

TECHNICAL NOTE

Stochastic Comparisons in Production
Yield Management

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Manufacturing firms routinely commit resources to increase yield rates through product- and process-improvement initiatives. Champions of such yield-improvement projects may assume that stochastically larger yield rates are beneficial. In this note, we show that this need not hold, even when the contingent production lot sizes are chosen optimally. We employ stochastic comparison techniques to show that a yield rate that is smaller in the convex order ensures higher expected profit, and we provide a distribution-free bound on the size of increase in expected profit. We also identify properties of yield-rate distributions that do make stochastically larger yield rates beneficial.

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1. Introduction

Consider a manufacturer that faces a random demand X , and that needs to decide how many blanks q to start to maximize its expected profit. Suppose that the process yield (i.e., the number of good units resulting from the q started) is given by $U \cdot q$, where U is a random variable called the yield rate. Let $c, p, h \geq 0$ denote, respectively, the per-item production, shortage, and holding costs, and let $r \geq 0$ be the per-item revenue. For a realization x of demand and u of yield rate, the manufacturer's profit is given by

$$\begin{aligned} \pi(q, u, x) &= r \min\{x, uq\} - cq - p(x - uq)^+ - h(uq - x)^+ \quad (1) \\ &= (r + h)x - q(hu + c) - (r + h + p)(x - uq)^+, \quad (2) \end{aligned}$$

where $(y)^+ = \max\{0, y\}$. To avoid cases when it is optimal to produce an "infinitely large" amount, we assume throughout this note that $E(U) > 0$ and $c + hE(U) > 0$. In the typical production environment $U \leq 1$, however, this is not a requirement of the model. The manufacturer seeks the production lot size q^* to maximize $E\pi(q, U, X)$.

In manufacturing industries, yield management refers to activities such as estimating parameters of the yield distribution, choosing optimal lot sizes, and determining appropriate recourse when yield is lower than expected. Yield management also encompasses efforts to alter yield rates via product- and process-improvement projects. Given that a firm has an opportunity to change its yield rate from U to \hat{U} , we investigate whether there exists an ordering of U and \hat{U} that will guarantee $\max_q E\pi(q, \hat{U}, X) \geq \max_q E\pi(q, U, X)$. Because many manufacturers assume

that a stochastic increase in yield rate is beneficial, we first present an example to show that their hypothesis is not always true. This example considers a discrete yield-rate distribution. Similar examples exist for continuous distributions.

For the example, we simplify the problem by assuming that demand is a constant x , and that the random yield rate U can assume finitely many values (say $\alpha_1, \dots, \alpha_n$) with $P(U = \alpha_i) = \eta_i$ for $i = 1, \dots, n$. In this case, the objective function can be expressed as

$$\begin{aligned} E\pi(q, U, x) &= (r + h)x - cq - hq \sum_{i=1}^n \alpha_i \eta_i \\ &\quad - (r + p + h) \sum_{i=1}^n (x - \alpha_i q)^+ \eta_i. \quad (3) \end{aligned}$$

Observe that (3) is piecewise linear in q with "kinks" at the points $\{x/\alpha_i\}$. Therefore,

$$\max_{q \geq 0} E\pi(q, U, x) = \max_{i=0,1,\dots,n} E\pi(q_i, U, x),$$

where $q_0 = 0$ and $q_i = x/\alpha_i$ for $i = 1, \dots, n$.

Consider the problem instance with $x = 100$, $r = \$10$, $p = \$6$, $c = \$2$, $h = \$0.5$, and $P(U = 0.6) = 0.4$, $P(U = 0.7) = 0.1$, and $P(U = 0.8) = 0.5$. Carrying out the simple calculations, it follows that the optimal lot size is $q^* = 100/0.6 \approx 167$, and the resulting optimal expected profit is \$657.50.

Next, suppose that the manufacturer manages, in some cases, to improve the realized yield when the process performance is worst, while leaving yield unaffected when performance is best. In particular, suppose that the new yield

rate \hat{U} has mass function $P(\hat{U} = 0.6) = P(\hat{U} = 0.7) = 0.25$ and $P(\hat{U} = 0.8) = 0.5$. Observe that $E(\hat{U}) = 0.725$ and $\text{Var}(\hat{U}) = 0.006875$. In comparison, $E(U) = 0.71$ and $\text{Var}(U) = 0.0089$. The new yield rate \hat{U} has higher mean and lower variance than does U , and moreover, it is straightforward to check that \hat{U} is stochastically larger than U (see Shaked and Shanthikumar 1994). In summary, the new yield rate “seems better.”

Direct calculations reveal that for yield rate \hat{U} , the optimal lot size is again $\hat{q}^* = 100/0.6 \approx 167$, which gives an optimal expected profit of \$656.25. Hence, the better yield-rate distribution gives lower optimal expected profit. Why does the manufacturer’s profit decline? Owing to the relationship between yield-rate distribution and the optimal production lot size, the improvement in yield-rate distribution does not affect the lot size. This leads to greater holding cost, because the number of unused items is stochastically larger. The problem goes away if there is no holding cost; i.e., $h = 0$.

The study of random yield problems has resulted in a substantial literature, a majority of which is reviewed by Yano and Lee (1995). Work that has appeared since 1995 is discussed in Bollapragada and Morton (1999). Most research treats yield rate as an exogenous parameter and focuses on the determination of optimal lot sizes. The stochastically proportional yield model, which we also utilize in this note, is the most widely-studied model; see Shih (1980), Gerchak et al. (1986), and Henig and Gerchak (1990). In another class of models, yield is specified directly, e.g., as a binomial random variable. The only study that considers the effect of stochastically ordered yield distributions is Gerchak and Henig (1994). It shows that when yield is binomial, stochastically larger yield results in greater expected profits.

There are few models in the literature that consider simultaneous yield rate and lot-size decisions. Gerchak and Parlar (1990) study the problem of jointly determining yield variability and lot sizes within the framework of a continuous-time model with known constant demand (EOQ framework). They assume that yield variability can be affected through investments. Several papers simultaneously consider quality control (which affects yield) and inventory policy; see Lee and Rosenblatt (1987) and Porteus and Argelus (1997) for details. Another context arises when there are multiple suppliers of the same component, each with a different yield distribution and per-unit price. Here, the effective yield observed by the buyer is determined by which suppliers are chosen and how much is ordered from each. Typically, the buyer benefits from diversification so long as per-unit prices are commensurate with yield distributions. A number of authors have studied this problem, including Anupindi and Akella (1993), Gerchak and Parlar (1990), and Henig and Levin (1990).

Outside the production yield management area, we find articles that show that stochastically improving a key input to an operational model need not produce an improvement

in the desired outcome. For example, Gerchak and Mossman (1992) show that risk pooling, which occurs when several independent random demands are aggregated into one, can increase inventories or move order quantities further from mean or median demand in a single-echelon inventory system. Song (1994) shows that stochastically larger lead times may be associated with smaller long-run average cost. Ridder et al. (1998) present examples to show that larger demand variability can lead to lower costs in a newsvendor framework. Gerchak and Golany (2000) study the trade-off between cost and productivity in formulating a hiring policy and show that increasing variability of the completed length-of-service distribution is beneficial when its mean is held constant. Cooper and Gupta (2004) show that stochastically larger demand in revenue management problems can lead to lower expected revenue.

The remainder of this note is organized as follows. Section 2 treats the single-period model in which the manufacturer faces a one-time demand. Section 3 considers its multiperiod analog in which demand and yield rates in different periods are assumed to be independent. Section 4 allows arbitrary dependence among the yield rates in different periods. Section 5 presents distribution-free upper and lower bounds on the expected profit, some numerical examples, and insights.

2. One-Period Model

We begin this section by reviewing some definitions from the theory of stochastic comparisons; see Shaked and Shanthikumar (1994) or Müller and Stoyan (2002) for extensive background. If A and B are random variables and $Ef(A) \leq Ef(B)$ for all nondecreasing functions f for which the expectations exist, then A is stochastically smaller than B ; written as $A \leq_{st} B$. An equivalent intuitive condition is that $P(A > x) \leq P(B > x)$ for all x . If $A \leq_{st} B$, then it immediately follows that $E(A) \leq E(B)$. If, instead, $Ef(A) \leq Ef(B)$ for all convex functions f for which the expectations exist, then A is smaller than B according to the convex order; written as $A \leq_{cx} B$. If the distribution functions of A and B are, respectively, $H_A(\cdot)$ and $H_B(\cdot)$, we shall also sometimes use the alternative notation $H_A \leq_{cx} H_B$ in place of $A \leq_{cx} B$. It can be shown that if $A \leq_{cx} B$, then $E(A) = E(B)$ and $\text{Var}(A) \leq \text{Var}(B)$.

Let F and G be the distribution functions of demand X and yield rate U , respectively. (Throughout this note, all random variables under consideration are assumed to be nonnegative and have finite means. Furthermore, we assume that yield rate and demand are independent.) We will be concerned with what happens when the yield rate is altered to \hat{U} with distribution function \hat{G} . Let $\pi^* = \max_{q \geq 0} E\pi(q, U, X)$ and $\hat{\pi}^* = \max_{q \geq 0} E\pi(q, \hat{U}, X)$. As shown in the previous section, it is possible to have $U \leq_{st} \hat{U}$ and $\pi^* > \hat{\pi}^*$; i.e., stochastically larger yield may have lower optimal expected profit.

Is there a relationship between U and \hat{U} that ensures an ordering between π^* and $\hat{\pi}^*$? From (2), it follows that

$\pi(q, u, x)$ is a concave function of u for each q and x . By conditioning on X , we see that if $\hat{U} \leq_{cx} U$, then $E\pi(q, \hat{U}, X) \geq E\pi(q, U, X)$ for all $q \geq 0$, and hence we have the following result.

PROPOSITION 1. *If $\hat{U} \leq_{cx} U$, then $\hat{\pi}^* \geq \pi^*$.*

On an intuitive level, Proposition 1 implies that a less-variable yield rate always gives higher expected profit. Expressions for the optimal cost and order quantity in EOQ-based random yield models (see Gerchak and Parlar 1990) show that a yield rate with a fixed (or larger) mean and lower variance ensures higher expected profit and a higher order quantity in the EOQ environment. In our setting, however, we need to have an ordering of yield rate distributions according to the stronger notion of variability captured by the convex order. As we saw in the example in the introduction, higher mean and lower variance are not sufficient to guarantee higher expected profits. Moreover, we will demonstrate in §5 that a less-variable yield rate need not correspond to a higher optimal order quantity.

For many common distributions, it is possible to identify an ordering of their parameters that is equivalent to a convex order. Table 1.1 of Müller and Stoyan (2002) provides examples including the uniform, gamma, Weibull, lognormal, and beta distributions. Within each of these families, if the parameters of two distributions are such that the two distributions have the same mean but one has a lower variance, then the one with the smaller variance is smaller in the convex order. Alternatively, if U is the “original” yield rate, and the new yield rate is $U(\alpha) = \alpha U + (1 - \alpha)E(U)$ for some $\alpha \in [0, 1]$, then $U(\alpha) \leq_{cx} U(1) \equiv U$, and $U(\alpha_1) \leq_{cx} U(\alpha_2)$ if $\alpha_1 \leq \alpha_2$. The new random variable $U(\alpha)$ is known as the mean-preserving transform of the original yield rate. Such constructs have been used in stylized models for generating insights (for example, see Gerchak and Mossman 1992 and references therein). Using this approach, it is possible to express investment costs and subsequent benefits as a function of the parameter α , and to carry out a cost-benefit analysis.

Although a stochastically larger yield rate does not, in general, increase optimal expected profit, there are additional conditions and special problem instances in which stochastically increasing the yield rate is desirable. We discuss some such cases in the remainder of this section. When yield rates are continuously distributed, Shih (1980) and Gerchak et al. (1986) have identified the equivalent of the following optimality condition:

$$\int_0^\infty \int_0^{x/q^*} u dG(u) dF(x) = \frac{hE(U) + c}{r + h + p}, \quad (4)$$

under the additional assumption, $(r + p)E(U) > c$. Let U and \hat{U} be two continuous yield-rate distributions such that $U \leq_{st} \hat{U}$. After writing out $E\pi(q^*, U, X)$ and $E\pi(\hat{q}^*, \hat{U}, X)$ as integrals, plugging (4) into these expressions, and going through several lines of simplifications,

we find that

$$\hat{\pi}^* - \pi^* = (r + h + p) \int_0^\infty x [G(x/q^*) - \hat{G}(x/\hat{q}^*)] dF(x). \quad (5)$$

Because $U \leq_{st} \hat{U}$ implies that $G(\cdot) \geq \hat{G}(\cdot)$, it follows that $G(x/q^*) \geq \hat{G}(x/q^*)$. If, moreover, $\hat{q}^* \geq q^*$, then $\hat{G}(x/q^*) \geq \hat{G}(x/\hat{q}^*)$ and the quantity in square brackets in (5) is nonnegative, making the entire expression nonnegative. Put differently, $U \leq_{st} \hat{U}$ and $\hat{q}^* \geq q^*$ are sufficient (though not necessary) conditions for profit improvement. However, we cannot a priori verify the sign of the expression on the right side of (5) without solving the problems with yield rates U and \hat{U} to obtain q^* and \hat{q}^* . Therefore, we next consider cases in which yield rates belong to specific distributional families.

PROPOSITION 2. *Suppose that for each $q \geq 0$, there exists \hat{q} so that $\hat{q}\hat{U}$ and qU have the same distribution and $\hat{q} \leq q$. Then, $\hat{\pi}^* \geq \pi^*$.*

PROOF. Fix $q \geq 0$, and let \hat{q} be as in the statement of the proposition. By (1) we have $E\pi(q, U, X) - E\pi(\hat{q}, \hat{U}, X) = c(\hat{q} - q) \leq 0$. So, $E\pi(q, U, X) \leq E\pi(\hat{q}, \hat{U}, X)$ and the result follows because q is arbitrary. \square

Proposition 2 states that if after changing from U to \hat{U} , it is possible to obtain the same yield distribution with a smaller lot size, then expected profit rises. This result is not only intuitive and easy to prove, it also allows us to compare yield-rate distributions within several common parametric families. The next paragraph contains some examples. In each case $U \leq_{st} \hat{U}$, $\text{Var}(U) \leq \text{Var}(\hat{U})$, and $\hat{\pi}^* \geq \pi^*$. That is, Proposition 2 is used in each instance to identify cases in which a larger and more variable yield-rate distribution is desirable (note the contrast to Proposition 1).

If $U \sim \text{Uniform}[0, \beta]$ and $\hat{U} \sim \text{Uniform}[0, \hat{\beta}]$ where $\beta \leq \hat{\beta}$, then take $\hat{q} = q\beta/\hat{\beta}$. If $U \sim \text{Weibull}(\alpha, \lambda)$ with density $g(u) = \lambda \alpha u^{\alpha-1} \exp(-\lambda u^\alpha)$ and $\hat{U} \sim \text{Weibull}(\alpha, \hat{\lambda})$ where $\lambda \geq \hat{\lambda}$, then take $\hat{q} = (\hat{\lambda}/\lambda)^{(1/\alpha)} q$. If $P(U = \beta) = \delta = 1 - P(U = 0)$ and $P(\hat{U} = \hat{\beta}) = \delta = 1 - P(\hat{U} = 0)$ where $\beta \leq \hat{\beta}$, then take $\hat{q} = q\beta/\hat{\beta}$. If $U \sim \text{Gamma}(\alpha, \lambda)$ with density $g(u) = \lambda^\alpha u^{\alpha-1} \exp(-\lambda u)/\Gamma(\alpha)$ and $\hat{U} \sim \text{Gamma}(\hat{\alpha}, \hat{\lambda})$ where $\alpha \leq \hat{\alpha}$ and $\lambda \geq \hat{\lambda}$, then we need a two-step argument. First, consider $\tilde{U} \sim \text{Gamma}(\hat{\alpha}, \hat{\lambda})$ where $\tilde{\lambda} = \hat{\alpha}\hat{\lambda}/\alpha$. In view of the fact that $\tilde{U} \leq_{cx} U$, we have $\tilde{\pi}^* \geq \pi^*$ where $\tilde{\pi}^*$ is the optimal objective function value for yield rate \tilde{U} . We next apply the proposition to compare optimal expected revenues for \tilde{U} and \hat{U} . For order quantity \tilde{q} , take $\hat{q} = \tilde{q}\hat{\lambda}/\tilde{\lambda}$. The proposition implies $\hat{\pi}^* \geq \tilde{\pi}^*$, and therefore $\hat{\pi}^* \geq \pi^*$.

Under the additional assumption of deterministic demand, the following proposition shows that if the yield-rate distribution is translated to the right, then the optimal expected profit increases. Hence, it provides another set of instances in which stochastically larger yield is desirable.

PROPOSITION 3. *For constant demand x , if $\hat{U} = U + \beta$, where $\beta \geq 0$ is a constant, then $\hat{\pi}^* \geq \pi^*$.*

PROOF. It suffices to show that there exists \hat{q} such that $E\pi(\hat{q}, \hat{U}, x) \geq E\pi(q^*, U, x)$. If $q^* = 0$, we can set $\hat{q} = 0$. Otherwise, from (2) and upon choosing \hat{q} such that $x/\hat{q} - \beta = x/q^*$, we obtain the following sequence of equalities:

$$\begin{aligned} & E\pi(\hat{q}, \hat{U}, x) - E\pi(q^*, U, x) \\ &= (c + hEU)(q^* - \hat{q}) - \beta h\hat{q} + (r + h + p) \\ &\quad \cdot [E(\min\{x, (U + \beta)\hat{q}\}) - E(\min\{x, Uq^*\})] \\ &= (c + hEU)(q^* - \hat{q}) - \beta h\hat{q} + (r + h + p) \\ &\quad \cdot \left[\beta\hat{q} + \hat{q}E\left(\min\left\{\frac{x}{\hat{q}} - \beta, U\right\}\right) - q^*E\left(\min\left\{\frac{x}{q^*}, U\right\}\right) \right] \\ &= (r + p)\beta\hat{q} - (q^* - \hat{q}) \\ &\quad \cdot \left[(r + p + h)E\left(\min\left\{\frac{x}{q^*}, U\right\}\right) - (c + hEU) \right] \\ &= \frac{(q^* - \hat{q})}{q^*} [rx - (r + h + p)E(\min\{x, Uq^*\}) \\ &\quad + q^*(c + hEU) + px] \\ &= \frac{(q^* - \hat{q})}{q^*} [rx - E\pi(q^*, U, x)] \geq 0. \end{aligned} \tag{6}$$

To obtain the second equality above, we have used the facts that $\min\{a, b + c\} = c + \min\{a - c, b\}$ and that for all $b > 0$, $\min\{ab, c\} = b \min\{a, c/b\}$. Similarly, the third and the fourth equalities follow from the relationship $x/\hat{q} - \beta = x/q^*$ and rearranging the resulting terms. The nonnegativity of the expression in (6) follows from the facts that $rx \geq \pi(q, u, x)$ for all $q, u, x \geq 0$ and $\hat{q} = (x/(x + \beta q^*)) \cdot q^* \leq q^*$. \square

An immediate corollary of the above proposition is that if the yield-rate distribution with the higher mean is smaller in the convex order than another yield-rate distribution after subtracting a constant from the former distribution to make the two means equal, then the former yield-rate distribution is guaranteed to yield higher expected profits. Hence, the following provides conditions under which a yield rate with higher mean and lower variance is desirable.

COROLLARY 1. For constant demand x , if there exists a U' such that $\hat{U} = U' + \beta$, where $\beta \geq 0$ is a constant, and $U' \leq_{cx} U$, then $\hat{\pi}^* \geq \pi^*$.

3. Multiperiod Model: Independent Yield Rates and Demands

We now generalize the model presented in §2 to a multiperiod setting. We consider a Markov-decision-process (MDP) formulation of the production yield management problem, which is similar to that of Henig and Gerchak (1990). However, unlike Henig and Gerchak, we carry out stochastic comparisons of the expected profit function under different yield-rate distributions.

We use notation from §2 with an additional subscript t to denote time. For a τ planning-period problem, random demand and yield are represented by $\mathbf{X} = (X_1, \dots, X_\tau)$ and $\mathbf{U} = (U_1, \dots, U_\tau)$. We assume that $\{X_t\}$ and $\{U_t\}$ are sequences of independent random variables, and that \mathbf{X} and \mathbf{U} are independent of each other. The term S_t is used to denote the inventory level at the beginning of period t (also called “state”). All demand that cannot be met is back-ordered. Thus, while s_1 is a parameter of the problem, $S_t = (U_{t-1}q_{t-1} + S_{t-1} - X_{t-1})$ is a function of the realizations u_{t-1} , s_{t-1} , and x_{t-1} of the corresponding random variables when $t \geq 2$. Note that when s_t is negative, it denotes back-ordered demand that is carried forward from period $t - 1$.

Analogous to §2, we use $\pi_t(s, q, u, x)$ to denote manufacturer’s profit in period t when state is s , order quantity is q , yield rate is u , and demand is x . In view of the independence assumptions, the conditional distribution of $\pi_t(S_t, q, U_t, X_t)$ depends on the history of the process up to time t only through S_t ; i.e., start-of-period inventory level is sufficient to describe the state of the process. The one-period profit function can be written as follows:

$$\begin{aligned} \pi_t(s, q, u, x) &= r_t \min\{x, uq + s\} - c_t q - p_t(x - uq - s)^+ \\ &\quad - h_t(uq + s - x)^+ \\ &= (r_t + h_t)x - q(h_t u + c_t) - h_t s \\ &\quad - (r_t + h_t + p_t)(x - uq - s)^+. \end{aligned}$$

As before, we assume that $c_t, p_t, h_t, r_t \geq 0$. With this assumption in hand, we can immediately confirm that $w_{t,s,q,x}^0(u) = \pi_t(s, q, u, x)$ is a concave function of u .

Let $v_t(\cdot)$ denote the value function of the MDP; informally, $v_t(s)$ represents the maximum expected revenue over periods $t, \dots, \tau + 1$ provided that the state just prior to period t is s . The value function can be iteratively determined via the optimality equations;

$$v_t(s) = \sup_{q \geq 0} \int_x \int_u [\pi_t(s, q, u, x) + \gamma v_{t+1}(uq + s - x)] \cdot dG_t(u) dF_t(x),$$

where G_t and F_t are the distributions of the yield rate and demand in period t , and $\gamma \in (0, 1]$ is the per-period discount rate. We assume that there is a concave function $v_{\tau+1}(s)$ that represents the salvage value of s units of net inventory at time $\tau + 1$. We also need additional mild restrictions on $v_{\tau+1}$ to avoid the possibility of infinite expected profit. For example, if $v_{\tau+1}(s) = \theta s$, it suffices to require that $E(U_t)\gamma^{\tau+1-t}\theta < c_t + E(U_t) \sum_{j=t}^{\tau} \gamma^{j-t} h_j$ for $t = 1, 2, \dots, \tau$. When the period- t order quantity is “very large,” it is unlikely that an additional marginal unit ordered in period t will ever be consumed. Hence, for high order quantities, the right side of the inequality is the total expected cost incurred by a marginal unit ordered in period t , and the left side is the expected benefit from that unit. Therefore, this condition ensures that the firm cannot

reap ever larger net benefits from ever larger orders. Similar conditions can be found for nonlinear concave $v_{\tau+1}$. We omit the details.

The main result of this section states that if for each $t = 1, \dots, \tau$, G_t is the distribution of the original yield rate and that through process-improvement efforts, we replace it by \widehat{G}_t (keeping other parameters unchanged), where \widehat{G}_t is smaller in the convex order than G_t , then the expected profits increase. Below, we let $\widehat{v}_t(s)$ denote the value function of the MDP with yield-rate distributions $\{\widehat{G}_t\}$.

PROPOSITION 4. *If $\widehat{G}_t \leq_{cx} G_t$ for all t , then $\widehat{v}_t(s) \geq v_t(s)$ for all t and s .*

PROOF. The proof is by (backwards) induction on t . The statement is true for $t = \tau + 1$.

A slight modification of Theorem 4 of Henig and Gerchak (1990) shows that $v_t(s)$ is concave in s . Therefore, the functions $w_{t,s,q,x}(u) = \gamma v_t(s + uq - x)$ and $\widehat{w}_{t,s,q,x}(u) = \gamma \widehat{v}_t(s + uq - x)$ are also concave in u . In view of the concavity of w^0 , we see that $w_{t,s,q,x}^0(u) + \widehat{w}_{t,s,q,x}(u)$ is concave, as well. Therefore, $\widehat{G}_t \leq_{cx} G_t$ implies that

$$\int_u [w_{t,s,q,x}^0(u) + \widehat{w}_{t,s,q,x}(u)] dG_t(u) \leq \int_u [w_{t,s,q,x}^0(u) + \widehat{w}_{t,s,q,x}(u)] d\widehat{G}_t(u). \quad (7)$$

For the inductive hypothesis, suppose that $v_{t+1}(s) \leq \widehat{v}_{t+1}(s)$ for all s . From the definitions above, this implies that $w_{t+1,s,q,x}(u) \leq \widehat{w}_{t+1,s,q,x}(u)$. With this in hand, we have

$$v_t(s) = \sup_{q \geq 0} \int_x \int_u [w_{t,s,q,x}^0(u) + w_{t+1,s,q,x}(u)] dG_t(u) dF_t(x) \leq \sup_{q \geq 0} \int_x \int_u [w_{t,s,q,x}^0(u) + \widehat{w}_{t+1,s,q,x}(u)] dG_t(u) dF_t(x) \quad (8)$$

$$\leq \sup_{q \geq 0} \int_x \int_u [w_{t,s,q,x}^0(u) + \widehat{w}_{t+1,s,q,x}(u)] d\widehat{G}_t(u) dF_t(x) \quad (9)$$

$$= \widehat{v}_t(s).$$

Above, inequality (8) follows from the inductive hypothesis, and inequality (9) follows from (7). This completes the proof. \square

An alternate proof can be obtained by checking the conditions in Theorem 5.2.28 of Müller and Stoyan (2002). Finally, if for each $q \geq 0$ and t there exists \widehat{q}_t so that $\widehat{q}_t \widehat{U}_t$ and qU_t have the same distribution and $\widehat{q}_t \leq q$, then arguing by induction, it is possible to develop a multiperiod analog of Proposition 2. We omit the details.

4. Multiperiod Model: Correlated Yield Rates

Production yield in different planning periods is often dependent through time. In this section, we therefore consider yield rates that are arbitrarily correlated. This generalization makes analysis more difficult because the MDP

framework of §3 is no longer applicable. Therefore, we are able to prove only a more restrictive result. We identify a stochastic ordering of yield rates that results in greater expected profit only in the situation when q_t is chosen independently of the realized values of demand and yield rates up to time t . This is reasonable when production quantities must be fixed (committed to) ahead of time. Note that this approach is different from §3, where period- t production quantities are chosen after demand and yield uncertainties in periods $1, \dots, t - 1$ are resolved.

Let $\pi_t(\mathbf{q}, \mathbf{u}, \mathbf{x})$ denote the period- t profit function when \mathbf{u} and \mathbf{x} are the vectors of realized values of \mathbf{U} and \mathbf{X} , and $\mathbf{q} = \{q_1, \dots, q_\tau\}$ is the vector of order quantities. Note that s_t is not an argument of the profit function, because all lot sizes are fixed at $t = 1$. Of course, the initial inventory s_1 is taken into account when selecting \mathbf{q} . Moreover, because we are now allowing correlated yield rates, the conditional distribution of the period- t profit depends on s_1 and on the entire history of the process up to and including time t . Analogous to Equation (1), $\pi_t(\mathbf{q}, \mathbf{u}, \mathbf{x}) = r_t \min\{x_t, u_t q_t + s_t\} - c_t q_t - p_t(x_t - u_t q_t - s_t)^+ - h_t(u_t q_t + s_t - x_t)^+$, which simplifies as described below:

$$\pi_t(\mathbf{q}, \mathbf{u}, \mathbf{x}) = \pi_t^1(\mathbf{q}, \mathbf{u}, \mathbf{x}) + \pi_t^2(\mathbf{q}, \mathbf{u}, \mathbf{x}) + \pi_t^3(\mathbf{q}, \mathbf{u}, \mathbf{x}), \quad (10)$$

where $\pi_t^1(\mathbf{q}, \mathbf{u}, \mathbf{x}) = -p_t x_t + ((r_t + p_t)u_t - c_t)q_t$, $\pi_t^2(\mathbf{q}, \mathbf{u}, \mathbf{x}) = (r_t + p_t)[\sum_{i=1}^{t-1}(u_i q_i - x_i) + s_1]$, and $\pi_t^3(\mathbf{q}, \mathbf{u}, \mathbf{x}) = -(r_t + p_t + h_t)[\sum_{i=1}^t(u_i q_i - x_i) + s_1]^+$ for all $1 \leq t \leq \tau$. In defining $\pi_t^2(\mathbf{q}, \mathbf{u}, \mathbf{x})$, we have used the convention that an empty sum equals zero.

For each fixed \mathbf{q} , yield vector \mathbf{u} , and demand \mathbf{x} , the τ planning-period profit function can be written as

$$\pi(\mathbf{q}, \mathbf{u}, \mathbf{x}) = \sum_{t=1}^{\tau} \gamma^{t-1} \pi_t(\mathbf{q}, \mathbf{u}, \mathbf{x}). \quad (11)$$

The manufacturer's problem is to choose a vector \mathbf{q} to maximize $E\pi(\mathbf{q}, \mathbf{U}, \mathbf{X})$, and to evaluate the impact of changing \mathbf{U} on $\sup_{\mathbf{q}} \{E\pi(\mathbf{q}, \mathbf{U}, \mathbf{X})\}$. Note that in Equation (11), the value/cost of any terminal inventory/shortage is not explicitly shown. Instead, end-of-planning-horizon salvage value, which is assumed linear in the leftover amount, is accounted for by adjusting h_τ . Similarly, any shortages in period τ are satisfied by a (possibly outsourced) special order with instantaneous delivery and perfect yield, and the period- τ shortage cost, p_τ , includes any additional cost/revenue resulting from this special order. Furthermore, as before, r_t , p_t , c_t , and h_t are assumed to be nonnegative for each $t = 1, \dots, \tau$.

In what follows, we require the concept of the positive-linear-combination convex order. For τ -dimensional random vectors \mathbf{V} and $\widehat{\mathbf{V}}$, we say that $\widehat{\mathbf{V}} \leq_{plcx} \mathbf{V}$ if $\mathbf{a} \cdot \widehat{\mathbf{V}} \leq_{cx} \mathbf{a} \cdot \mathbf{V}$ for all $\mathbf{a} = (a_1, \dots, a_\tau)$ such that $a_i \geq 0$ for $i = 1, \dots, \tau$, where $\mathbf{a} \cdot \mathbf{V} = \sum_{i=1}^{\tau} a_i V_i$ (see Scarsini 1998, Müller and Stoyan 2002, and Corbett and Rajaram 2002 for more on the *plcx* order). Because the composition of a convex function with

a linear function is a convex function, it follows that if vectors are comparable in the convex order, then they are also comparable in the positive-linear-combination convex order.

To link our development with the positive-linear-combination convex order, we will show how to express the profit function in terms of compositions of concave functions and positive linear combinations. Consider the functions

$$\xi_{t,x,q}^1(z) = -p_t x_t + z - c_t q_t, \tag{12}$$

$$\xi_{t,x,q}^2(z) = (r_t + p_t) \left[z - \sum_{i=1}^{t-1} x_i + s_1 \right], \tag{13}$$

$$\xi_{t,x,q}^3(z) = -(r_t + p_t + h_t) \left[z - \sum_{i=1}^t x_i + s_1 \right]^+. \tag{14}$$

Observe for each \mathbf{x}, \mathbf{q} , and t that (12)–(14) are concave functions (from \mathcal{R} to \mathcal{R}) of z . We shall also use the following τ -dimensional vectors. Let $\mathbf{a}^{1,t,q}$ be the vector with all zeros, except with $(r_t + p_t)q_t$ in the t th position. Let $\mathbf{a}^{2,t,q} = (q_1, \dots, q_{t-1}, 0, \dots, 0)$ and $\mathbf{a}^{3,t,q} = (q_1, \dots, q_t, 0, \dots, 0)$. With these definitions in hand, we see that

$$\pi_t^i(\mathbf{q}, \mathbf{u}, \mathbf{x}) = \xi_{t,x,q}^i(\mathbf{a}^{i,t,q} \cdot \mathbf{u}). \tag{15}$$

Using the independence of \mathbf{X} and \mathbf{U} , we now have from (10)–(15) that

$$\begin{aligned} E[\pi(\mathbf{q}, \mathbf{U}, \mathbf{X}) \mid \mathbf{X} = \mathbf{x}] &= \sum_{t=1}^{\tau} \gamma^{t-1} [E \xi_{t,x,q}^1(\mathbf{a}^{1,t,q} \cdot \mathbf{U}) + E \xi_{t,x,q}^2(\mathbf{a}^{2,t,q} \cdot \mathbf{U}) \\ &\quad + E \xi_{t,x,q}^3(\mathbf{a}^{3,t,q} \cdot \mathbf{U})]. \end{aligned}$$

If $\hat{\mathbf{U}} \leq_{plcx} \mathbf{U}$, then $E \xi_{t,x,q}^i(\mathbf{a}^{i,t,q} \cdot \hat{\mathbf{U}}) \geq E \xi_{t,x,q}^i(\mathbf{a}^{i,t,q} \cdot \mathbf{U})$ for each i . Therefore, $E[\pi(\mathbf{q}, \hat{\mathbf{U}}, \mathbf{X}) \mid \mathbf{X} = \mathbf{x}] \geq E[\pi(\mathbf{q}, \mathbf{U}, \mathbf{X}) \mid \mathbf{X} = \mathbf{x}]$, and integrating over \mathbf{x} gives the following result.

PROPOSITION 5. *If \mathbf{U} and $\hat{\mathbf{U}}$ are two yield vectors such that $\hat{\mathbf{U}} \leq_{plcx} \mathbf{U}$, then $E\pi(\mathbf{q}, \hat{\mathbf{U}}, \mathbf{X}) \geq E\pi(\mathbf{q}, \mathbf{U}, \mathbf{X})$ for each fixed \mathbf{q} . In particular, this means that $\sup_{\mathbf{q}} \{E\pi(\mathbf{q}, \hat{\mathbf{U}}, \mathbf{X})\} \geq \sup_{\mathbf{q}} \{E\pi(\mathbf{q}, \mathbf{U}, \mathbf{X})\}$.*

Note that the above parallels Proposition 1 of §2. (In one dimension, \leq_{cx} and \leq_{plcx} are the same.) Proposition 5 holds regardless of the dependence structure of \mathbf{X} , and of \mathbf{U} , although \mathbf{X} must be independent of \mathbf{U} .

5. Insights and Examples

In this section, we provide distribution-free bounds on the expected profit in a single-period problem. These bounds can help a company evaluate the maximum benefit possible from yield improvement efforts, even when it knows only the mean yield rate and not the entire yield-rate distribution. We also present examples that show the net change in expected profits and optimal lot size as a function of a parameter of the yield-rate distribution.

Suppose that the manufacturer with a one-period demand would like to know the magnitude of gain in expected profits that can come from convex-ordered yield-rate improvements. To this end, let \mathbb{M}_μ denote the collection of distribution functions G for which $\int u dG(u) = \mu$. In addition, let $\mathbb{M}_\mu^{[0,1]}$ be those distributions in \mathbb{M}_μ for which $G(u) = 0$ for $u < 0$, and $G(1) = 1$. Let $\underline{G}_\mu(\cdot)$ be the point mass at μ ; that is, $\underline{G}_\mu(u) = 0$ if $u < \mu$, and $\underline{G}_\mu(u) = 1$ otherwise. Similarly, let \bar{G}_μ be the distribution that assigns probability $1 - \mu$ to the value 0, and probability μ to the value 1. As described in Example 1.10.5 of Müller and Stoyan (2002), \underline{G}_μ is the minimal element of \mathbb{M}_μ with respect to the convex order, and likewise, \bar{G}_μ is the maximal element of $\mathbb{M}_\mu^{[0,1]}$ with respect to the convex order. This means that if $G \in \mathbb{M}_\mu$, then $\underline{G}_\mu \leq_{cx} G$. Similarly, if $G \in \mathbb{M}_\mu^{[0,1]}$, then $G \leq_{cx} \bar{G}_\mu$. To see this directly, suppose that $A \sim G \in \mathbb{M}_\mu^{[0,1]}$, $B \sim \underline{G}_\mu$, and $\bar{B} \sim \bar{G}_\mu$. Then, for any convex function $f(\cdot)$, we have that $Ef(B) = f(\mu) \leq Ef(A) = Ef(A \cdot 1 + (1 - A) \cdot 0) \leq E[Af(1) + (1 - A)f(0)] = \mu f(1) + (1 - \mu)f(0) = Ef(\bar{B})$. The first inequality is Jensen’s inequality; the second follows from the convexity of f and the fact that $P(A \in [0, 1]) = 1$.

Consider functions $\pi_U(q, \mu)$ and $\pi_L(q, \mu)$, as defined below:

$$\begin{aligned} \pi_U(q, \mu) &= (r + h)E(X) - q(h\mu + c) \\ &\quad - (r + h + p)E[(X - \mu q)^+] \quad \text{and} \\ \pi_L(q, \mu) &= (r + h)E(X) - q(h\mu + c) \\ &\quad - (r + h + p)E[(1 - \mu)X + \mu(X - q)^+]. \end{aligned}$$

(The subscript U for “upper bound” should not be confused with yield rate.) For the ensuing discussion, suppose that we have an initial yield rate $U \sim G \in \mathbb{M}_\mu^{[0,1]}$. Then, $\pi_U(q, \mu)$ is the expected profit for a manufacturer whose yield-rate distribution changes from G to \underline{G}_μ . This is the highest possible expected profit among all yield-rate distributions that have a mean yield rate of μ , given that the manufacturer uses a production lot of size q . Similarly, $\pi_L(q, \mu)$, which corresponds to \bar{G}_μ , is a lower bound on expected profit. Put differently, $\pi_L(q, \mu) \leq E\pi(q, U, X) \leq \pi_U(q, \mu)$.

Let $\pi_L(\mu) = \max_q \pi_L(q, \mu)$ and $\pi_U(\mu) = \max_q \pi_U(q, \mu)$. Similarly, let q_L^* and q_U^* denote the optimal lot sizes corresponding to the lower-bounding and the upper-bounding expected-revenue functions, respectively. That is, $q_L^* = \arg\{\max_q \pi_L(q, \mu)\}$ and $q_U^* = \arg\{\max_q \pi_U(q, \mu)\}$. It therefore follows that

$$\begin{aligned} \pi_L(q, \mu) &\leq \pi_L(\mu) \leq E\pi(q_L^*, U, X) \leq E\pi(q^*, U, X) \quad \text{and} \\ E\pi(q, U, X) &\leq E\pi(q^*, U, X) \leq \pi_U(q^*, \mu) \leq \pi_U(\mu). \end{aligned}$$

Putting the above inequalities together, we see that $\pi_L(\mu) \leq E\pi(q^*, U, X) \leq \pi_U(\mu)$. Note that the bounds $\pi_L(\mu)$ and $\pi_U(\mu)$ are independent of yield-rate distribution and only require the mean yield rate μ as input data. Furthermore, for continuous demand distributions, it is easy to verify that $q_L^* = F^{-1}(\rho)$ and $q_U^* = \mu^{-1}F^{-1}(\rho)$, where $\rho = (r - c)/\mu +$

$p)/(r + h + p)$ is the ratio of the per-unit shortage cost to the sum of shortage and overage cost. Because $G \in \mathbb{N}_\mu^{[0,1]}$ implies $\mu \leq 1$, it immediately follows that $q_L^* \leq q_U^*$. The maximum increase in expected profits resulting from yield improvement efforts, when mean yield rate remains fixed at μ , is $\pi_U(\mu) - \pi_L(\mu)$. This bound can be significantly improved if we know the current yield-rate distribution by using $E\pi(q^*, U, X)$ instead of $\pi_L(\mu)$.

We have carried out some numerical experiments to illustrate the effect of changing yield-rate variability. In these experiments, four different uniform yield-rate distributions are considered, and each is subjected to a mean-preserving transformation. Figure 1 shows the percent improvement for different yield-rate distributions as a function of $1 - \alpha$, where α is the parameter of the yield-rate $U(\alpha) = \alpha U + (1 - \alpha)E(U)$. When the original yield rate U is uniformly distributed on $[a, b]$, then $U(\alpha)$ is uniformly distributed on $[(1 - \alpha)b/2 + (1 + \alpha)a/2, (1 - \alpha)a/2 + (1 + \alpha)b/2]$. Both have the same mean; however, $U(\alpha)$ is defined over a narrower range. The data for Figure 1 is as follows. Demand is exponentially distributed with rate 0.1, which means that the mean and variance of demand are 10 and 100, respectively. Revenue/cost parameters are $r = 30$, $p = 6$, $c = 2$, and $h = 0.5$. Distribution-free bounds on the expected profit are $\pi_L(\mu) = 52.9$ and $\pi_U(\mu) = 165.8$. Note that the percent gain in expected profit exhibits diminishing returns. A quick calculation reveals that the percent reduction in coefficient of variation of yield rate for the cases shown in Figure 1 is precisely $100(1 - \alpha)$. So, the results in Figure 1 are in direct agreement with those shown in Figure 2 of Rajaram and Karmarkar (2002).

In Figure 2, the optimal production lot size is plotted against $1 - \alpha$. The original yield-rate distribution U is uniformly distributed over $[0, 1]$, so $E(U) = 0.5$, $\text{Var}(U) = 0.083$, and the demand is gamma distributed with parameters 3 and 0.1, so that $E(X) = 30$ and $\text{Var}(X) = 300$.

Figure 1. Percent improvement in expected profit as a function of $1 - \alpha$.

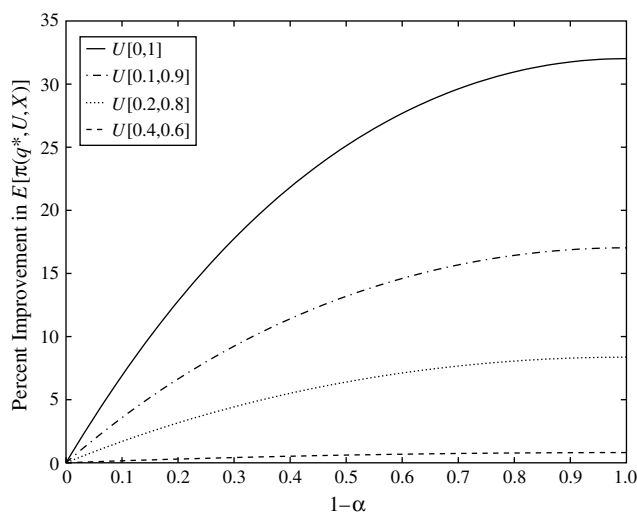
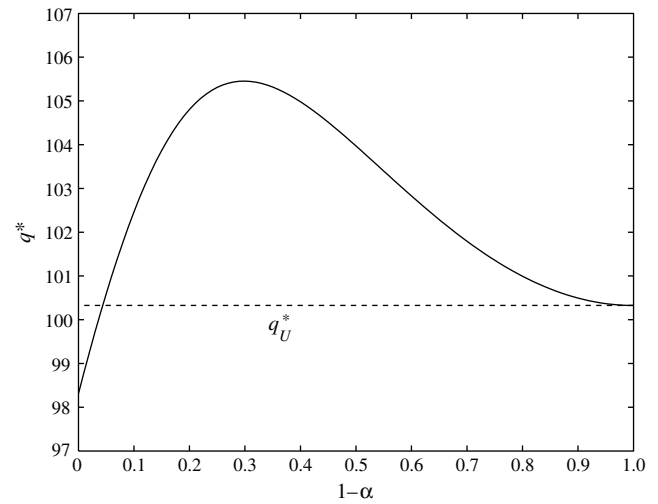


Figure 2. Optimal order quantity as a function of $1 - \alpha$.



Other parameters used in this figure are the same as in Figure 1. This shows that the optimal lot size is not necessarily monotone in α , where α is a measure of yield-rate variability. Moreover, the optimal lot size, q_U^* , corresponding to \underline{G}_μ , is not the largest value of the production lot size amongst all convex-ordered yield-rate distributions. Note that the optimal lot sizes are known to be nonmonotone in demand variability (see Gerchak and Mossman 1992). However, the relationship between lot sizes and yield-rate variability appears not to have been studied before, except in the EOQ setting as described earlier. We do not show $q_L^* = 50.2$ on this plot, because it is considerably smaller than the other values.

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