=t+1}^{L} \widetilde{y}_s[F(\min\{\widetilde{y}_t, \ldots, \widetilde{y}_{s-1}\}) - F(\widetilde{y}_s)]^+ \\
&= \widetilde{y}_t[1 - G(\widetilde{y}_t)] + \alpha \sum_{s=t+1}^{L} \widetilde{y}_s[F(\widetilde{y}_{s-1}) - F(\widetilde{y}_s)] .
\end{aligned} \tag{16}$$

The final equality above holds because $\widetilde{y}$ is decreasing.

Hence, to complete the proof, it suffices to show that $\varphi_{i(t)}(y) \le \varphi_t(\widetilde{y})$ for $t = 1, \dots, L$. To this end, fix $t \in \{1, \dots, L\}$ and consider (15) with $i = i(t)$:

$$\varphi_{i(t)}(y) = y_{i(t)}[1 - G(y_{i(t)})] + \alpha \sum_{j=i(t)+1}^{L} y_j [F(\min\{y_{i(t)}, \dots, y_{j-1}\}) - F(y_j)]^+ . \tag{17}$$

Notice that the jth term in the summation in (17) is non-zero only if $y_j < \min\{y_{i(t)}, \dots, y_{j-1}\}$, that is only if y_j is the strict minimum of $\{y_{i(t)}, \dots, y_j\}$. (Figure 3 depicts an example with the following bookkeeping.) Let $j(1) < j(2) < \cdots < j(N)$ denote those times $j \in \{i(t) + 1, \dots, L\}$ at which $y_j < \min\{y_{i(t)}, \dots, y_{j-1}\}$. By (17) we have

$$\varphi_{i(t)}(y) = y_{i(t)}[1 - G(y_{i(t)})] + \alpha \sum_{\ell=1}^{N} y_{j(\ell)}[F(y_{j(\ell-1)}) - F(y_{j(\ell)})] \tag{18}$$

where we take $j(0) = i(t)$ so $y_{j(0)} = y_{i(t)} = \widetilde{y}_t$.

Recall that $\widetilde{y}$ contains the prices of y rearranged in decreasing order. Therefore, $\{y_{j(1)}, \dots, y_{j(N)}\} \subset \{\widetilde{y}_{t+1}, \dots, \widetilde{y}_L\}$. That is, $y_{j(\ell)} = \widetilde{y}_{z(\ell)}$ for some $\{z(\ell)\}$ with $t + 1 \le z(1) < z(2) < \cdots < z(N) \le L$. Let $z(0) = t$. Hence, for the ℓth term in the sum in (18), we have

$$\begin{aligned} y_{j(\ell)}[F(y_{j(\ell-1)}) - F(y_{j(\ell)})] = \widetilde{y}_{z(\ell)}[F(\widetilde{y}_{z(\ell-1)}) - F(\widetilde{y}_{z(\ell)})] &= \widetilde{y}_{z(\ell)} \sum_{s=z(\ell-1)+1}^{z(\ell)} [F(\widetilde{y}_{s-1}) - F(\widetilde{y}_s)] \\ &\le \sum_{s=z(\ell-1)+1}^{z(\ell)} \widetilde{y}_s [F(\widetilde{y}_{s-1}) - F(\widetilde{y}_s)] \end{aligned}$$

where the inequality holds because $\widetilde{y}$ is decreasing.

Plugging back into (18) and then using (16) yields

$$\begin{aligned} \varphi_{i(t)}(y) &\le y_{i(t)}[1 - G(y_{i(t)})] + \alpha \sum_{\ell=1}^{N} \sum_{s=z(\ell-1)+1}^{z(\ell)} \widetilde{y}_s [F(\widetilde{y}_{s-1}) - F(\widetilde{y}_s)] \\ &= \widetilde{y}_t [1 - G(\widetilde{y}_t)] + \alpha \sum_{s=t+1}^{z(N)} \widetilde{y}_s [F(\widetilde{y}_{s-1}) - F(\widetilde{y}_s)] \\ &\le \widetilde{y}_t [1 - G(\widetilde{y}_t)] + \alpha \sum_{s=t+1}^{L} \widetilde{y}_s [F(\widetilde{y}_{s-1}) - F(\widetilde{y}_s)] = \varphi_t(\widetilde{y}) . \end{aligned}$$

This completes the proof. □

Proof of Proposition 3. Consider a sequence $q \in B^n(L)$. By Lemma C.1, $q \in C^n(L)$. Then $q = (s_0, s_1, \dots, s_n)$ where $s_i = (q_{\tau(i)}, q_{\tau(i)+1}, \dots, q_{\tau(i+1)-1})$ for $i = 0, \dots, n$.

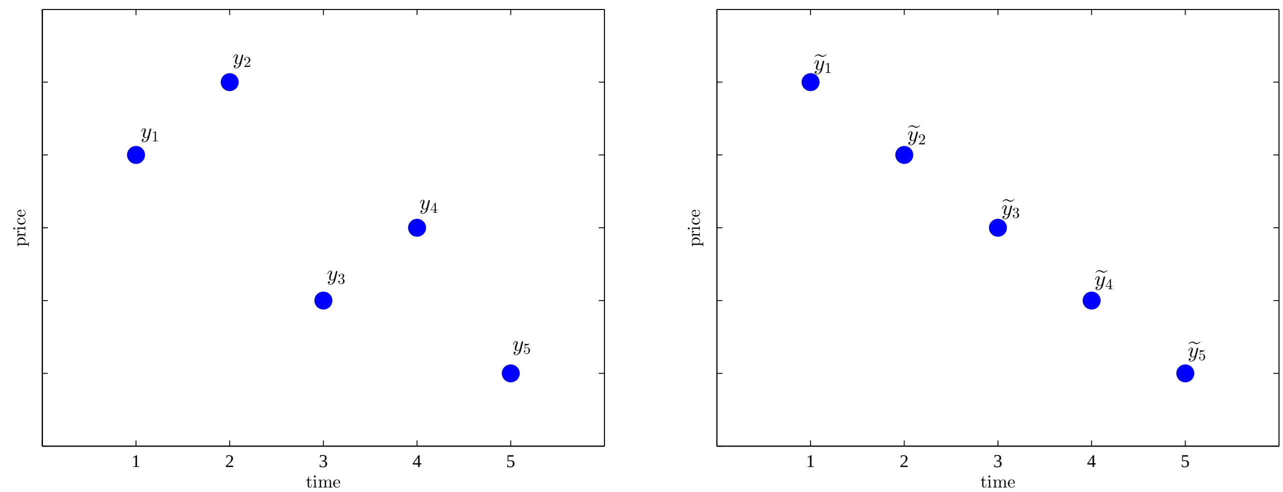

Figure 3: The left panel shows a policy y and the right panel shows policy $\widetilde{y}$ obtained by rearranging the elements of y from largest to smallest. In this example, for $t = 1$, we have $i(1) = 2$ and $N = 2$ with $j(1) = 3$, $j(2) = 5$, $z(1) = 4$, $z(2) = 5$. For $t = 2$, we have $i(2) = 1$ and $N = 2$ with $j(1) = 3$, $j(2) = 5$, $z(1) = 4$, $z(2) = 5$. For $t = 3$, we have $i(3) = 4$ and $N = 1$ with $j(1) = 5$ and $z(1) = 5$. For $t = 4$, we have $i(4) = 3$ and $N = 1$ with $j(1) = 5$ and $z(1) = 5$. For $t = 5$, we have $i(5) = 5$ and the summation in (17) is empty.

In subsequence s_1, we know by (13) that there exists $m(1) \in \{0, \ldots, \min\{k-2, \tau(2)-\tau(1)-2\}\}$ such that

$$\min\{q_{\tau(1)}, q_{\tau(1)+1}, \ldots, q_{\tau(1)+m(1)}\} \geq q_{\tau(1)-1} > q_{\tau(1)+m(1)+1} \geq \cdots \geq q_{\tau(2)-1}.$$

Let $(q_{i(1)}, q_{i(2)}, \ldots, q_{i(m(1)+1)})$ be the re-arrangement of $q_{\tau(1)}, \ldots, q_{\tau(1)+m(1)}$ from largest to smallest, and consider another sequence $\widetilde{q}$ such that

$$\widetilde{q}_t = \begin{cases} q_t & \text{if} \quad t \leq \tau(1) - 1 \\ q_{i(1)} & \text{if} \quad t = \tau(1) \\ \vdots & \\ q_{i(m(1)+1)} & \text{if} \quad t = \tau(1) + m(1) \\ q_t & \text{if} \quad t \geq \tau(1) + m(1) + 1 \end{cases}$$

Observe that $\widetilde{q}_{\tau(1)} \geq \widetilde{q}_{\tau(1)+1} \geq \cdots \geq \widetilde{q}_{\tau(1)+m(1)} \geq \widetilde{q}_{\tau(1)-1} > \widetilde{q}_{\tau(1)+m(1)+1} \geq \cdots \geq \widetilde{q}_{\tau(2)-1}$. The difference between sequences q and $\widetilde{q}$ is that subsequence s_1 in q is replaced by a reordered decreasing sequence in $\widetilde{q}$. Let $\bar{s}_1 = (\widetilde{q}_{\tau(1)}, \ldots, \widetilde{q}_{\tau(1)+m(1)}, \ldots, \widetilde{q}_{\tau(2)-1})$. Then $\widetilde{q} = (s_0, \bar{s}_1, s_2, \ldots, s_n)$.

Next we will show $V_L(q) \leq V_L(\widetilde{q})$.

Since $\widetilde{q}_t = q_t$ for $t \leq \tau(1) - 1$, we have

$$\rho_t(\widetilde{q}) = \rho_t(q) \quad \text{for} \quad t \leq \tau(1) - 1 \tag{19}$$

Since s_0 is decreasing, we have

$$\begin{aligned} q_{\tau(1)+m(1)+1} &= \min\{q_1, \ldots, q_{\tau(1)+m(1)+1}\} \\ &= \widetilde{q}_{\tau(1)+m(1)+1} = \min\{\widetilde{q}_1, \ldots, \widetilde{q}_{\tau(1)+m(1)+1}\} \end{aligned} \tag{20}$$

By (20) and the fact that $\widetilde{q}_t = q_t$ for $t \geq \tau(1) + m(1) + 1$, we have

$$\rho_t(\widetilde{q}) = \rho_t(q) \quad \text{for} \quad t \geq \tau(1) + m(1) + 2 \tag{21}$$

Let $y = (q_{\tau(1)}, q_{\tau(1)+1}, \ldots, q_{\tau(1)+m(1)}, q_{\tau(1)+m(1)+1})$, $\widetilde{y} = (\widetilde{q}_{\tau(1)}, \widetilde{q}_{\tau(1)+1}, \ldots, \widetilde{q}_{\tau(1)+m(1)}, \widetilde{q}_{\tau(1)+m(1)+1})$, and recall that $\widetilde{q}_{\tau(1)+m(1)+1} = q_{\tau(1)+m(1)+1}$.

Consider policy q. Since $\min\{q_{\tau(1)}, q_{\tau(1)+1}, \ldots, q_{\tau(1)+m(1)}\} \geq q_{\tau(1)-1}$, and $q_{\tau(1)+m(1)+1} < q_{\tau(1)-1}$, the customers who initially arrive before period $\tau(1)$ will not buy anything in periods $\tau(1), \tau(1) + 1, \ldots, \tau(1) + m(1)$ and then some of them will purchase in period $\tau(1) + m(1) + 1$. We use X to denote the revenue accrued in periods $\tau(1), \ldots, \tau(1)+m(1)+1$ from sales to customers who arrived in time $\tau(1) - 1$ or earlier. By the preceding comment, such revenues are received only in period $\tau(1) + m(1) + 1$. Hence

$$\sum_{t=\tau(1)}^{\tau(1)+m(1)+1} \rho_t(q) = V_{L(y)}(y) + X$$

where $L(y) = m(1) + 2$ and

$$X = \alpha q_{\tau(1)+m(1)+1} \sum_{i=m(1)+2}^{k} \Big[F(\min\{q_{\tau(1)+m(1)+1-i}, \ldots, q_{\tau(1)-2}, q_{\tau(1)-1}\}) - F(q_{\tau(1)+m(1)+1})\Big]^+$$

Similarly,

$$\sum_{t=\tau(1)}^{\tau(1)+m(1)+1} \rho_t(\widetilde{q}) = V_{L(\widetilde{y})}(\widetilde{y}) + \widetilde{X}$$

where $L(\widetilde{y}) = m(1) + 2$ and

$$\widetilde{X} = \alpha \widetilde{q}_{\tau(1)+m(1)+1} \sum_{i=m(1)+2}^{k} \Big[F(\min\{\widetilde{q}_{\tau(1)+m(1)+1-i}, \ldots, \widetilde{q}_{\tau(1)-2}, \widetilde{q}_{\tau(1)-1}\}) - F(\widetilde{q}_{\tau(1)+m(1)+1})\Big]^+$$

By the definition of $\widetilde{q}$, it is easy to see $X = \widetilde{X}$. Lemma C.2 implies $V_{L(y)}(y) \leq V_{L(\widetilde{y})}(\widetilde{y})$ because $L(y) = L(\widetilde{y}) = m(1) + 2 \leq k$. Thus

$$\sum_{t=\tau(1)}^{\tau(1)+m(1)+1} \rho_t(q) \leq \sum_{t=\tau(1)}^{\tau(1)+m(1)+1} \rho_t(\widetilde{q}) \tag{22}$$

Therefore, by (19), (21), and (22) it follows that $V_L(q) \leq V_L(\widetilde{q})$.

Continuing in this fashion for subsequences $s_2, s_3, \ldots, s_n$, we obtain a sequence $\bar{q} = (s_0, \bar{s}_1, \ldots, \bar{s}_n) \in E^n(L)$ such that $V_L(q) \leq V_L(\bar{q})$. □

D Proof of Proposition 4

Consider an arbitrary $q \in E^n(L)$ and define

$$M(i) = \min\{t \in \{t(i)+1, \ldots, t(i+1)-1\} : q_t < q_{t(i)-1}\} - t(i) - 1 \text{ for } i = 1, \ldots, n.$$

The time $t(i) + M(i) + 1$ is the first time in $\{t(i)+1, \ldots, t(i+1)-1\}$ that the price drops below $q_{t(i)-1}$. Immediately after time $t(i)-1$, there are $M(i)+1$ consecutive prices greater than or equal to $q_{t(i)-1}$. The length of sequence s_i is $t(i+1)-t(i)$, and there is at least one price lower than $q_{t(i)-1}$ in s_i by condition 3 in the definition of $E^n(L)$; therefore, we have $t(i+1) - t(i) - [M(i)+1] \geq 1$. Thus $M(i) \leq t(i+1) - t(i) - 2$. By condition 4 in the definition, $t(i) + M(i) + 1 \leq t(i) + k - 1$, and thus $M(i) \leq k - 2$. Hence,

$$M(i) \in \{0, \ldots, \min\{k-2, t(i+1) - t(i) - 2\}\}. \tag{23}$$

Moreover,

$$q_{t(i)} \geq q_{t(i)+1} \geq \cdots \geq q_{t(i)+M(i)} \geq q_{t(i)-1} > q_{t(i)+M(i)+1} \geq q_{t(i)+M(i)+2} \geq \cdots \geq q_{t(i+1)-1}.$$

For policy $q \in E^n(L)$, consider another policy q' with length $L-1$ as follows.

$$q'_j = \begin{cases} q_j & \text{for} \quad 1 \leq j \leq t(n) - 1 \\ q_{j+1} & \text{for} \quad t(n) \leq j \leq L - 1 \end{cases} \tag{24}$$

Intuitively, q' is constructed from q by removing price $q_{t(n)}$ from q and shifting all prices originally in periods $t(n)+1, \ldots, L$ one period earlier. We also consider another policy $q'' = \psi(q)$ with length $L+1$ as follows.

$$q''_j = \begin{cases} q_j & \text{for} \quad 1 \leq j \leq t(n) \\ q_{j-1} & \text{for} \quad t(n)+1 \leq j \leq L+1 \end{cases} \tag{25}$$

If we want to emphasize the dependence of q'' on q, we will use the notation $\psi(q)$ to denote the sequence defined in (25). Intuitively, q'' is constructed from q by inserting a copy of price $q_{t(n)}$ in period $t(n)+1$ and shifting those prices originally in periods $t(n)+1, \ldots, L$ one period later. Define

$$\Delta_1(q) = V_L(q) - V_{L-1}(q')$$
$$\Delta_2(q) = V_{L+1}(q'') - V_L(q).$$

It may be helpful to refer to Figure 4 to visualize the arguments in the proof of the following lemma.

Lemma D.1 *For any $q \in E^n(L)$ with $t(n) + M(n) \geq k+1$, we have $\Delta_1(q) = \Delta_2(q)$.*

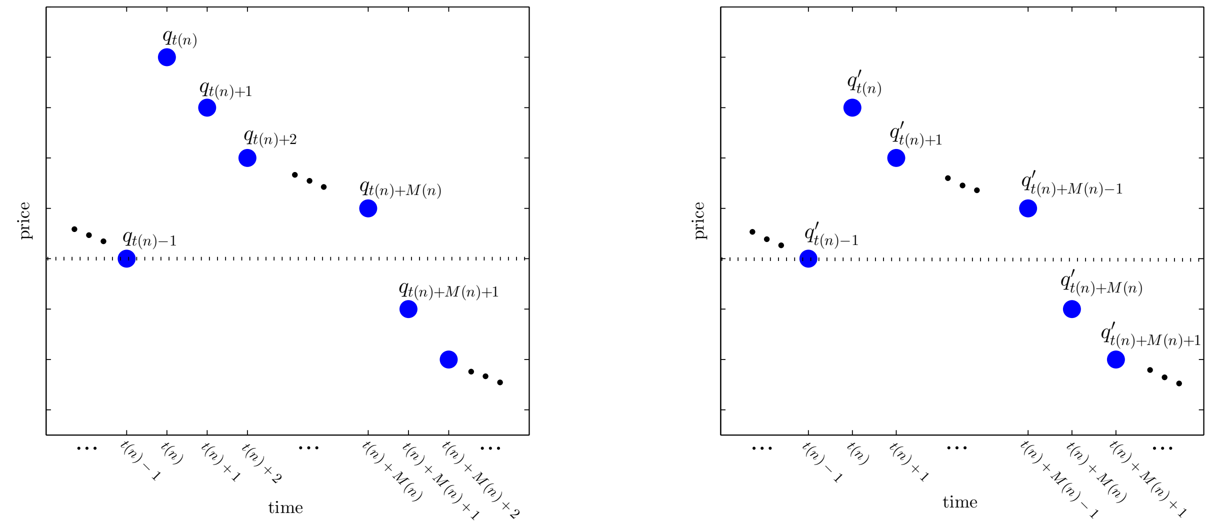

Figure 4: The left panel shows a portion of an element q of some $E^n(L)$ and the right panel shows a portion of q' as defined in (24). The policy q' is constructed from q by removing price $q_{t(n)}$ from q and shifting all prices originally in periods $t(n)+1, \ldots, L$ one period earlier. The dashed line through $q_{t(n)-1}$ and $q'_{t(n)-1}$ may be helpful for understanding the bookkeeping in the proof of Lemma D.1.

Proof. Fix $q \in E^n(L)$. Consider q' as defined in (24). To simplify our notation let $t = t(n) = t(n|q)$ and $M = M(n) = M(n|q)$. As in the proof of Lemma C.2 we have $V_L(q) = \sum_{i=1}^{L} \varphi_i^L(q)$, where $\varphi_i^L(q)$ is defined in (14) to be the revenue obtained from customers who arrive in period i when we use policy q. Likewise, we have $V_{L-1}(q') = \sum_{i=1}^{L-1} \varphi_i^{L-1}(q')$.

From the definition of q' and (14), it follows immediately that $\varphi_{i+1}^L(q) = \varphi_i^{L-1}(q')$ for $i = t, \ldots, L-1$. In addition, $\phi_i^L(q) = \varphi_i^{L-1}(q')$ for $i = 1, \ldots, t+M-k-1$. To see this, observe that a customer who arrives in a period $i \in \{1, \ldots, t+M-k-1\}$ sees the same sequence of prices in periods $i, \ldots, t-1$ under q' as he would under q. Moreover, under both q and q', no prices in periods $t, \ldots, i+k$ fall below the price at time $t-1$ because $i+k \leq (t+M-k-1)+k = t+M-1$. (Recall the definitions of t, M, and q'.) Hence, no customer that arrives in $i \in \{1, \ldots, t+M-k-1\}$ buys after time $t-1$ under policy q or q'.

Therefore,

$$\Delta_1(q) = V_L(q) - V_{L-1}(q') = \varphi_t^L(q) + \sum_{i=t+M-k}^{t-1} \left[\varphi_i^L(q) - \varphi_i^{L-1}(q') \right].$$

Similarly, we have that $\varphi_{i+1}^{L+1}(q'') = \varphi_i^L(q)$ for $i = t+1, \ldots, L$ and $\varphi_i^{L+1}(q'') = \varphi_i^L(q)$ for $i = 1, \ldots, t+M-k$, so

$$\Delta_2(q) = V_{L+1}(q'') - V_L(q) = \varphi_{t+1}^{L+1}(q'') + \sum_{i=t+M-k+1}^{t} \left[\varphi_i^{L+1}(q'') - \varphi_i^L(q) \right]$$

Using the definitions of t, M, k, q', and q'', it can now be checked using (14) that $\varphi_t^L(q) = \varphi_{t+1}^{L+1}(q'')$ and $\varphi_i^L(q) - \varphi_i^{L-1}(q') = \varphi_{i+1}^{L+1}(q'') - \varphi_{i+1}^L(q)$ for $i = t+M-k, \ldots, t-1$. Therefore, $\Delta_1(q) = \Delta_2(q)$. □

Recall from Section 5 that $E = \cup_{L=1}^{\kappa} \cup_{n=1}^{L-1} E^n(L) = \cup_{n=1}^{\kappa-1} \cup_{L=n+1}^{\kappa} E^n(L)$. For $n = 1, \ldots, \kappa-1$, let $E^n = \cup_{L=1}^{\kappa} E^n(L)$ so that $E = \cup_{n=1}^{\kappa-1} E^n$. Define

$$\mathcal{E}_1 = \cup_{n=2}^{\kappa-1} E^n$$
$$\mathcal{E}_2 = \{q \in E^1 : M(n) \neq 0 \text{ and } \max\{q_{t(n)+1}, \ldots, q_{t(n)+M(n)}\} > q_{t(n)-1}\}$$
$$\mathcal{E}_3 = \{q \in E^1 : M(n) \neq 0 \text{ and } q_{t(n)+1} = \cdots = q_{t(n)+M(n)} = q_{t(n)-1}\}$$
$$\mathcal{E}_4 = \{q \in E^1 : M(n) = 0\}.$$

It is not hard to see that

$$E = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3 \cup \mathcal{E}_4.$$

In addition, let

$$\mathcal{E}_3^* = \{q \in \mathcal{E}_3 : M(n) < k-2\}.$$

Observe that if $q \in \arg\max\{v(p) : p \in E\}$, then $q \in E$ and hence $q \in E^n$ for some $n = 1, \ldots, \kappa-1$. We will use the following lemma, which describes properties of such a q if it belongs to $\mathcal{E}_1$, $\mathcal{E}_2$, or $\mathcal{E}_3$.

Lemma D.2 *Consider $q \in \arg\max\{v(p) : p \in E\}$. Suppose $q \in E^n$ and $t(n) + M(n) \geq k+1$.*

1. *If $q \in \mathcal{E}_1 \cup \mathcal{E}_2$, then $v(q'') \geq v(q)$. If, in addition, $q'' \in E$, then $q'' \in \arg\max\{v(p) : p \in E\}$.*

2. *If $v(q) > v_1(x^*)$ where $x^* \in \arg\max\{x[1-G(x)], x \in \mathcal{P}\}$, then $q \notin \mathcal{E}_3 - \mathcal{E}_3^*$.*

Proof. Let $L = L(q)$.

Part 1: Suppose $q \in \mathcal{E}_1 \cup \mathcal{E}_2$. Consider the sequence q' defined by (24). Observe that $q' \in E$, and therefore $v_{L-1}(q') \leq v_L(q)$ by the optimality of q. (To emphasize how the condition $q \in \mathcal{E}_1 \cup \mathcal{E}_2$ is used in this argument, note that if $q \in \mathcal{E}_3 \cup \mathcal{E}_4$ then q' is decreasing and hence not in E.) We have

$$\begin{aligned}
\Delta_1(q) &= V_L(q) - V_{L-1}(q') && \text{[by the definition of } \Delta_1(q)\text{]} \\
&= L \cdot v_L(q) - (L-1)v_{L-1}(q') \\
&= (L-1)[v_L(q) - v_{L-1}(q')] + v_L(q) \\
&\geq v_L(q). && (26)
\end{aligned}$$

Consequently,

$$\begin{aligned}
v_{L+1}(q'') &= \frac{1}{L+1} V_{L+1}(q'') \\
&= \frac{1}{L+1}\Big[V_L(q) + \Delta_2(q)\Big] && \text{[by the definition of } \Delta_2(q)] \\
&= \frac{1}{L+1}\Big[V_L(q) + \Delta_1(q)\Big] && \text{[by Lemma D.1]} \\
&\geq \frac{1}{L+1}\Big[V_L(q) + v_L(q)\Big] && \text{[by (26)]} \qquad (27) \\
&= v_L(q).
\end{aligned}$$

Part 2: Suppose $v(q) > v_1(x^*)$. Suppose for a contradiction that $q \in \mathcal{E}_3 - \mathcal{E}_3^*$. Then, q is such that $n = 1$, $M(n) = k - 2$, and $q_{t(n)+1} = \cdots = q_{t(n)+M(n)} = q_{t(n)-1}$. Construct the sequence $q^\dagger \in \mathcal{E}_3^* \subset E$ defined as follows,

$$q_t^\dagger = \begin{cases} q_t & \text{for} \quad t = 1, \ldots, t(n) + M(n) - 1 \\ q_{t+1} & \text{for} \quad t = t(n) + M(n), \ldots, L - 1. \end{cases}$$

We will show $v(q^\dagger) > v(q)$, which contradicts $q \in \arg\max\{v(p) : p \in E\}$.

From the definition of $q^\dagger$ we have $\rho_t(q^\dagger) = \rho_t(q)$ for $t = 1, \ldots, t(n) + M(n) - 1$. Moreover,

$$\rho_t(q) = q_t\big\{1 - G(q_t) + \alpha k[F(q_{t-1}) - F(q_t)]\big\} \qquad \text{for } t = t(n) + M(n), \ldots, L \qquad (28)$$

$$\rho_t(q^\dagger) = q_{t+1}\big\{1 - G(q_{t+1}) + \alpha k[F(q_t) - F(q_{t+1})]\big\} \qquad \text{for } t = t(n) + M(n), \ldots, L - 1 \qquad (29)$$

because $t(n) + M(n) - 1 \geq k$ and both $(q_{t(n)}, q_{t(n)+1}, \ldots, q_L)$ and $(q^\dagger_{t(n)}, q^\dagger_{t(n)+1}, \ldots, q^\dagger_{L-1})$ are decreasing. Thus by (28)–(29) it follows that $\rho_t(q^\dagger) = \rho_{t+1}(q)$ for $t = t(n) + M(n), \ldots, L - 1$. Hence

$$V_L(q) - V_{L-1}(q^\dagger) = \rho_{t(n)+M(n)}(q) = q_{t(n)+M(n)}[1 - G(q_{t(n)+M(n)})] \leq v_1(x^*). \qquad (30)$$

The last inequality holds by the definition of x^*. Thus,

$$\begin{aligned}
v_{L-1}(q^\dagger) - v_L(q) &= \frac{V_{L-1}(q^\dagger)}{L-1} - \frac{V_L(q)}{L} \\
&= \frac{V_L(q) - L[V_L(q) - V_{L-1}(q^\dagger)]}{L(L-1)} \\
&\geq \frac{V_L(q) - L \cdot v_1(x^*)}{L(L-1)} && \text{[by (30)]} \\
&= \frac{v_L(q) - v_1(x^*)}{L-1} \\
&> 0. && \text{[by supposition]}
\end{aligned}$$

Hence $v(q^\dagger) > v(q)$, which is a contradiction. □

Proof of Proposition 4. Let $D_0 = \cup_{L=1}^{\kappa+k} D_0(L)$ where $D_0(L) = \{q \in \mathcal{P}^L : q_1 \geq q_2 \geq \cdots \geq q_L\}$. Observe that $D \subset D_0$. The set D_0 differs from D in two ways: elements of D_0 can have regeneration points and elements of D_0 can be slightly longer than those of D. Any element $x \in D_0$ with say ℓ regeneration points can be expressed as $(x^1, x^2, \ldots, x^\ell, x^{\ell+1})$ where $x^1, \ldots, x^{\ell+1} \in D$ and the average revenue of x is a convex combination of the average revenues of $x^1, \ldots, x^{\ell+1}$. In particular, $v(x) = \sum_{i=1}^{\ell+1} \lambda_i v(x^i)$ where $\lambda_i = L(x^i)/L(x)$. Hence, $d \in \arg\max_{p \in D} v(p)$ satisfies

$$v(d) \geq v(x) \quad \text{for all } x \in D_0. \tag{31}$$

Consider $q \in \arg\max_{p \in E} v(p)$. Let $L = L(q)$. We have $q \in E$ and therefore $q \in E^n(L)$ for some n. If $v_L(q) \leq v_1(x^*)$ where $x^* \in \arg\max\{x[1 - G(x)], x \in \mathcal{P}\}$, then we are done because $x^* \in D(1) \subset D$. Therefore, to complete the proof we hereafter assume $v_L(q) > v_1(x^*)$.

We consider two cases.

Case 1: $t(n) + M(n) \geq k + 1$. We have two subcases, 1A and 1B below.

Subcase 1A: $q \in \mathcal{E}_1 \cup \mathcal{E}_2$.

By Lemma D.2, we have that $v(q'') \geq v(q)$.

Below we will use the notation $M(n|q)$ and $t(n|q)$ to emphasize the dependence upon q. We will consider two possibilities, (i) and (ii). Recall (23).

(i) Suppose that $M(n|q) = k - 2$. (Here $M(n|q) = k - 2 - l$ with $l = 0$.) In this case $t(n|q)$ is an F-regeneration point of the policy $q^1 = \psi(q) = q''$ as defined in (25). To see why this is so, note that $t(n|q) + M(n|q) + 1 = t(n|q) + k - 1$ because $M(n|q) = k - 2$. In addition, $q_{t(n|q)+i} \geq q_{t(n|q)-1}$ for $i = 0, \ldots, M(n|q)$, because $q \in E^n(L)$. It follows by construction of q^1 that

$$q^1_{t(n|q)-1} \leq \min\{q^1_{t(n|q)}, \ldots, q^1_{t(n|q)+k-1}\}$$

and hence $t(n|q)$ is an F-regeneration point of q^1.

(ii) Suppose that $M(n|q) < k-2$. Then $M(n|q) = k-2-l$ for some $l \in \{1, \ldots, k-2\}$. Consider $q^1 = \psi(q)$. Then $q^1 \in E^n(L+1)$ with $t(n|q^1) = t(n|q)$ and $M(n|q^1) = M(n|q) + 1$. Consider the policy $q^2 = \psi(q^1) = \psi(\psi(q))$. If $l = 1$, then q^2 has an F-regeneration point at $t(n|q)$.

For general $l \in \{1, \ldots, k-2\}$, consider $q^{l+1} \in \mathcal{P}^{L+l+1}$ defined as follows: let $q^0 = q$, and $q^i = \psi(q^{i-1}) \in \mathcal{P}^{L+i}$ for $i = 1, \ldots, l+1$. Intuitively, q^i is obtained from q by inserting i new copies of price $q_{t(n|q)}$ into q at time periods $t(n|q) + 1, \ldots, t(n|q) + i$ and shifting all prices in q that appeared after $t(n|q)$ "to the right" by i time periods. By construction $t(n|q)$ is an F-regeneration point of q^{l+1}. (It now may be helpful to refer back to Figure 2 in Section 5. In the example depicted there with $k = 5$, the left panel shows q and the right panel shows q^{l+1} where $l+1 = L' = 2$.) Note also that $q^i \in E^n(L+i)$ for $i = 0, \ldots, l$. Moreover, $t(n|q^i) = t(n|q^{i-1}) = t(n|q)$ and $M(n|q^i) = M(n|q^{i-1}) + 1 = M(n|q) + i$ for $i = 1, \ldots, l$.

We can now combine (i) and (ii) to see that if $M(n|q) = k-2-l$ for some $l \in \{0, \ldots, k-2\}$ (which exhausts all possibilities because $q \in E^n(L)$), then q^{l+1} has an F-regeneration point at time $t(n|q)$. In addition, in either subcase, by construction, $q^i_{t(n|q^i)} = q_{t(n|q)}$ for $i = 0, \ldots, l$. By Lemma D.2, we have

$$v_{L+l+1}(q^{l+1}) \geq v_L(q). \tag{32}$$

Since $t(n|q)$ is an F-regeneration point of q^{l+1}, we can decompose q^{l+1} into two independent subsequences $p^1 = (q^{l+1}_1, \ldots, q^{l+1}_{t(n|q)-1})$ and $p^2 = (q^{l+1}_{t(n|q)}, \ldots, q^{l+1}_{L+l+1})$ with $L(p^1) = t(n|q) - 1$ and $L(p^2) = L + l - t(n|q) + 2$ so that $q^{l+1} = (p^1, p^2)$ and

$$\begin{aligned} v_{L+l+1}(q^{l+1}) &= \frac{1}{L+l+1}\Big[\sum_{i=1}^{t(n|q)-1} \rho_t(q^{l+1}) + \sum_{i=t(n|q)}^{L+l+1} \rho_t(q^{l+1})\Big] \\ &= \frac{L(p^1)}{L+l+1} v(p^1) + \frac{L(p^2)}{L+l+1} v(p^2). \end{aligned} \tag{33}$$

Observe that

$$p^1 \in \begin{cases} E^{n-1}(t(n|q)-1) & \text{if} \quad q \in \mathcal{E}_1 \\ D_0(t(n|q)-1) & \text{if} \quad q \in \mathcal{E}_2 \end{cases} \tag{34}$$

and $p^2 \in D_0$. If $q \in \mathcal{E}_1$, then $p^1 \in E$ by (34) and therefore $v(q) \geq v(p^1)$. Combining this with (32) and (33), we have $v(p^2) \geq v(q)$. Hence, $v(d) \geq v(p^2) \geq v(q)$. If $q \in \mathcal{E}_2$, then p^1 and p^2 are both decreasing and again by (32) and (33), we have $\max\{v(p^2), v(p^1)\} \geq v(q)$. Hence, $v(d) \geq v(q)$.

Subcase 1B: $q \in \mathcal{E}_4 \cup \mathcal{E}_3$.

We have $v_L(q) > v_1(x^*)$, so Lemma D.2 implies $q \in \mathcal{E}_4 \cup \mathcal{E}_3^*$. We consider (i) $k > 2$ and (ii) $k = 2$ separately.

(i) Suppose $k > 2$. Consider sequence q'' as defined in (25). It is easy to see $q'' \in E$. Thus, $v(q'') \leq v(q)$ by the definition of q.

$$\begin{aligned} \Delta_2(q) &= V_{L+1}(q'') - V_L(q) \\ &= (L+1)v(q'') - Lv(q) \\ &= L \cdot (v(q'') - v(q)) + v(q'') \\ &\leq v(q'') \leq v(q) \end{aligned} \tag{35}$$

Now, consider sequence q' with length $L-1$ as defined in (24). It is easy to see that $q' \in D_0$

and consequently $v(d) \geq v(q')$. We have

$$\begin{aligned}
v_{L-1}(q') &= \frac{1}{L-1} V_{L-1}(q') = \frac{1}{L-1}[V_L(q) - \Delta_1(q)] \\
&= \frac{1}{L-1}[Lv(q) - \Delta_2(q)] && \text{[By Lemma D.1]} \\
&\geq \frac{1}{L-1}[Lv(q) - v(q)] && \text{[By (35)]} \\
&= v(q).
\end{aligned}$$

Thus we have proved $v(d) \geq v(q') \geq v(q)$.

(ii) Suppose $k = 2$. Then $\mathcal{E}_3 = \emptyset$. Hence, we need only consider $q \in \mathcal{E}_4$. Suppose for a contradiction that $v(q) > v(d)$. Consider sequence q' with length $L-1$ as defined in (24). It is easy to see that $q' \in D_0$. Hence, $v(q') \leq v(d) < v(q)$. In addition,

$$\Delta_1(q) = V_L(q) - V_{L-1}(q') = Lv(q) - (L-1)v(q') = (L-1)(v(q) - v(q')) + v(q) \geq v(q). \tag{36}$$

Consider sequence q'' as defined in (25). Note that q'' has an F-regeneration point at time $t(1)$. We can decompose q'' into two independent decreasing sequences $q'' = (p^1, p^2)$ where $p^1 = (q''_1, \ldots, q''_{t(1)-1})$ and $p^2 = (q''_{t(1)}, \ldots, q''_{L+1})$. And

$$v(q'') = \frac{t(1)-1}{L+1} v(p^1) + \frac{L+2-t(1)}{L+1} v(p^2) \leq v(d) < v(q). \tag{37}$$

The last inequality is the supposition we made in hope of producing a contradiction.

On the other hand,

$$\begin{aligned}
v(q'') &= \frac{1}{L+1} V(q'') = \frac{1}{L+1}[V(q) + \Delta_2(q)] \\
&= \frac{1}{L+1}[V(q) + \Delta_1(q)] && \text{[By Lemma D.1]} \\
&\geq \frac{1}{L+1}[V(q) + v(q)] && \text{[By (36)]} \\
&= v(q),
\end{aligned} \tag{38}$$

which contradicts (37).

Case 2: $t(n) + M(n) \leq k$.

Write $q = (p^1, p^2)$, where $p^1 = (q_1, \ldots, q_{t(n)+M(n)+1})$ and $p^2 = (q_{t(n)+M(n)+2}, \ldots, q_L)$. Reorder p^1 into a decreasing sequence $p^3 = (q_{i(1)}, \ldots, q_{i(t(n)+M(n)+1)})$. Then consider another sequence p^0 defined by $p^0 = (p^3, p^2)$. Then $p^0 \in D_0$.

Since $t(n) + M(n) \leq k$, it follows that $t(n) + M(n) + 1 \leq k + 1$. Hence, by Lemma C.2, we have

$$\sum_{t=1}^{t(n)+M(n)+1} \rho_t(p^0) \geq \sum_{t=1}^{t(n)+M(n)+1} \rho_t(q) \tag{39}$$

Since $q_{t(n)+M(n)+1} = \min\{q_1, \dots, q_{t(n)+M(n)+1}\} = p^0_{t(n)+M(n)+1} = \min\{p^0_1, \dots, p^0_{t(n)+M(n)+1}\}$, and $q_j = p^0_j$ for $j \geq t(n) + M(n) + 1$, we have

$$\rho_t(p^0) = \rho_t(q) \quad \text{for } t = t(n) + M(n) + 2, \dots, L. \tag{40}$$

Thus by (39) and (40), we have $V(p^0) \geq V(q)$. Hence $v(p^0) \geq v(q)$. Moreover, $v(d) \geq v(p^0)$ by (31) because $p^0 \in D_0$. Therefore, $v(d) \geq v(q)$. This completes the proof. □

E Proof of Proposition 5

Lemma E.1 *For any policy $y = (y_1, \dots, y_{M+1}) \in \mathcal{P}^{M+1}$ with $y_1 \geq y_2 \geq \dots \geq y_{M+1}$ and $M \leq k$, consider the policy $\check{y}$ defined by $\check{y}_t = y_{t+1}$ for $t = 1, \dots, M$. Then*

$$V_{M+1}(y) - V_M(\check{y}) \; = \; y_1[1 - G(y_1)] + \alpha \sum_{t=1}^{M} y_{t+1}[F(y_t) - F(y_{t+1})] \; = \; \varphi_1^{M+1}(y),$$

where $\varphi_1^{M+1}(y)$ is defined in (14).

Proof. Both y and $\check{y}$ are decreasing, so by the definition of $\rho_t(\cdot)$,

$$\begin{aligned}
\rho_{t+1}(y) &= y_{t+1}\big\{1 - G(y_{t+1}) + \alpha t[F(y_t) - F(y_{t+1})]\big\} \\
\rho_t(\check{y}) &= \check{y}_t\big\{1 - G(\check{y}_t) + \alpha(t-1)[F(\check{y}_{t-1}) - F(\check{y}_t)]\big\} \\
&= y_{t+1}\big\{1 - G(y_{t+1}) + \alpha(t-1)[F(y_t) - F(y_{t+1})]\big\}
\end{aligned}$$

for $t = 1, \dots, M$. Thus $\rho_{t+1}(y) - \rho_t(\check{y}) = \alpha y_{t+1}[F(y_t) - F(y_{t+1})]$ for $t = 1, \dots, M$. Therefore,

$$\begin{aligned}
V_{M+1}(y) - V_M(\check{y}) &= \rho_1(y) + \sum_{t=1}^{M} \big[\rho_{t+1}(y) - \rho_t(\check{y})\big] \\
&= y_1[1 - G(y_1)] + \alpha \sum_{t=1}^{M} y_{t+1}[F(y_t) - F(y_{t+1})]
\end{aligned}$$

□

Proof of Proposition 5. Consider $L \geq k + m$ and $q \in \arg\max\{v_L(p) : p \in D(L)\}$. We will establish that there exists a policy $q^\circ \in D(k+m-1) \cup D(1)$ such that $v(q^\circ) \geq v(q)$, from which the proposition follows. To this end, observe that it must be that $q_j = q_{j+1}$ for some $j \in \{k, \dots, L-1\}$ because there are m prices in $\mathcal{P}$ and q is decreasing. (For a policy of length at least $k + m$, at least one price must appear multiple times in period k or later. The policy q is decreasing, so such a price must appear in consecutive periods.)

Let $x^* \in \arg\max\{v_1(x) : x \in D(1)\} = \arg\max\{x[1 - G(x)] : x \in \mathcal{P}\}$. If $v_L(q) < v_1(x^*)$, then we are done. Therefore we just need to consider the case that

$$v_L(q) \geq v_1(x^*). \tag{41}$$

Consider the sequence $q^\dagger \in D(L-1) \subset D$ as follows:

$$q_t^\dagger = \begin{cases} q_t & \text{for} \quad t = 1, \ldots, j \\ q_{t+1} & \text{for} \quad t = j+1, \ldots, L-1 \end{cases}$$

From the definition of $q^\dagger$ we have

$$\sum_{t=1}^{j} \rho_t(q^\dagger) = \sum_{t=1}^{j} \rho_t(q). \tag{42}$$

Moreover,

$$\rho_t(q) = q_t\big\{1 - G(q_t) + \alpha k[F(q_{t-1}) - F(q_t)]\big\} \qquad \text{for } t = j+1, \ldots, L \tag{43}$$

$$\rho_t(q^\dagger) = q_{t+1}\big\{1 - G(q_{t+1}) + \alpha k[F(q_t) - F(q_{t+1})]\big\} \qquad \text{for } t = j+1, \ldots, L-1 \tag{44}$$

because $j \geq k$ and both q and $q^\dagger$ are decreasing. Thus by (43) and (44) it is easy to see that $\rho_t(q^\dagger) = \rho_{t+1}(q)$ for $t = j+1, \ldots, L-1$, and hence

$$\sum_{t=j+1}^{L-1} \rho_t(q^\dagger) = \sum_{t=j+2}^{L} \rho_t(q)\,. \tag{45}$$

Therefore, by (42) and (45)

$$V_L(q) - V_{L-1}(q^\dagger) = \rho_{j+1}(q) = q_{j+1}[1 - G(q_{j+1})] \leq v_1(x^*). \tag{46}$$

The last inequality holds by the definition of x^*. Thus,

$$\begin{aligned} v_{L-1}(q^\dagger) - v_L(q) &= \frac{V_{L-1}(q^\dagger)}{L-1} - \frac{V_L(q)}{L} \\ &= \frac{V_L(q) - L[V_L(q) - V_{L-1}(q^\dagger)]}{L(L-1)} \\ &\geq \frac{V_L(q) - L \cdot v_1(x^*)}{L(L-1)} && \text{[by (46)]} \\ &= \frac{v_L(q) - v_1(x^*)}{L-1} \\ &\geq 0 && \text{[by (41)]} \end{aligned}$$

If $L - 1 = k + m - 1$ we are done with $q^\circ = q^\dagger$. Otherwise we can remove a repeated price from $q^\dagger$ in some period later than k while improving (or keeping the same) the average revenue as above. We continue in this fashion until we arrive at q° with length $k + m - 1$ as desired. □

F Proof of Proposition 6

Proof. We will prove by induction that $v_j(q^j) \le v_{j+1}(q^{j+1})$ for $j = 1, \dots, k$ when $v_1(q^1) \le v_2(q^2)$.

When $j = 1$, we have $v_1(q^1) \le v_2(q^2)$.

For any $j \in \{2, \dots, k\}$, suppose for the inductive hypothesis that

$$v_{j-1}(q^{j-1}) \le v_j(q^j). \tag{47}$$

To complete the proof, we will show that $v_j(q^j) \le v_{j+1}(q^{j+1})$. By (14), we have

$$\varphi_1^j(q^j) = q_1^j[1 - G(q_1^j)] + \alpha \sum_{t=2}^{j} q_t^j [F(q_{t-1}^j) - F(q_t^j)]$$

Let $\widehat{q}^{j-1} = (q_2^j, \dots, q_j^j)$. Then $\widehat{q}^{j-1} \in D(j-1)$ and

$$\begin{aligned}
v_{j-1}(\widehat{q}^{j-1}) - \varphi_1^j(q^j) &\le v_{j-1}(q^{j-1}) - \varphi_1^j(q^j) \\
&= v_{j-1}(q^{j-1}) - [V_j(q^j) - V_{j-1}(\widehat{q}^{j-1})] && \text{[by Lemma E.1]} \\
&\le v_{j-1}(q^{j-1}) + V_{j-1}(q^{j-1}) - V_j(q^j) \\
&= j v_{j-1}(q^{j-1}) - j v_j(q^j) \\
&\le 0 && \text{[by (47)]} \qquad (48)
\end{aligned}$$

The preceding is the key piece of the inductive step mentioned in the main text after Proposition 6. To see this, note that $v_{j-1}(\widehat{q}^{j-1}) = \frac{1}{j-1}\sum_{t=1}^{j-1} \varphi_t^{j-1}(\widehat{q}^{j-1}) = \frac{1}{j-1}\sum_{t=2}^{j} \varphi_t^j(q^j)$, which is the average revenue per period over periods $2, \dots, j$ accrued under policy q^j from just those customers who initially arrive in periods $2, \dots, j$. We now have

$$\begin{aligned}
v_j(q^j) &= \frac{1}{j} V_j(q^j) = \frac{1}{j}\left[\varphi_1^j(q^j) + V_{j-1}(\widehat{q}^{j-1})\right] \\
&= \frac{1}{j}\left[\varphi_1^j(q^j) + (j-1) v_{j-1}(\widehat{q}^{j-1})\right] \\
&\le \varphi_1^j(q^j) \qquad \text{[by (48)]} \qquad (49)
\end{aligned}$$

Consider $\widetilde{q}^{j+1} \in D(j+1)$ as follows.

$$\widetilde{q}_t^{j+1} = \begin{cases} q_1^j & \text{for} \quad t = 1 \\ q_{t-1}^j & \text{for} \quad t = 2, \dots, j+1 \end{cases}$$

Then

$$\begin{aligned}\varphi_1^{j+1}(\widetilde{q}^{j+1}) &= \widetilde{q}_1^{j+1}[1-G(\widetilde{q}_1^{j+1})] + \alpha\Big\{\widetilde{q}_2^{j+1}[F(\widetilde{q}_1^{j+1}) - F(\widetilde{q}_2^{j+1})] + \sum_{t=3}^{j+1}\widetilde{q}_t^{j+1}[F(\widetilde{q}_{t-1}^{j+1}) - F(\widetilde{q}_t^{j+1})]\Big\} \\ &= q_1^j[1-G(q_1^j)] + \alpha\sum_{t=2}^{j} q_t^j[F(q_{t-1}^j) - F(q_t^j)] \\ &= \varphi_1^j(q^j) \geq v_j(q^j) \qquad \text{[by (49)]} \end{aligned} \tag{50}$$

Hence

$$v_{j+1}(\widetilde{q}^{j+1}) = \frac{1}{j+1}V_{j+1}(\widetilde{q}^{j+1}) = \frac{1}{j+1}\Big[\varphi_1^{j+1}(\widetilde{q}^{j+1}) + V_j(q^j)\Big] = \frac{1}{j+1}\Big[\varphi_1^{j+1}(\widetilde{q}^{j+1}) + jv_j(q^j)\Big] \geq v_j(q^j)$$

by (50). Therefore, we have $v_{j+1}(q^{j+1}) \geq v_{j+1}(\widetilde{q}^{j+1}) \geq v_j(q^j)$, which completes the inductive step.

We have now proved that $v_1(q^1) \leq v_2(q^2) \leq \cdots \leq v_{k+1}(q^{k+1})$. Consequently, $v_{k+1}(q^{k+1}) \geq \max\{v(q) : q \in \cup_{L=1}^k D(L)\}$ where $q^{k+1} \in D(k+1)$, from which the second statement in the proposition follows. If $v_1(q^1) < v_2(q^2)$, then the above argument is easily modified to show that $v_1(q^1) < v_2(q^2) < \cdots < v_{k+1}(q^{k+1})$. □

G Proof of Proposition 7

Our proof of Proposition 7 uses a discretization approach similar to one found in Besbes and Lobel (2015). To begin, let

$$\begin{aligned} D_0(L) &= \{q \in [0,\bar{P}]^L : q_1 \geq q_2 \geq \cdots \geq q_L\} \\ D(L) &= \{q \in D_0(L) : q \text{ has no F-regeneration points}\}. \end{aligned}$$

Throughout this section, we use $D(L)$ to denote the set of decreasing price sequences with no F-regeneration points where the individual prices are selected from the continuous price set $[0,\bar{P}]$. Note that this is slightly different from $D(L)$ as used Section 4, where prices are selected from a finite price set.

For $L \geq 1$, let $d^L \in \arg\max\{v_L(q) : q \in D_0(L)\}$. Consider the problem $\sup\{v_2(q) : q \in [0,\bar{P}]^2\}$. Since $v_2(\cdot)$ is continuous and $[0,\bar{P}]^2$ is compact, the supremum is a maximum. By Lemma C.2 (which holds when $\mathcal{P} = [0,\bar{P}]$) there is a decreasing sequence that is optimal. Therefore $d^2 = (d_1^2, d_2^2) \in \arg\max\{v_2(q) : q \in [0,\bar{P}]^2\}$, where $d_1^2 \geq d_2^2$.

Let $\widehat{\mathcal{P}}(n) = \{d_1^2, d_2^2\} \cup \{0, \frac{1}{n}, \frac{2}{n}, \ldots, V^n - \frac{2}{n}, V^n - \frac{1}{n}, V^n\}$ where $V^n = \lfloor n\bar{P}\rfloor/n$. Observe that $\widehat{\mathcal{P}}(n)$ is a finite set.

Define

$$\widehat{D}_0(n,L) = \{q \in \widehat{\mathcal{P}}(n)^L : q_1 \geq q_2 \geq \cdots \geq q_L\}$$
$$\widehat{D}(n,L) = \{q \in \widehat{D}_0(n,L) : q \text{ has no F-regeneration points}\}$$

Observe that d^2 is an element of $\widehat{D}_0(n,2)$ for all n by construction of $\widehat{\mathcal{P}}(n)$. Note also that $\widehat{D}_0(n,L)$ and $\widehat{D}(n,L)$ are finite sets, and therefore the supremum of any real-valued function taken over those sets is in fact a maximum.

Before we prove Proposition 7, we state two lemmas. Proofs of the lemmas appear at the end of this section, after the proof of Proposition 7.

Lemma G.1 *Suppose that $|F(x)-F(y)| \leq A|x-y|$ and $|G(x)-G(y)| \leq A|x-y|$ for all $x, y \in [0, \bar{P}]$ for some finite A; that is, $F(\cdot)$ and $G(\cdot)$ are Lipschitz continuous on $[0, \bar{P}]$. Then $|H(p) - H(q)| \leq C \sup_t |p_t - q_t|$ for any $p, q \in [0, \bar{P}]^\infty$ where C is a finite constant.*

Lemma G.2 *Suppose that $F(\cdot)$ and $G(\cdot)$ are Lipschitz continuous on $[0, \bar{P}]$ with continuous and strictly positive derivatives on $(0, \bar{P})$. Then, there exists a finite integer c, which does not depend upon n, with $c \geq 2$ such that for all $L > c$, the following inequality holds:*

$$\max_q \{v(q) : q \in \widehat{D}_0(n,L)\} \leq v_2(d^2) \quad \textit{for all } n \geq 1.$$

Proof of Proposition 7. Consider a sequence of pricing policies $\{p^n\} \in \mathcal{P}^\infty = [0, \bar{P}]^\infty$ such that

$$H(p^n) \geq H^* - \frac{1}{n} \tag{51}$$

where $H^* = \sup_p \{H(p) : p \in [0, \bar{P}]^\infty\}$.

Let $\widehat{p}^n = (\widehat{p}^n_1, \widehat{p}^n_2, \dots)$ where $\widehat{p}^n_t$ is equal to p^n_t rounded to the closest element in the set $\widehat{\mathcal{P}}(n)$. Lemma G.1 implies that

$$|H(p^n) - H(\widehat{p}^n)| \leq C \sup_t |p^n_t - \widehat{p}^n_t|, \tag{52}$$

where C is a finite constant. By (51) and (52), we have $H(\widehat{p}^n) \geq H^* - \frac{C+1}{n}$.

Define $H_n = \sup_p \{H(p) : p \in \widehat{\mathcal{P}}(n)^\infty\}$. By definition, $H_n \geq H(\widehat{p}^n)$ because $\widehat{p}^n \in \widehat{\mathcal{P}}(n)^\infty$. So $H_n \geq H^* - \frac{C+1}{n}$. Given any $\varepsilon > 0$, it follows that $H_n \geq H^* - \varepsilon$ for $n > n(\varepsilon) = \frac{C+1}{\varepsilon}$. Therefore, $\liminf_n H_n \geq H^* - \varepsilon$. This holds for any $\varepsilon > 0$, and so

$$\liminf_n H_n \geq H^*\,. \tag{53}$$

Moreover,

$$H_n = \max_q \{H(p) : p = (q, q, q, \dots), q \in \cup_{L=1}^{|\widehat{\mathcal{P}}(n)|+k-1} \widehat{D}(n, L)\} \tag{54}$$

by Theorem 1 (and its proof, which shows that the decreasing cycles in the theorem have no F-regeneration points when viewed in isolation).

We now take a brief detour and observe that $\cup_{L=1}^{M} D(L) \subset \cup_{L=1}^{M} D_0(L)$ for any M. Any element $x \in \cup_{L=1}^{M} D_0(L)$ with say ℓ regeneration points can be expressed as $(x^1, x^2, \dots, x^\ell, x^{\ell+1})$ where $x^1, \dots, x^{\ell+1} \in \cup_{L=1}^{M} D(L)$ and the average revenue of x is $v(x) = \sum_{i=1}^{\ell+1} \lambda_i v(x^i)$ for some $\{\lambda_i\}$. Hence, for any $d \in \arg\max_{p \in \cup_{L=1}^{M} D_0(L)} v(p)$ we have

$$v(d) \leq v(y) \quad \text{for some } y \in \cup_{L=1}^{M} D(L). \tag{55}$$

By (54), we now have

$$\begin{aligned}
H_n &= \max_q \{v(q) : q \in \cup_{L=1}^{|\widehat{\mathcal{P}}(n)|+k-1} \widehat{D}(n, L)\} && \text{[by Lemma 1]} \\
&\leq \max_q \{v(q) : q \in \cup_{L=1}^{|\widehat{\mathcal{P}}(n)|+k-1} \widehat{D}_0(n, L)\} && [\widehat{D}(n, L) \subset \widehat{D}_0(n, L)] \\
&\leq \max_q \{v(q) : q \in \cup_{L=1}^{c} \widehat{D}_0(n, L)\} && \text{[by Lemma G.2 and } d^2 \in \widehat{D}_0(n, 2)] \\
&\leq \sup_q \{v(q) : q \in \cup_{L=1}^{c} D_0(L)\} && [\widehat{D}_0(n, L) \subset D_0(L)] \\
&= \max_q \{v(q) : q \in \cup_{L=1}^{c} D_0(L)\} && [\cup_{L=1}^{c} D_0(L) \text{ is compact and } v(\cdot) \text{ is continuous}] \\
&= \max_q \{v(q) : q \in \cup_{L=1}^{c} D(L)\} && \text{[by (55)].}
\end{aligned} \tag{56}$$

As an aside, we note that the second inequality above becomes an equality for n so large that $|\widehat{\mathcal{P}}(n)| + k - 1 \geq c$.

By (56) and Lemma 1, we see that

$$H_n \leq \max_q \{H(p) : p = (q, q, q, \dots), q \in \cup_{L=1}^{c} D(L)\}. \tag{57}$$

By (53) and (57) it follows that

$$\max_q \{H(p) : p = (q, q, q, \dots), q \in \cup_{L=1}^{c} D(L)\} \geq H^*$$

and hence

$$\max_q \{H(p) : p = (q, q, q, \dots), q \in \cup_{L=1}^{c} D(L)\} = H^*. \tag{58}$$

Therefore there exists a decreasing cyclic policy that is an optimal solution to (2) when $\mathcal{P} = [0, \bar{P}]$. In particular, any policy that attains the maximum in the optimization problem on the left side of (58) solves (2) for $\mathcal{P} = [0, \bar{P}]$. □

Proof of Lemma G.1. Consider $p = (p_1, p_2, \dots) \in \mathcal{P}^\infty$ and $q = (q_1, q_2, \dots) \in \mathcal{P}^\infty$. We have

$$
\begin{aligned}
|H(p) - H(q)| &= \Big| \liminf_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \rho_t(p) - \liminf_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \rho_t(q) \Big| \\
&\le \limsup_{T\to\infty} \Big| \frac{1}{T}\sum_{t=1}^{T} [\rho_t(p) - \rho_t(q)] \Big| \\
&\le \sup_t \Big| \rho_t(p) - \rho_t(q) \Big| \\
&\le \sup_t \Big| p_t[1 - G(p_t)] - q_t[1 - G(q_t)] \Big| \\
&\quad + \alpha \sum_{i=1}^{k} \sup_t \Big| p_t[F(\min\{p_{t-i}, \dots, p_{t-1}\}) - F(p_t)]^+ - q_t[F(\min\{q_{t-i}, \dots, q_{t-1}\}) - F(q_t)]^+ \Big|
\end{aligned}
$$

Now we will evaluate the terms in the final expression above.

$$
\begin{aligned}
\Big| p_t[1 - G(p_t)] - q_t[1 - G(q_t)] \Big| &= \Big| p_t[G(q_t) - G(p_t)] + [1 - G(q_t)](p_t - q_t) \Big| \\
&\le \Big| p_t[G(q_t) - G(p_t)] \Big| + \Big| [1 - G(q_t)](p_t - q_t) \Big| \\
&\le p_t A \Big| p_t - q_t \Big| + [1 - G(q_t)] \Big| p_t - q_t \Big| \\
&\le (\bar{P}A + 1) \Big| p_t - q_t \Big|
\end{aligned}
$$

By the same process, we have

$$
\Big| p_t[1 - F(p_t)] - q_t[1 - F(q_t)] \Big| \le (\bar{P}A + 1) \Big| p_t - q_t \Big|.
$$

Similarly,

$$
\begin{aligned}
&\Big| q_t[1 - F(\min\{q_{t-i}, \dots, q_{t-1}\})] - p_t[1 - F(\min\{p_{t-i}, \dots, p_{t-1}\})] \Big| \\
&\qquad \le \Big| q_t[F(\min\{p_{t-i}, \dots, p_{t-1}\}) - F(\min\{q_{t-i}, \dots, q_{t-1}\})] \Big| + \Big| [1 - F(\min\{p_{t-i}, \dots, p_{t-1}\})](q_t - p_t) \Big| \\
&\qquad \le \bar{P}A \sup_j \Big| p_j - q_j \Big| + \Big| q_t - p_t \Big|
\end{aligned}
$$

So

$$
\begin{aligned}
&\Big| p_t[F(\min\{p_{t-i}, \dots, p_{t-1}\}) - F(p_t)]^+ - q_t[F(\min\{q_{t-i}, \dots, q_{t-1}\}) - F(q_t)]^+ \Big| \\
&\quad \le \Big| p_t[F(\min\{p_{t-i}, \dots, p_{t-1}\}) - F(p_t)] - q_t[F(\min\{q_{t-i}, \dots, q_{t-1}\}) - F(q_t)] \Big| \\
&\quad = \Big| p_t[1 - F(p_t)] - p_t[1 - F(\min\{p_{t-i}, \dots, p_{t-1}\})] - q_t[1 - F(q_t)] + q_t[1 - F(\min\{q_{t-i}, \dots, q_{t-1}\})] \Big| \\
&\quad \le \Big| p_t[1 - F(p_t)] - q_t[1 - F(q_t)] \Big| + \Big| q_t[1 - F(\min\{q_{t-i}, \dots, q_{t-1}\})] - p_t[1 - F(\min\{p_{t-i}, \dots, p_{t-1}\})] \Big| \\
&\quad \le 2(\bar{P}A + 1) \sup_j \Big| p_j - q_j \Big|
\end{aligned}
$$

Therefore, taking $C = (1 + 2k\alpha)(\bar{P}A + 1)$ we have

$$\Big|H(p) - H(q)\Big| \leq \sup_t \Big\{(\bar{P}A + 1)\big|p_t - q_t\big|\Big\} + \alpha k \sup_t \Big\{2(\bar{P}A + 1)\sup_j \big|p_j - q_j\big|\Big\} \leq C \sup_t \big|p_t - q_t\big|.$$

□

Proof of Lemma G.2. It suffices to show $\sup_q\{v_L(q) : q \in D_0(L)\} \leq v_2(d^2)$, because $\widehat{D}_0(n, L) \subset D_0(L)$.

Consider $L \geq k + 1$ and $d = d^L \in \arg\max_{q \in D_0(L)} v_L(q)$. Then

$$\begin{aligned}
V_L(d) &= \sum_{i=1}^{L} \rho_i(d) \\
&= d_1[1 - G(d_1)] + \sum_{i=2}^{k} d_i\Big\{1 - G(d_i) + \alpha(i-1)[F(d_{i-1}) - F(d_i)]\Big\} \\
&\qquad + \sum_{i=k+1}^{L} d_i\Big\{1 - G(d_i) + \alpha k[F(d_{i-1}) - F(d_i)]\Big\} \\
&= \sum_{i=1}^{k} \Big\{d_i[1 - G(d_i)] + \alpha \sum_{j=i+1}^{L} d_j[F(d_{j-1}) - F(d_j)]\Big\} + \sum_{i=k+1}^{L} d_i[1 - G(d_i)] \\
&\leq k\bar{P} + (L - k)v_1(d^1).
\end{aligned}$$

The last inequality holds because each item inside the large curly braces is bounded above by $\bar{P}$ and because d^1 maximizes $v_1(x) = x[1 - G(x)]$ over $x \in [0, \bar{P}]$. It follows that

$$v_L(d^L) \leq v_1(d^1) + \frac{k(\bar{P} - v_1(d^1))}{L}. \tag{59}$$

Note that $G(\bar{P}) = 1$, so $v_1(\bar{P}) = 0$. Thus, $d^1 \in (0, \bar{P})$. For $x \in [d^1, \bar{P}]$, let

$$R(x) = v_2(x, d^1) - v_1(d^1) = \frac{1}{2}\Big\{x[1 - G(x)] + \alpha d^1[F(x) - F(d^1)] - d^1[1 - G(d^1)]\Big\}$$

Observe that $R(d^1) = 0$. Moreover, $R'(x) = \frac{1}{2}\{1 - G(x) - xg(x) + \alpha d^1 f(x)\}$ where $f(x) = F'(x)$ and $g(x) = G'(x)$, and $R'(x)$ is continuous. The price d^1 maximizes $x[1 - G(x)]$, and hence by the first order condition, we have $1 - G(x) - xg(x)|_{x=d^1} = 0$. Therefore, $R'(d^1) = \frac{1}{2}\alpha d^1 f(d^1) > 0$ where the inequality holds because $f(\cdot)$ is strictly positive. It follows that there must exist some $p \in (d^1, \bar{P}]$ such that $R(p) > 0$. Hence, for such a p we have $v_2(p, d^1) > v_1(d^1)$. Therefore, $v_2(d^2) > v_1(d^1)$ where $d^2 \in \arg\max\{v_2(q) : q \in D_0(2)\}$. It now follows by (59) that $v_L(d^L) \leq v_2(d^2)$ for all L sufficiently large. Hence, there exists some $c < \infty$ such that $v_L(d^L) = \sup_q\{v_L(q) : q \in D_0(L)\} \leq v_2(d^2)$ when $L > c$. □