MATHMATICAL MODELLING OF THE EFFECT OF SOLE ELASTICITY DISTRIBUTION ON PRONATION

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Abstract—A coronal plane model of a distributed elastic sole has been proposed and analyzed with respect to the effects of different medial-lateral elasticity distribution on pronation under quasi-static conditions. The distributed model consists of an array of linear vertical line springs. Under minimum energy assumption, the behavior of the top surface of the interface under resultant force and moment loading was shown to be equivalent to that of a rigid-body mechanism under the same loading. The model was then combined with a rigid-link model of the lower limb. Expressions that describe the relationship of the interface aggregate parameters with pronation and the center of pressure were obtained. These expressions were confirmed by an experiment in which the elastic distribution in the interface was systematically varied and the pronation angle and the center of pressure measured. The model has the potential of being a useful analytical tool in the design of elastic soles in running shoes.

INTRODUCTION

The relative displacements between the shank and the foot occur at two joints, the subtalar joint and the talocrural joint (the ankle). Pronation and supination refer to the rotation about the subtalar joint. Because of the spatial orientation of the subtalar joint axis, pronation motion is a combination of eversion, dorsiflexion and abduction of the foot relative to the shank, and supination embodies the motions in the other polarities in these orthogonal directions (Vogelbach and Combs, 1977). However, the major component of the pronation equivalent axis is in the anteroposterior direction, making eversion the major component of the pronation motion. For this reason pronation has been approximated as the eversion rotation in the coronal plane.

Natural motion of the subtalar joint serves two purposes (Clarke et al., 1983): (i) to allow the foot to adapt to uneven terrain, and (ii) to decrease the impact forces at heel strike by prolonging the duration of the impact phase. The latter is often accomplished by pronation since the initial contact between the foot and the ground is usually on the lateral side (Nigg, 1987). Excessive pronation, however, has been related to several running injuries (Jernick and Heifitz, 1979). These injuries include that of the Achilles tendon, ankle ligaments, the knee and those of the ‘shin splint’ type. The etiology of these pains and injuries is speculated to be the excessive eccentric stretching of tendons, ligaments and muscles which control pronation (Luethi and Stacof, 1987) as well as kinematic alterations creating extra torsional moment at the knee joint (Clements et al., 1981). For these reasons, the fields of sports medicine and athletic footwear design and manufacturing have taken serious efforts in reducing excessive pronation in footwear products.

Nigg et al. (1987) proposed that pronation occurs in heel strike running because the lateral position of the point of action of the ground reaction force relative to the foot causes the foot to rotate in the pronation direction. Thus, they reckon that by reducing the moment arm of the force, pronation can be decreased. This is achieved by a more curved outsole of the lateral side (lateral wedge) or by using a double-density midsole in which the more compliant material is on the lateral side. The former method is said to decrease the moment arm by making the initial contact point more medial, and the use of more compliant sole materials on the lateral side reduces the moment arm through a greater deformation on the lateral side, resulting in a contact geometry resembling that of a curved surface.

There are, however, no quantitative suggestions as to what the distribution of elastic materials should be in order to decrease or increase pronation by a certain amount. Elastic materials are being used in athletic midsoles for two reasons (Nigg, 1986): (i) to better distribute the pressure under the foot, and (ii) to store the mechanical energy at heel strike, and restore it to the runner in order to improve the energy efficiency of running. A better understanding of the quantitative relationship between the distribution of elastic materials in the midsole and pronation is important, so that when midsoles become more compliant to satisfy other shoe design objectives, excessive pronation can still be avoided. In this paper, we shall present a mathematical relationship between the distribution of elastic midsole materials and pronation. To do this, we shall first develop a load/deformation relationship of the sole and then apply it to a model of the lower limb. We shall also present the dual result of the prediction
of the center of pressure which provides an additional means to evaluate the model and suggests an interesting design methodology. A static experiment will then be performed to test this relationship under a controlled variation of the sole elasticity distribution. Finally, the validity of the relationship will be discussed in light of the experimental results and design suggestions will be given.

THE ELASTIC MODEL OF THE INTERFACE

The coronal cross-section of an elastic interface representing the sole of the shoe is described in Fig. 1. The elastic properties of the interface are modelled as a distribution of linear line springs in the plane. The interface is being loaded by a combined force and moment representing the generalized loading at heel strike. In this section, we derive the mechanical admittance of the interface, i.e. the relationship between the loading efforts and the geometric change. This will be combined with a model of the lower limb in the next section in order to quantify the effects of sole elasticity distribution on pronation.

To obtain the mechanical admittance, we assume the following:

- quasi-static loading condition;
- the problem is confined to the coronal plane;
- shear deformation of the sole is negligible;
- the line springs in Fig. 1 are independent.

Coronal analysis is justified because the major component of pronation takes place in this plane. We neglect the horizontal, shear deformation of the sole because (i) it contributes to pronation only indirectly compared to the contribution arising from the vertical deformation of the sole, and (ii) the peak horizontal force is only about 5% of the peak value of the vertical component during running (Cavanagh, 1987).

The boundary conditions for the model are described in Fig. 2. These are:

1. the bottom surface of the interface undergoes no displacements since the sole is in contact with the ground;
2. there are no boundary forces on the medial and lateral borders of the sole;
3. the foot applies a distribution of forces on the top surface of the interface, giving rise to a pair of specified resultant force and moment.

In general, the resultant force has both horizontal and vertical components. However, since we neglect any horizontal deformation, the only relevant loading on the sole consists of the resultant vertical force and a moment.

Because the form of the force distribution function is not specified, the use of the resultant force and moment as a boundary condition leads to multiple solutions for the geometry of the loaded interface. In order to achieve a single solution, we used an energy
minimization approach while constraining the force at any point within the sole model to be non-tensile.

Energy minimization approach

A mechanical system set in motion tends to decrease its potential energy and converge to a local minimum. This principle can be applied to the elastic sole under the loading of the resultant forces and moment. Out of all the possible configurations (i.e. the displacements of the springs describing the contour of the upper boundary of the sole) which satisfy the prescribed physical constraints and the boundary conditions, the minimum energy configuration is that configuration of the sole which has the least potential energy.

The model of the sole described in Fig. 1 consists of a distribution of vertical linear line springs within a width of \((b_0-a_0)\). At any position \(x, a_0<x<b_0\), the distributed stiffness is \(k(x)\) such that if the compression is \(z(x)\) then the compressive force in the interval \(x\) to \(x+\delta x\) is \(k(x)z(x)\delta x\). Using the method of Lagrange multipliers to incorporate the two resultant force/moment equality constraints and one compressive loading inequality constraint, and minimizing the elastic energy of the sole, the minimum energy state \(z_s(x)\) was found. For a resultant foot/sole compressive force \(F_z\) and moment \(M\) the solution is given by the following equations (see the Appendix for details):

\[
\begin{align*}
Z^*(x) &= \max\{\lambda_1 + \lambda_2 x, 0\}, \quad \text{for all } x \in [a_0, b_0] \quad (1)
\end{align*}
\]

(i.e. the minimum energy compression is given either as a linear function of the distance \(x\) when it corresponds to positive compression, or zero), where

\[
\begin{align*}
\begin{bmatrix}
\lambda_1 \\
\lambda_2
\end{bmatrix} &= \frac{1}{KS_2-S_1^2} \begin{bmatrix}
S_2 & -S_1 \\
-S_1 & K
\end{bmatrix} \begin{bmatrix}
F_z \\
M
\end{bmatrix} \quad (2)
\end{align*}
\]

\[
\begin{align*}
K &= \int_{a_0}^{b_0} k(x) \, dx, \quad (3)
S_1 &= \int_{a_0}^{b_0} k(x)x \, dx, \quad (4)
S_2 &= \int_{a_0}^{b_0} k(x)x^2 \, dx, \quad (5)
\end{align*}
\]

\[
\begin{align*}
\int_{a_0}^{b_0} k(x)z_s(x) \, dx - F_z &= 0, \quad (6)
\int_{a_0}^{b_0} k(x)z_s(x) \, dx - M &= 0. \quad (7)
\end{align*}
\]

\([a, b]\) is the range of \(x\) for which the part of the solution \(z^*(x)>0\), i.e. the sole is compressed.

When the region of integration \([a, b]=[a_0, b_0]\), i.e. the whole width of the sole is under compression, we note the following.

- For an interface which has a linear unloaded surface, the minimum-energy approximation of the top surface due to a resultant compressive force and moment loading is also linear. One transformation that a plane undergoes which allows a plane to remain planar is a rigid-body transformation. It can be shown (Li, 1989) that the transformation of a top surface loaded by a resultant vertical force and moment under minimum-energy approximation can be described by a subset of the two-dimensional rigid-body motion. In particular, the Lagrange multipliers \(\lambda_1\) and \(\lambda_2\) describe the rigid-body vertical translation and rotation about the origin of the top surface resulting from the force and moment loading.

- The deformations described by \(\lambda_1, \lambda_2\) are linearly dependent on the applied resultant force \(F_z\) and moment \(M\). This recalls the behavior of a linear spring, and the matrix in equation (2) can be considered as an admittance matrix which relates the loadings and the displacements linearly.

- The admittance matrix is constant with respect to either loading or deformation, and depends only on the aggregate material properties of the interface and the choice of the origin of the coordinate system.

These three characteristics together suggest a physically realizable equivalent mechanism for the sole describing the minimum-energy configuration within the range in which the admittance matrix is constant. The first characteristic suggests that the mechanism has a rigid body as the interface; the second suggests the use of linear elastic springs; the third suggests that the parameters of the mechanism are specific to the material properties of the sole but not to the loading.

The proposed mechanism shown in Fig. 3 consists of a rigid slab supported on a vertical linear line spring \(K_v\), at a pivot point \(CP_o\), and a torsional linear spring \(K_t\), which acts at the pivot point.

The admittance matrix depends on the lateral translation of the coordinate system because \(S_1\) and \(S_2\) in equation (2) depend on the choice of the origin of the coordinate system. It can be shown, however, that the admittance matrix can be diagonalized via a lateral translation of the origin. Furthermore, the pair of diagonal elements of the translated matrix are invariant to the coordinate system in which \(S_1\) and \(S_2\) are expressed. It can further be proven (Li, 1989) that the top surface of the sole in the minimum-energy state and the proposed mechanism are equivalent, with the appropriate vertical translational and rotational spring constants \(K_v, K_t\) and the lateral position of the pivot point \(CP_o\). The diagonal elements of the diagonalized admittance matrix are in fact the admittances of the two springs in the equivalent mechanism and the translated origin is the position of the pivot. The parameters of the mechanism are determined from the distributed elastic coefficients using the following equations:

\[
K_v = K, \quad (8)
\]

\[
K_t = \frac{KS_2-S_1^2}{K}. \quad (9)
\]

\(*CP_o\) indicates the location of the center of the pressure needed for the top surface of the sole to remain horizontal.
Equivalent mechanism consisting of a rigid slab supported by a vertical linear line spring and a linear torsion spring which captures the behavior of the surface of the distributed sole model under minimum-energy configuration. \( K_s \) represents the stiffness of the vertical spring; \( K_t \) represents the stiffness of the torsional spring; \( CP_o \) represents the \( x \)-coordinate of the center of pressure when the slab stays horizontal (the pivot point).

\[
CP_o = \frac{S_s}{K_s}.
\]

\( K_s, S_s, S_t \) are defined in equations (3)-(5). Note that although \( K_s, S_s, S_t \) are coordinate-dependent, \( K_s \) and \( K_t \) are not. The solid model of the elastic interface in Fig. 3, characterized by \( CP_o, K_s, K_t \), will therefore be used as a functional equivalent of the original sole.

**MODELLING AND ANALYSIS OF THE LOWER LIMB**

The posterior view of the model of the right lower limb is described in Fig. 4. The coronal plane model consists of the elastic interface (the shoe or sole), the foot, and the shank. The elastic interface is represented by the two-degrees-of-freedom (vertical and rotational) mechanism developed in the previous section. The vertical and rotational stiffnesses are \( K_v \) and \( K_r \) as defined in equations (8) and (9), respectively. We denote the displacement from the interface pivot to the ankle joint by \( r \) expressed in a body reference frame embedded in the foot.

The foot is modelled by a rigid segment resting on the mechanism. The shank is modelled as a straight rigid link connected to the foot via a revolute joint representing the combined ankle/subtalar joint. The loading applied by the rest of the body on the shank is summarized by a force \( F \) and a moment \( M \) acting at the knee. A constant ankle stiffness \( K_a \) and a joint neutral angle \( \theta_{n} \) are assumed such that the torque at the ankle is \( K_a (\theta_{p} - \theta_{n}^\circ) \). The value of \( K_a \) is based on a linearization of the ankle stiffness function given in Chen et al. (1988) or Mizrahi et al. (1990). This model will be analyzed under quasi-static conditions.

We define the pronation angle as \( \theta_{p} = \theta_{a} - \theta_{f} - \pi/2 \), where \( \theta_{a} \) and \( \theta_{f} \) represent the orientation of the shank and the foot, respectively: \( \theta_{a} < 0 \) refers to pronation and \( \theta_{f} > 0 \) refers to supination. Under quasi-static conditions, the force transmitted across the ankle to the foot is the same as the force applied to the shank from the rest of the body. The moment transmitted across the ankle is related to \( \theta_{a}, \theta_{f} \) and the ankle spring stiffness. Treating the rigid slab of the equivalent mechanism and the foot as a 'free body', and considering its angular equilibrium, \( \theta_{p} \) can be solved in two steps: (i) linearizing the equilibrium equation and solving for \( \theta_{a} \), the rotation of the foot; and then (ii) applying the kinematic relationship between \( \theta_{a}, \theta_{f} \) and \( \theta_{p} \) to solve for \( \theta_{p} \).

The equilibrium equation is:

\[
K_s \left( \frac{\pi}{2} - \theta_{f} \right) - K_v \theta_{f} + [R(\theta_{f}) \mathbf{r}] \times \mathbf{F} = 0, \quad (11)
\]

where

\[
R(\theta_{f}) = \begin{bmatrix}
\cos(\theta_{f}) & -\sin(\theta_{f}) \\
\sin(\theta_{f}) & \cos(\theta_{f})
\end{bmatrix}
\]

is the rotation matrix which transforms the vector \( \mathbf{r} \)
Effect of sole elasticity distribution

originally in the body coordinate system into the inertial coordinate system, and \( \mathbf{a} \times \mathbf{b} \) denotes the two-dimensional vector product between the vectors \( \mathbf{a} \) and \( \mathbf{b} \).

Linearizing equation (11) about \( \theta_i = 0 \) and solving for a perturbation in \( \theta_i \) results in the following equation:

\[
\theta_i = \frac{K_s(\theta_a - \pi/2 - \theta^0 - \theta_i) + r \times F}{K_s + K_r + r \cdot F},
\]

where \( r \cdot F \) is the scalar product between vectors \( r \) and \( F \). Substituting the kinematic relationship between the pronation, shank and foot angles results in the following expression for the pronation angle:

\[
\theta_p = \frac{K_s\theta^0 + (K_r + r \cdot F)(\theta_i - \pi/2) + (X_a - CP_0)F_z}{K_s + K_r + r \cdot F}. \quad (13)
\]

Remarks:
- active torques in the ankle can be accounted for by modifying the neutral angle \( \theta^0 \) and the ankle stiffness \( K_s \) (Hogan, 1980). This would be reflected in a bias in the total pronation \( \theta_p \);
- although it is assumed that horizontal loading does not affect sole deformation directly, it nevertheless has an effect on the pronation angle due to its effect on the moment exerted on the sole;
- the term \( r \cdot F \) is usually negative and represents a decrease in the apparent interface rotational stiffness;
- the vertical and horizontal forces, the shank angle and the stiffness are sufficient to impose a solution for \( \theta_p \), i.e. the moment effects transferred from the upper body to the ankle do not enter the expression. We shall discuss this point further in the Discussion section.

An expression for the center of pressure \( CP \) can also be obtained by considering the ratio between the vertical component of the loading force and the moment on the interface:

\[
CP = \frac{K_sCP_0 + (K_r + r \cdot F)X_a}{(K_s + K_r + r \cdot F)} - \frac{K_s(K_r + r \cdot F)}{(K_s + K_r + r \cdot F)F_z} (\theta_i - \frac{\pi}{2} - \theta^0)\quad (14)
\]

Note that equation (14) for calculating \( CP \) is very similar in its form to equation (13), that was used to calculate \( \theta_p \). In particular, note that both include the sums of a weighted mean and a bias term. They are in fact duals to the same problem with \( \theta^0 \) and \( CP_0 \), \( \theta \), and \( X_a \) playing equivalent roles. This structure provides an additional means to experimentally test the model.

Equation (14) can be rewritten as one single weighted mean in the following manner:

\[
CP = \frac{K_sCP_0 + (K_r + r \cdot F) [X_a - K_s/F_z (\theta_i - \pi/2 - \theta^0)]}{(K_s + K_r + r \cdot F)} \quad (15)
\]

Equations (13) and (14) [or (13) and (15)] summarize the effects of the elastic interface on pronation and the location of the center of pressure.

**EXPERIMENTAL EVALUATION**

In this section we present a static standing experiment in which we systematically varied the distribution of elastic materials under the foot of the subject, and measured the resulting pronation angles and the location of the center of pressure. This procedure was done for the purpose of comparing theoretical predictions arising from equations (13) and (14) and the actual measurements.

**Methodology**

The subject stood wearing sneakers, with the feet at a fixed distance (approximately 0.75 m) apart for all the trials and conditions in the experiment (see Fig. 5). This was done by marking the footprints of both feet on one piece of paper. This requirement ensured that shank angles were relatively constant for all the conditions throughout the experiment. The subject was also requested to maintain an upright posture and to look forward.

![Composite Foam](https://via.placeholder.com/150)

**Fig. 5.** A composite foam is constructed by joining two pieces of foam of different stiffnesses side by side in the coronal section. By varying the position of the border relative to the foot, different stiffness distributions are simulated.
Each foot was supported on a piece of foam. The elastic distribution of the foam under the left foot was fixed, whereas a composite foam was used to support the right foot. The composite foam was constructed by joining two pieces of foam of different stiffnesses side by side. The elastic distribution was altered by moving laterally and medially the border at which the pieces are joined relative to the right foot, thereby generating a different elastic distribution under the right foot (see Fig. 5).

$$K_r = \frac{K_s}{L^3} = \frac{S_2 - S_1}{KL} = \frac{(x - 1)^2 \mathcal{L}^2 + 2(x - 1)(2 \mathcal{L}^2 - 3 \mathcal{L} + 3) \mathcal{L} + 1}{12[(x - 1) \mathcal{L} + 1]},$$

$$CP_0 = \frac{S_1}{KL} = \frac{(x - 1)}{2[(x - 1) \mathcal{L} + 1]},$$

$$K_r = \frac{K}{L \mathcal{K}_{right}} = \frac{(x - 1)}{2 \mathcal{L} + 1}.$$

Kinematic data were collected using the WA-TRACK system, an integration of the optoelectronic spatial measurement system WATSMART (Northern Digital, Waterloo, Canada) and the rigid-body motion analysis software TRACK (MIT, Cambridge, MA). Angular displacements were decomposed using the method of Grood and Suntay (1983) and Siegler et al. (1988). Based on those definitions, pronation was identified as the eversion of the foot with respect to the shank, i.e., the angular displacement of the shank with respect to the foot in the direction of the long axis of the foot. The subject was requested to stand with the right foot on a force platform (AMTI, Newton, MA, U.S.A.) which recorded the six components of the reaction force and moment. The medial-lateral coordinate of the center of pressure relative to the origin of the force platform was computed from the ratio between the anteroposterior moment and the vertical reaction force. Because the sensor origin was located inside the platform, the center of pressure was translated to the surface of the platform along the direction of the resultant force. Because the sensor origin was located inside the platform, the center of pressure was translated to the surface of the platform along the direction of the resultant force.

Each trial required the subject to stand steadfastly in the specified posture for 5 s and the pronation angle and the center of pressure for each condition was averaged over the period. The experiment was carried out at the following 10 conditions: defining the distance of the border between the two foams from the left edge of the footprint as $x$ and the width of the foot as $L$, then $\mathcal{L} = x/L$. The stiffness per unit area for the softer foam was 425 kN m$^{-2}$ and the stiffness ratio between the two foams was 19.36. Thus, $x = 19.36$ when the stiffer material was on the medial side and $x = 1/19.36$ when it was on the lateral side. The dimensionless quantities $K_r/(L^3K_{med}) (K_{med})$, the stiffness per unit width for the more compliant material and $CP_0/L$ are plotted against $\mathcal{L}$ in Fig. 6.

In generating predictions of the pronation angle and the center of pressure locations from equations (13) and (14), we have assumed that (i) the foot measured $20 \times 9.0$ cm, (ii) the ankle joint was located at halfway medial-laterally and 3 cm above the sole (see Fig. 5), (iii) the shank angle (15°) and neutral pronation angle $\theta^o$ were constant, and (iv) the ankle joint stiffness* was equal to 40 N m rad$^{-1}$.

Results

A comparison of the pronation angle and the center of pressure location between the predictions made by equations (13) and (14) and the actual measurements is shown in Fig. 7. When the more compliant material was introduced on the lateral side, $x = 19.36$, $\mathcal{L}$ varied from 1–0 in Fig. 7(b), pronation angle decreased by more than 3° before increasing again 4° higher than it began with. The center of pressure, on the other hand, underwent an initial lateral migration and then a medial migration before ending up at a point more lateral than when $\mathcal{L} = 1$ [Fig. 7(d)]. As the more compliant material was introduced on the medial side [$x = 1/19.36$, $\mathcal{L} = 0–1$ in Fig. 7(a)], the measured pronation angle increased by 10° and then decreased to a value 3° higher than it began with. The center of pressure pattern in Fig. 7(c) exhibited a general lateral migration.

Experimental parameters

We denote the stiffnesses of the materials on the medial and lateral sides of the composite foam by $K_{med}$ and $K_{med}$. The aggregate stiffness, the first and second moments of the stiffness of the composite foam (assuming a rectangular foot interface contact area) were derived using equations (3)-(5):

$$K = (K_{med} - K_{lateral})x + K_{lateral}L,$$

$$S_1 = \frac{1}{2}(K_{med} - K_{lateral})x^2 + \frac{1}{2}K_{lateral}L^2,$$

$$S_2 = \frac{1}{3}(K_{med} - K_{lateral})x^3 + \frac{1}{3}K_{lateral}L^3.$$

Defining $\mathcal{L}^2 := x/L$, $\mathcal{L} = K_{med}/K_{lateral}$ applying equations (8)-(10), the dimensionless parameters for the lumped-sole model for the interface in these experiments were calculated:

$$K_r = \frac{K_{med}}{L^3} = \frac{S_2 - S_1}{KL} = \frac{(x - 1)^2 \mathcal{L}^2 + 2(x - 1)(2 \mathcal{L}^2 - 3 \mathcal{L} + 2) \mathcal{L} + 1}{12[(x - 1) \mathcal{L} + 1]},$$

$$CP_0 = \frac{S_1}{KL} = \frac{(x - 1)}{2[(x - 1) \mathcal{L} + 1]},$$

$$K_r = \frac{K}{L \mathcal{K}_{right}} = \frac{(x - 1)}{2 \mathcal{L} + 1}.$$

*This value was estimated from the elastic stiffness in Mizrahi et al. (1990).
Fig. 6. Lumped-model parameters of the sole as the elastic composition of the foam changes. The stiffer foam is on the medial side when \( \alpha = 19.36 \), and on the lateral side when \( \alpha = 1/19.36 \). \( K_r/(K_{rot}L^2) \) describes the dimensionless rotational stiffness of the model, and \( CP_o/L \) describes the dimensionless pivot location. \( z' \) defines the border line between the two pieces of foam.

Fig. 7. Measured and predicted pronation angles \( \theta_z \) and the center of pressure location \( CP \) as a function of \( z' \). Pronation angle \( \theta_z \) vs \( z' \): (a) \( \alpha = 1/19.36 \); (b) \( \alpha = 19.36 \). Center of pressure location \( CP \) vs \( z' \): (c) \( \alpha = 1/19.36 \); (d) \( \alpha = 19.36 \). Solid lines are measurements, dotted lines are predictions. Note the small changes in the pronation angle for most of the range when \( \alpha > 1 \). A constant offset of 3° for the pronation curve and -6 cm for the CP curve have been included in the respective plots.
A theoretical model of the elastic interface has been presented with the aim of predicting the effect of the interface on pronation. We now explore the limitations arising from our assumptions, and compare the theoretical predictions with the experimental measurements. The center of pressure location, being a dual variable to the pronation angle, will also be compared to give further testing of the model.

Some of the assumptions we made cannot be fully justified. In particular, the ankle stiffness, which was assumed constant, has been shown to depend on the load (Hunt, 1982), the displacement (Kearney and Hunter, 1982) and the degree of antagonistic muscle action on the joint (Hogan, 1980). Moreover, the published values of the ankle–subtalar stiffness vary from 0 to 100 N m rad⁻¹ (Chen et al., 1988; Mizrahi et al., 1990); thus, our estimated value (40 N m rad⁻¹) may be widely wrong. Furthermore, since muscle activities were not monitored during the different trials, neither the active torque, nor the degree of antagonism was guaranteed constant. Also, the presence of the arch on the medial side of the foot should be further discussed. First, the pure moment loading transferred through the shank was not included in the analysis. Such a moment would affect the neutral pronation angle, and since the shank and thigh positions were kept constant during the experiment, we assumed that the pure moment loading was also constant. The incorporation of the offset value into the theoretical pronation angle (Fig. 7) was meant to address that issue.

Second, the moment effects from the upper body transferred to the ankle do not enter explicitly into the expression for \( \theta_p \). Since this moment must equal the sum of the passive \([K_p(\theta_p - \theta_0^p)]\) and the active moments, we expect a part of this to be balanced out by the active moments, resulting in a bias in \( \theta_0^p \) and \( \theta_p \). Because \( \theta_0^p \) and, hence, the active ankle torque is assumed constant while \( \theta_p \) changes under different conditions in our experiment, the loading of the more proximal links (i.e. the shank and the thigh) will change. By considering the moment balance equations for the shank or the thigh, we see that these segments will not generally be in equilibrium if the subject is to impose arbitrary moment loading at the hip. Since in our static experiment, the segments are clearly in equilibrium, the moment loading at the hip cannot be arbitrary and indeed we anticipate small changes in this loading between different conditions. This situation is consistent with the general principle that one cannot impose both displacement and effort variables in the same degree of freedom. In the rectilinear degrees of freedom, we have imposed the effort variables (horizontal and vertical forces) while in the rotational degree of freedom a displacement variable (shank angle) was imposed. We believe that because of the slower dynamics of the leg compared to the foot (due to large differences in the inertial properties) the imposition of the shank angle is more realistic.

Despite these limitations, the measured pronation angle [Figs 7(a) and 7(b)] follows the same trend as the theoretical prediction. The CP measurements and the predictions also agree qualitatively [Figs 7(c) and 7(d)]. In both cases, the mathematical model is accurate with respect to both the presence of various features, as well as the overall range of change that was introduced in the pronation angles and the center of pressure. Moreover, the fact that both the pronation angle and the center of pressure predictions, which are duals according to our model, follow the experimental results further suggests that the structure of the model is consistent with reality.

In order to gain insights into the roles of the lumped parameters \((K_p, CP_p)\), the trends in Fig. 7 are discussed in light of equations (13) and (15). First we note that the second term in equation (13) which is a bias, is proportional to \((X_p - CP_p)\) and inversely proportional to \(K_p\). This term causes \( \theta_p \) to follow the trend of \((-CP_p)\), especially for small \(K_p\). This is evidenced by the correspondence between the local minimum and maximum of \( \theta_p \) curves in Fig. 7 and those in the respective \( CP_p \) curves [Figs 6(c) and 6(d)]. Second, the first term in equation (13) represents a weighted mean between the shank angle \((\theta_p - \pi/2)\) and the neutral angle \(\theta_0^p\). This term converges to \(\theta_0^p\) as \(K_p \to 0\), and to \((\theta_p - \pi/2)\) as \(K_p \to \infty\). This is evident in Fig. 7(b), where changes in \( CP_p \) are more gradual. The interplay between these two terms can be seen in the asymmetry of the magnitudes of changes between \(\alpha \to 1\) and \(\alpha < 1\) for large \(K_p\); when \(\alpha \to 1\) both terms affect \(\theta_p\) in the same direction, whereas when \(\alpha > 1\) they act in a different direction, resulting in much smaller changes in the pronation angle [Fig. 7(b)].

Similarly, we observe that the expression in equation (15) is a weighted mean between \([X_p - (K_p/F_p)](\theta_p - \pi/2 - \theta_0^p)\) and \(CP_p\). This implies that \( CP_p \) tends to follow the trend of \( CP_p \) for small \(K_p\), but moves medially for large and increasing \(K_p\), since \([X_p - (K_p/F_p)](\theta_p - \pi/2 - \theta_0^p)\) lies on the medial side of the foot in this experiment. Again, the maximum and minimum in the CP curve and the asymmetry are an evidence of the interplay between these two effects.

The roles of \(K_p\) and \(CP_p\) can also be attributed to two distinct physical factors. Because \(K_p\) and \(K_s\) are in series with one another, changing \(K_p\) thus changes the overall admittance of the system. Therefore, as the total admittance decreases, the ankle joint does not displace as much. The second factor is the moment arm by which the loading force causes the interface to
rotate, altering the pronation angle. Since the moment arm is mediated by \( CP_0 \), changing the elastic distribution of the support will change the value of \( CP_0 \) and as a result the pronation angle is also altered.

We shall now discuss the use of the model in affecting and understanding designs of elastic soles to prevent excessive pronation. Excessive pronation is conjectured to increase the risk of injuries by excessively stressing the ligaments and muscles in the lower limb. Chen et al. (1988) demonstrated that the flexibility of the ankle complex is increased by 30% when the anterior talofibular ligaments (ATFL) are sectioned. This implies that the ATFL are responsible for roughly 30% of the stiffness and, hence, carry about the same amount of the load in the composite ankle. Therefore, excessive stress on the ligaments corresponds directly to the situation when the ankle spring is overly activated. With this reasoning in mind, the goal of the design translates to making \( cp \) as close to the neutral angle as possible by manipulating the design parameters \( CP_0 \) and \( K_r \). Because \( K_r \) and \( CP_0 \) are aggregate parameters of the sole elastic distribution, their specifications do not define the design uniquely but present constraints that the elastic distributions have to satisfy.

The model suggests that in order to prevent excessive pronation, \( CP_0 \) should lie medial or close to \( X_a \) and \( K_r \) should be small. By making \( K_r \) small, we shift the weighted mean in equation (13) towards \( T_0 \). However, we do not expect \( K_r \) to have decreased so much as to eliminate pronation completely. By placing \( CP_0 \) medial to the ankle joint we would enable the downward force to rotate the foot, cancelling the remaining pronation. However, if \( CP_0 \) cannot be placed medial to \( X_a \), the moment effect of \( F_t \) would increase pronation. Similarly, if the foot is expected to be in the supinated position, when \( CP_0 \) is medial to the ankle joint, the moment effect would further increase supination. In this case, \( K_r \) should be large such that the effects of the otherwise undesirable moment are diminished.

Current schemes to control pronation are (Nigg, 1987): (i) the use of double-density foam, with the softer material on the lateral side; (ii) the use of medial support on the posterior side of the arch; and (iii) the introduction of a negative flare or a rounded heel on the lateral border. In light of the model presented in this paper, the described double-density midsole corresponds to an interface with \( CP_0 \) on the medial side; an arch support increases the stiffness of the foot medially, which again shifts \( CP_0 \) medially; a rounded heel allows the foot to rotate and a negative flare causes the rolling point, which is analogous to \( CP_0 \) of our model, to be placed more medially. Therefore, all the three methods translate similarly to a change in the location of \( CP_0 \), a change that is indeed advocated by this model. The rotation due purely to the geometry of the shoe is different from that due to an elastic interface in that the amount of rotation induced in the latter depends on the loading force whereas that in the former is independent of the external loading. Since pronation is itself load-dependent (Vogelbach and Combs, 1977), a compensation which is also load-dependent would decrease the risks of overcompensation.

Nigg et al. (1987) reported that decreasing the midsole hardness also decreases pronation. They suggested that since the application of force at initial contact occurs on the lateral side of the heel region, the larger deformation experienced by a softer midsole results in a shorter moment arm with respect to the subtalar joint. Our model supports this via the \( CP \) argument—as the midsole hardness* decreases, the ability of the sole to attract \( CP \) towards a medially placed \( CP_0 \) improves, decreasing the moment arm. The fact that a decreasing stiffness increases the admittance of the sole provides also a kinematic explanation to the situation: since the sum of the pronation angle and the rotation of the sole is given by the trajectory of the shank angle \( \theta_a \) (as \( \theta_a = \pi/2 = \theta_f + \theta_t \)), when the sole admittance increases, more of the displacement is taken up in the deformation of sole and less by the ankle joint. The first explanation is applicable when a loading force is imposed and the second when the kinematics of the shank angle are imposed. Thus, our model provides a framework in which both situations can be analyzed.

The parameter \( CP_0 \) besides being a point about which the interface is expected to rotate, has another significance: for a given vertical force the elastic energy stored in the interface is smallest when the force is applied at \( CP_0 \). Earlier we explained that \( CP \) has a tendency to converge to \( CP_0 \) [equation (15)]. Thus, if it is desired that \( CP \) should follow some trajectory during stance, an elastic interface that has the desired property may be designed by prescribing \( CP_0 \). The coronal-plane model presented here essentially combines the effect of the sole elasticity in the anteroposterior direction for the whole length of the foot. By considering the variation of the region of support in the anteroposterior direction during stance, the medial–lateral position of \( CP_0 \) will also vary, thus providing a criterion for the prescription of the elastic distribution in both the medial–lateral and the anteroposterior directions.

**Conclusion**

A model of an elastic interface has been developed to study the effects of midsole elastic distribution on pronation under quasi-static conditions in the coronal plane. The relationship successfully predicted the effects on pronation and the center of pressure movement in an experiment in which the elastic distribution was varied. The proposed model has the potential of

*We assume that the midsole stiffness is monotonic with the hardness.
being used as a quantitative tool for the design of elastic soles for running shoes.

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REFERENCES


APPENDIX

We wish to determine the configuration \( z^* \) which minimizes the total elastic energy subject to the constraint that the interface force and moment are \( F_i \) and \( M_i \leq 0 \). Thus,

\[
\min J = \int_{a_0}^{b_0} \frac{1}{2} k(v)^2(v)dv,
\]

s.t. \( F_i = \int_{a_0}^{b_0} k(x)z(x)dx \),

\[
M_i = \int_{a_0}^{b_0} k(x)z(x)x dx.
\]

According to the Kuhn–Tucker conditions, the optimal function \( z^*(\cdot) \) is given by

\[
\int_{a_0}^{b_0} [k(x)z^*(x) - \lambda_1 k(x) - \lambda_2 k(x) - \mu(x)]\delta x, dx = 0,
\]

\[
z^*(x) \mu(x) = 0.
\]

where \( \mu(x) > 0 \) when the inequality constraint is active; \( \mu(x) = 0 \) otherwise.

For each \( x \in [a_0, b_0] \) there are two alternatives: (i) \( \mu(x) > 0 \) and \( z^*(x) = 0 \) (active constraint); or (ii) \( \mu(x) = 0 \) and \( z^*(x) > 0 \) (inactive constraint).

Therefore, for each \( x \), either

\[
z^*(x) = (\lambda_1 + \lambda_2 x),
\]

or

\[
-\mu(x)/k(x) = (\lambda_1 + \lambda_2 x).
\]

The second alternative is applicable only when \( \lambda_1 + \lambda_2 x > 0 \). When it is strictly less than 0, the first alternative is not applicable since \( z(x) \) would be less than zero; when it is zero, \( z^*(x) = 0 \). Hence,

\[
z^*(x) = \max (\lambda_1 + \lambda_2 x, 0).
\]

Denoting the region \( x \in (a_0, b_0) \) as \( z(x) > 0 \), in order to satisfy the equality constraints,

\[
z^*(x) = \max \{\lambda_1 + \lambda_2 x, 0\}, \forall x \in (a_0, b_0),
\]

where \( \lambda_1 \), \( \lambda_2 \) are given by equations (2)–(7).