On the Strictly Complementary Slackness Relation in Linear Programming

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Abstract

Balinski and Tucker introduced in 1969 a special form of optimal tableaus for LP, which can be used to construct primal and dual optimal solutions such that the complementary slackness relation holds strictly. In this paper, first we note that using a polynomial time algorithm for LP Balinski and Tucker's tableaus are obtainable in polynomial time. Furthermore, we show that, given a pair of primal and dual optimal solutions satisfying the complementary slackness relation strictly, it is possible to find a Balinski and Tucker's optimal tableau in *strongly* polynomial time. This establishes the equivalence between Balinski and Tucker's format of optimal tableaus and a pair of primal and dual solutions to satisfy the complementary slackness relation strictly. The new algorithm is related to Megiddo's strongly polynomial algorithm that finds an optimal tableau based on a pair of primal and dual optimal solutions.

Key words: Linear programming, the simplex tableau, primal and dual solutions, complementary slackness, strongly polynomial-time.

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1 Introduction

Consider the following linear program

$$\begin{array}{lll}(P) & \min & c^T x \\ & \text{s.t.} & Ax = b \\ & & x \geq 0\end{array}$$

where A is an $m \times n$ matrix, $b \in \mathcal{R}^m$ and $c \in \mathcal{R}^n$, and its dual problem

$$\begin{array}{lll}(D) & \max & b^T y \\ & \text{s.t.} & y^T A \leq c^T.\end{array}$$

Assume that both (P) and (D) have optimal solutions, and $rank\{A\} = m$.

It is well known that for optimal solutions of (P) and (D), the complementary slackness relation holds (see Chvátal [2]). Moreover, the following stronger statement can be proved.

Lemma 1 *There exist optimal solutions to (P) and to (D) such that*

$$x_i^*(c_i - y^{*T} A_i) = 0 \text{ and } x_i^* + (c_i - y^{*T} A_i) > 0$$

for every $i = 1, 2, \cdots, n$.

Proof.
See Goldman and Tucker [4], Dantzig [3] and Schrijver [9].

□

To distinguish from the normal complementary slackness relation, which does not exclude the possibility that both x_i^* and $(c_i - y^{*T} A_i)$ are zero, we call the relation stated in Lemma 1 the *strictly complementary slackness relation.* Clearly, the concept of the strictly complementary slackness relation is interesting only for degenerate problems. Although the existence of a strictly complementary primal-dual pair is easily derived using Farkas' Lemma (cf. [9]), it is important to know how such a pair of solutions can be constructed. In 1969, Balinski and Tucker [1] showed a constructive way to get such strictly complementary pair in finite time. In their approach, a special format of the optimal tableau for linear programming was investigated. We will refer to this type of optimal tableau as *the Balinski-Tucker tableau* in this paper.

The recent development of the interior point methods for linear programming has opened new areas for research. In [6] an attempt was made to base the duality theory and the sensitivity analysis entirely on the interior point methodology. Typically, solutions produced by primal-dual interior point algorithms converge to the analytical center of the relative interior of the optimal face. Compared to the classical simplex method, where only basic feasible solutions are searched, this non-vertex property of the interior point methods was regarded as a disadvantage at early stages. To get a true vertex optimal solution from an approximative solution, a purification procedure can be used (cf. [7] and [12]). However, there are differences between a vertex optimal solution and an optimal basic solution, for there can be many bases corresponding to the same vertex. In [8], Megiddo showed that knowing only a primal optimal solution does not help in general to find an optimal basis which should be both primal and dual feasible. However, in the same paper he proved that if a primal optimal solution and a dual optimal solution are available simultaneously, then finding an optimal basis can be done in strongly polynomial time.

In this paper, we will show that knowing non-vertex primal and dual optimal solutions (in the degenerate case) actually we get more information. More precisely, we present strongly polynomial-time algorithms which can be used to construct "lexicographically feasible" extremal directions around the purified vertices. As a consequence, if we have both primal- and dual optimal solutions belonging to the relative interior of the respective optimal face, then a Balinski-Tucker tableau can be constructed in strongly polynomial time. In some sense, this result establishes that a Balinski-Tucker tableau and a pair of primal- and dual- optimal solutions in the relative interior of the optimal faces are equivalent. We remark also that a Balinski-Tucker tableau can be obtained in polynomial time in the *weak* sense, using a polynomial algorithm for LP. Our analysis is closely related to Megiddo's algorithm that finds an optimal basis using a pair of primal-dual solutions.

We organize this paper as follows. In the next section we discuss the properties of solutions satisfying the strictly complementary slackness relation. In the same section, the Balinski-Tucker tableau will be introduced, and its basic properties discussed. In Section 3, Megiddo's algorithm for finding an optimal basis will be included. New algorithms for finding Balinski-Tucker's tableau and the correctness proofs will be presented in Section 4. Finally, we conclude the paper in Section 5.

Before proceeding we mention the notations we use in this paper. We denote matrices by capital letters, and vectors by lower case letters with subscript denoting the coordinate. Index sets are denoted either by capital letters or by Greek letters. For a matrix $A = (a_1, \cdots, a_n)$ and an index set I, A_I denotes the submatrix of A whose columns belong to I, i.e. $A_I = \{a_j : j \in I\}$. The same rule applies to vectors. To ease the notation we do not distinguish a_j and A_j. A basis for the problem (P), denoted by B, is a maximal subset of $\{1, 2, \cdots, n\}$ such that the corresponding columns in A_B are independent. The complementary set $N := \{1, 2, \cdots, n\} \setminus B$ is called nonbasis. Finally, a tableau $T(B)$ associated with a basis B is a matrix given as follows:

$$T(B) = \begin{pmatrix} c_N^T - c_B^T A_B^{-1} A_N & -z_0 \\ A_B^{-1} A_N & A_B^{-1} b \end{pmatrix}$$

where $z_0 = c_B^T A_B^{-1} b$. If $A_B^{-1} b \geq 0$ we call the tableau *primal feasible* and we call it *dual feasible* if $c^T - c_B^T A_B^{-1} A \geq 0^T$. We call in this paper the matrix $A_B^{-1} A$ a *sub-tableau*.

2 Strictly Complementary Slackness and Balinski and Tucker's Tableau

Solutions for the primal and for the dual problem satisfying the strictly complementary slackness relation can be characterized by their topological position in the optimal faces. Let

$$\mathcal{F}_P = \text{ optimal solution set of (P)}$$

$$\mathcal{F}_D = \text{ optimal solution set of (D)}$$

and

$$\overset{o}{\mathcal{F}}_P = \text{ relative interior of } \mathcal{F}_P$$

$$\overset{o}{\mathcal{F}}_D = \text{ relative interior of } \mathcal{F}_D.$$

The definition of relative interiors can be found, e.g., in [9].

We have the following result.

Lemma 2 *Solutions x^* and y^* satisfy Lemma 1 if and only if $x^* \in \overset{o}{\mathcal{F}}_P$ and $y^* \in \overset{o}{\mathcal{F}}_D$.*

Proof.
See Corollary 2.1 of [5].

□

Remark 1 By the definition, a single point also belongs to its 0-dimensional relative interior.

To see how solutions satisfying Lemma 1 can be obtained we introduce a special tableau form used by Balinski and Tucker in [1]. First we note the following result proved in [1].

Lemma 3 *Starting from an arbitrary tableau (matrix) it is always possible to get one of the following two tableau forms using pivot operations:*

$$\text{(i)} \quad \begin{array}{|cccc|} \hline \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \oplus & \oplus & \cdots & \oplus \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \hline \end{array}$$

$$\text{(ii)} \quad \begin{array}{|ccccc|} \hline \cdot & \cdot & \ominus & \cdot & \cdot \\ \cdot & \cdot & \ominus & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & \ominus & \cdot & \cdot \\ \hline \end{array}$$

where $\oplus$ stands for either a positive number or zero, and $\ominus$ for a negative number or zero.

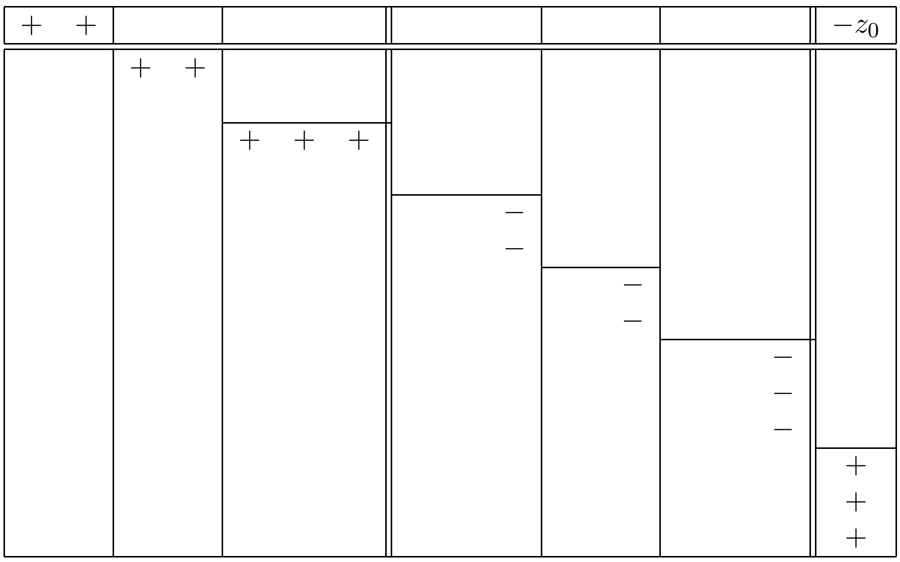
Proof.

See Corollary 1 of [1]. For an alternative proof we can also use the criss-cross method as follows (cf. [10] or [11]). Fix one row and one column. If there is no negative elements in the row or no positive elements in the column, then stop. Otherwise, among the negative elements in the row and the positive elements in the column, select the one with the minimal index. Suppose it is a row element (for the other case follow the similar procedure). Then for the column corresponding to the chosen element find positive elements. If there is none, stop. Otherwise select the one with the smallest index. Pivot on this position. Since this criss-cross method guarantees no repetition of the bases, after a finite amount of pivoting steps, we will have to terminate with a tableau belonging to one of the two above mentioned formats.

□

Using this lemma we are now able to present the following result of Balinski and Tucker [1].

Theorem 1 *There exists an optimal tableau in the following format*



where + stands for a positive number and − for a negative number, and all the unspecified numbers above the staircase are zeros.

Proof.

Applying Lemma 3 recursively on the north-east corner of the matrix which is not yet covered by positive rows or negative columns or the positive elements of the right-hand-side vector, the theorem easily follows.

□

Remark 2 Balinski-Tucker type tableaus are not unique in general.

Remark 3 A Balinski-Tucker tableau can also be called a *lexicographically optimal tableau.* Information about the dimension of the optimal faces is easily seen from a Balinski-Tucker tableau. The number of columns corresponding to the negative part of the staircase (columns between the two vertical double-lines in the displayed tableau), except for the basic ones, is equal to the dimension of the primal optimal face. The number of the rows corresponding to the positive part of the staircase equals the dimension of the dual optimal face.

Interestingly, using a polynomial algorithm for LP we have the following result.

Lemma 4 *It is possible to construct a Balinski-Tucker tableau in polynomial time.*

Proof.
We need only to show that the tableau forms displayed in Lemma 3 can indeed be obtained in polynomial time. To see this, for a given sub-tableau $A_B^{-1}A$ we fix one arbitrary column, say $A_B^{-1}A_n$. Consider now the following linear program:

$$\begin{array}{ll} \min & 0^T x \\ \text{s.t.} & A_B^{-1}\bar{A}x = -A_B^{-1}A_n \\ & x \geq 0 \end{array}$$

where $\bar{A}$ is $A_{B \cup N \setminus \{n\}}$. The dual problem is:

$$\begin{array}{ll} \max & -(A_B^{-1}A_n)^T y \\ \text{s.t.} & y^T A_B^{-1}\bar{A} \leq 0^T. \end{array}$$

Clearly the dual has a feasible solution $y = 0$. Now apply a polynomial time algorithm to solve the primal problem. There are two possible cases: 1) the primal problem has an optimal basic solution reported by the algorithm; and 2) the primal problem has no feasible solution, and a dual unbounded extremal direction is reported. In both cases, the required tableau format given in Lemma 3 would follow. □

For a pair of strictly complementary primal and dual solutions we have the following important property.

Lemma 5 *For any* $x^* \in \overset{o}{\mathcal{F}}_P$ *and* $y^* \in \overset{o}{\mathcal{F}}_D$*, the index sets* $I := \{i : x_i^* > 0\}$ *and* $J := \{j : c_j - {y^*}^T A_j > 0\}$ *form a unique partition of* $\{1, 2, \cdots, n\}$ *which is independent of* x^* *and* y^**.*

Proof.
See, e.g., [6]. □

It is easily seen that if a Balinski-Tucker tableau is available, then a pair of optimal solutions for (P) and (D) satisfying the strictly complementary slackness relation can be obtained as shown in the following theorem. Remark that due to Lemma 2 this pair of optimal solutions belong to the relative interior of the corresponding optimal faces.

Theorem 2 *If a Balinski-Tucker tableau is given, then a pair of strictly complementary optimal solutions for (P) and (D) can be found in strongly polynomial time.*

Proof.
See Section 4 of Balinski and Tucker [1].

□

The main purpose of this paper is to show that the reverse of the above theorem is also true. Namely, we are going to show that if such a pair of strictly complementary solutions for (P) and (D) are known, then we can construct a Balinski-Tucker tableau in strongly polynomial time. Before doing this, we first introduce in the next section Megiddo's algorithm which is useful for our analysis.

3 Megiddo's Algorithm

Megiddo [8] showed the following. To find an optimal tableau (basis), given an optimal solution x^* of (P), is in general as difficult as solving (P) from scratch. However, if *a pair* of optimal solutions x^* (of (P)) and y^* (of (D)) are available, it is possible to construct an optimal tableau (basis) in strongly polynomial time. We shall now present Megiddo's strongly polynomial-time algorithm for finding an optimal basis.

Megiddo's Algorithm

- Input: $\bar{x} \in \mathcal{F}_P$ and $\bar{y} \in \mathcal{F}_D$.
- Output: An optimal basis B and the tableau $T(B)$.

Step 1 Let $X := \{i : \bar{x}_i > 0\}$ and $Y := \{j : c_j - \bar{y}^T A_j = 0\}$ (hence $X \subseteq Y$ due to the complementarity).

Step 2 If columns of A_X are linearly independent, go to Step 4.

Step 3 Choose $j \in X$ such that A_j is linearly dependent on $A_{X \setminus j}$. Find w such that $A_j = A_{X \setminus j} w$ and let

$$d_k := \begin{cases} 1, & \text{for } k = j \\ -w_k, & \text{for } k \in X \setminus j \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$t := \min\{\frac{\bar{x}_i}{d_i} : \text{ for } i \text{ such that } d_i > 0\}$$

and $\bar{x} := \bar{x} - td$ and $X := \{i : \bar{x}_i > 0\}$. Go to Step 2.

Step 4 If there is $j \in Y$ such that A_j is linearly independent of A_X, then $X := X \cup j$ and repeat Step 4.

Step 5 If $rank\{A_X\} = m$, let $B := X$; construct the corresponding tableau $T(B)$ and stop.

Step 6 Let $j \notin X$ and A_j is linearly independent of A_X. Solve

$$\begin{array}{rcl} z^T A_Y & = & 0^T \\ z^T A_j & = & 1. \end{array}$$

Let

$$t := \min\{\frac{c_k - \bar{y}^T A_k}{z^T A_k} : \text{ for } k \text{ such that } z^T A_k > 0\}.$$

Let $\bar{y} := \bar{y} + tz$ and $Y := \{j : c_j - \bar{y}^T A_j = 0\}$. Go to Step 4.

Theorem 3 *Megiddo's algorithm is correct and runs in strongly polynomial time.*

Proof.

Clearly, during the procedure, $\bar{x}$ and $\bar{y}$ stay feasible and satisfy the complementary slackness relation and therefore remain optimal. Upon termination, the set X is a basis with $\bar{x}$ the corresponding optimal basic solution and $\bar{y}$ the corresponding dual optimal basic solution. The strong polynomiality follows immediately after noticing that each time at Step 3, $| X |$ strictly decreases, and at Step 6, $| Y |$ strictly increases. □

4 The New Algorithms

In this section we shall present three new algorithms. The last algorithm is to construct a Balinski-Tucker tableau, and it is actually a summary of the first two algorithms and Megiddo's algorithm. We start our discussion by introducing some notations.

For a vector v in $\mathcal{R}^n$, we denote its positive support by $\sigma(v) := \{i : v_i > 0\}$, and its negative support by $\chi(v) := \{i : v_i < 0\}$, and the support by $\pi(v) := \sigma(v) \cup \chi(v)$.

For a feasible basis B, let the corresponding basic solution of (P) be $\bar{x}$. Clearly, $\bar{x}_B = A_B^{-1} b$ and $\bar{x}_N = 0$. We call a direction $r \in \mathcal{R}^n$ a *primal extremal direction w.r.t.* B (a circuit) if and only if $Ar = 0$ and there is $j \in N$ with $r_j > 0$ and $\pi(r) \subseteq B \cup j$. Similarly, if we let the corresponding dual basic solution be $\bar{y}$, then $\bar{y}^T A_B = c_B^T$. We call a direction $d \in \mathcal{R}^m$ a *dual extremal direction w.r.t.* B if and only if there is $i \in B$ such that $d^T A_i < 0$ and $\pi(d^T A) \subseteq N \cup i$. It is well known that the primal extremal directions are extended nonbasic column vectors in the sub-tableau and the dual extremal directions are the row vectors in the sub-tableau. In case of degeneracy, however, it may happy that none of extremal directions are feasible. Note that extremal directions are related to the basis. In fact, for degenerate problems, finding a tableau with feasible extremal direction is as difficult as solving a linear program. To link extremal directions with Balinski and Tucker's tableau, we note the following lemma.

Lemma 6 *Suppose we have an optimal tableau $T(B)$ with primal optimal basic solution $\bar{x}$ and dual optimal basic solution $\bar{y}$. Moreover, suppose that there exist some primal extremal directions $r^{(1)}, r^{(2)}, \cdots, r^{(\nu)}$ w.r.t. B, and some dual extremal directions $d^{(1)}, d^{(2)}, \cdots, d^{(\tau)}$ w.r.t. B, in such a way that:*

1). $\bar{x}+\sum_{j=1}^{t} \delta_j r^{(j)} \in \mathcal{F}_P$ for $t = 1, 2, \cdots, \tau-1$, where δ_j are some positive numbers;

2). $\bar{y}+\sum_{j=1}^{s} \epsilon_j d^{(j)} \in \mathcal{F}_D$ for $s = 1, 2, \cdots, \nu-1$, where ϵ_j are some positive numbers;

3). $\bar{x} + \sum_{j=1}^{\tau} \delta_j r^{(j)} \in \overset{o}{\mathcal{F}}_P$;

4). $\bar{y} + \sum_{j=1}^{\nu} \epsilon_j d^{(j)} \in \overset{o}{\mathcal{F}}_D$.

Then the tableau is a Balinski-Tucker tableau.

Proof.
From 1) and 2) we know that

$$\chi(r^{(t)}) \subseteq \sigma(\bar{x} + \sum_{j=1}^{t-1} \delta_j r^{(j)})$$

for $t = 1, 2, \cdots, \tau$ and,

$$\sigma(d^{(s)^T} A) \subseteq \sigma(c^T - (\bar{y} + \sum_{j=1}^{s-1} \delta_j d^{(j)})^T A)$$

for $s = 1, 2, \cdots, \nu$.

By definition, it follows from 3) and 4) that

$$\sigma(\bar{x} + \sum_{j=1}^{\tau} \delta_j r^{(j)}) \text{ and } \sigma(c^T - (\bar{y} + \sum_{j=1}^{\nu} \delta_j d^{(j)})^T A)$$

form the optimal partition for strictly complementary pairs. Now put columns in the sub-tableau corresponding to $r^{(1)}, r^{(2)}, \cdots, r^{(\tau)}$ in the most right part of the sub-tableau with the right-to-left order, and arrange rows in $T(B)$ corresponding to $d^{(1)}, d^{(2)}, \cdots, d^{(\nu)}$ in the upper part of the sub-tableau from top to bottom.

With this construction it is clear that $T(B)$ is a Balinski-Tucker tableau.

□

The goal of the next two algorithms is in fact to find a basis set and extremal directions which can fulfill conditions of Lemma 6. Observe that if we have a vertex optimal solution and another optimal solution, then by connecting them we get a feasible direction for the vertex solution. Purify this direction to the extremal ones we obtain at least one feasible extremal direction. Applying this procedure repeatedly for a sequence of shrinking subproblems we finally get a set of needed extremal directions. Since we are interested in extremal directions induced in one tableau, the basic index must be recorded in the procedure in a consistent way. Another important observation is that finding primal extremal directions and finding dual ones can be done separately, due to the complementarity. The algorithms below are based on these observations. The primal part and the dual part are separately treated in two algorithms to ease the presentation. In the primal algorithm we call an index set I a *circuit* if the columns in A_I form a minimal dependent set, namely by deleting a certain column amongst them, the remaining ones will become independent.

The Primal Algorithm (PA)

- Input: $\bar{x} \in \overset{o}{\mathcal{F}}_P$.
- Output: A classification of indices in $\sigma(\bar{x})$ to either B or N; and a set of primal extremal directions $r^{(l)}$.

Step 0 Use Megiddo's algorithm to obtain a primal vertex optimal solution x^*.

Denote $I := \sigma(\bar{x})$ and $d := \bar{x} - x^*$.

Let $B := \{i : i \in \sigma(x^*)\}$, $N := \emptyset$ and $l := 1$.

Step 1 If $I = B \cup N$, stop.

Step 2 Let $K := I \setminus (B \cup N)$ and $d' := d$.

Step 3 If $B \cup K$ is not a circuit, go to Step 4.

Let

$$\lambda := \min\{\frac{d_i}{d'_i} : i \in K\}$$

and

$$\begin{aligned} d &:= d - \lambda d' \\ K' &:= \{i : d_i = 0 \text{ and } i \in K\}. \end{aligned}$$

Take any $i_l \in K'$ and let

$$\begin{aligned} r^{(l)} &:= d', \\ N &:= N \cup i_l, \\ B &:= B \cup K \setminus i_l, \\ l &:= l + 1. \end{aligned}$$

Go to Step 1.

Step 4 Find $f \in \mathcal{R}^n$ such that

- $Af = 0$;
- $\pi(f) \subseteq B \cup K$;
- $f_K \not\leq 0$ and f_K independent of d'_K.

Let

$$\lambda := \min\{\frac{d'_i}{f_i} : f_i > 0 \text{ and } i \in K\}$$

and

$$\begin{array}{rcl} d' & := & d' - \lambda f \\ K & := & K \setminus \{i : d'_i = 0\}. \end{array}$$

Go back to Step 3.

Remark 4 We may call a cycle "Step 3 – Step 4 – Step 3" an *inner-loop* and a cycle from Step 1 to the next Step 1 an *outer-loop*. It is clear that the nonbasis set N is monotonically expanding after performing one outer-loop, and the working set K is decreasing during each inner-loop. In any loop, for $i \notin \sigma(x^*)$ we have $d_i \geq 0$, $\sigma(d') \subseteq \sigma(d)$ and $\pi(d') \subseteq B \cup K$ which means that at the beginning of an outer-loop, $d_N = d'_N = 0$. During an inner-loop, $d'_K > 0$. At the end of an outer-loop we have $d_{K'} = 0$. Because at Step 4 f_K is chosen to be independent of d'_K, and so when finishing Step 4 we always have $K \neq \emptyset$. Moreover, because at the end of Step 4 we have $A_{B\cup K} d'_{B\cup K} = 0$ and $d'_K > 0$, so we conclude that columns in the matrix $A_{B\cup K}$ are not independent. At the end of an outer-loop (Step 3) we have $d'_{i_l} > 0$ and this shows that by removing A_{i_l} the remaining columns in $A_{B\cup K}$ are independent. Finally, we remark that when the algorithm is terminated we have $\pi(d) \subseteq \sigma(x^*)$.

Now we prove the following result.

Theorem 4 *For the algorithm (PA) the following hold:*

1). *The algorithm runs in strongly polynomial time;*
2). *All the produced directions $r^{(l)}$ $(1 \leq l \leq \tau)$ are primal extremal directions;*
3). *There exist positive numbers δ_l $(1 \leq l \leq \tau)$ with $1 \gg \delta_1 \gg \delta_2 \gg \cdots \gg \delta_\tau > 0$ such that*

$$x^* + \sum_{l=1}^{t} \delta_l r^{(l)} \in \mathcal{F}_P$$

for $1 \leq t \leq \tau - 1$ and

$$x^* + \sum_{l=1}^{\tau} \delta_l r^{(l)} \in \overset{o}{\mathcal{F}}_P .$$

Proof.

The strong polynomiality of the algorithm is easy to see, because after each inner-loop, the set K strictly expands, and after each outer-loop, the set $B \cup N$ strictly expands. To prove that the directions $r^{(l)}$ $(1 \leq l \leq \tau)$ are extremal directions defined on the same tableau, we will have to check that for every $r^{(l)}$, $\pi(r^{(l)}) \subseteq B \cup i_l$. Notice that we have $d_N = 0$ after performing one outer-loop, where N is the set of nonbasic variables so far, and so the elements in N will not contribute as nonzero elements in the later constructed extremal directions. This proves the above statement. The last part of the theorem follows from the relation

$$\chi(r^{(t)}) \subseteq \sigma(x^*) \cup \bigcup_{l=1}^{t-1} \sigma(r^{(l)})$$

which can be proved by induction on t for $t = 1, 2, \cdots, \tau$. Remark that at the end of the t-th outer-loop we have $r_K^{(t)} > 0$ and $\pi(r^{(t)}) \subseteq B \cup K$.

Moreover,

$$\sigma(x^*) \cup \bigcup_{l=1}^{\tau} \sigma(r^{(l)}) = I,$$

and so the theorem is proved.

□

The second algorithm treats the dual part. It works in a similar way as its primal counterpart.

The Dual Algorithm (DA)

- Input: $\bar{y} \in \overset{o}{\mathcal{F}}_D$.
- Output: A classification of indices in $\sigma(c^T - \bar{y}^T A)$ to either B or N; and a set of dual extremal directions $d^{(l)}$.

Step 0 Use Megiddo's algorithm to obtain a dual vertex optimal solution y^*.
Denote $J := \sigma(c^T - \bar{y}^T A)$, $I := \{1, 2, \cdots, n\} \setminus J$ and $z := y^* - \bar{y}$.
Let $N := \{j : \ j \in \sigma(c^T - y^{*T} A)\}$, $B := \emptyset$ and $l := 1$.

Step 1 If $rank\{A_{I \cup B}\} = m$, let $N := J \setminus B$, stop.

Step 2 Let $K := \emptyset$ and $z' := z$.

Step 3 If $rank\{A_{I \cup B \cup K}\} < m - 1$, go to Step 4.
Let

$$\lambda := \min\{\frac{z^T A_j}{z'^T A_j} : j \in J \setminus (B \cup N \cup K)\}$$

and

$$\begin{aligned} z &:= z - \lambda z' \\ K' &:= \{i : \ z^T A_j = 0 \text{ and } i \in J \setminus (B \cup N \cup K)\}. \end{aligned}$$

Take any $j_l \in K'$ and let

$$\begin{aligned} d^{(l)} &:= -z', \\ B &:= B \cup j_l, \\ N &:= N \cup K \setminus j_l, \\ l &:= l + 1. \end{aligned}$$

Go to Step 1.

Step 4 Find $u \in \mathcal{R}^m$ such that

- $u^T A_{I \cup B \cup K} = 0$;
- $u^T A_{J \setminus N} \not\leq 0^T$ and $u^T A_{J \setminus N}$ independent of $z'^T A_{J \setminus N}$.

Let

$$\lambda := \min\{\frac{z'^T A_j}{u^T A_j} : u^T A_j > 0 \text{ for } j \in J \setminus N\}$$

and

$$\begin{aligned} z' &:= z' - \lambda u \\ K &:= K \setminus \{j : z'^T A_j = 0\}. \end{aligned}$$

Go back to Step 3.

Remark 5 Similar to Algorithms (PA), now the basis set B is strictly expanding in an outer-loop and K is strictly expanding in an inner-loop. Because $u \neq 0$ and $u^T A_{I \cup B \cup K} = 0$ at the end of one inner-loop, so that $rank\{A_{I \cup B \cup K}\} < m$ after one inner-loop. Other properties of Algorithm (PA) also hold here in a parallel way (see Remark 4).

Now we have the similar result as for Algorithm (PA).

Theorem 5 *For the algorithm (DA) the following hold:*

1). The algorithm runs in strongly polynomial time;

2). All the produced directions $d^{(l)}$ $(1 \leq l \leq \nu)$ are dual extremal directions;

3). There exist positive numbers ϵ_l $(1 \leq l \leq \nu)$ with $1 \gg \epsilon_1 \gg \epsilon_2 \gg \cdots \gg \epsilon_\nu > 0$ such that

$$y^* + \sum_{l=1}^{s} \epsilon_l d^{(l)} \in \mathcal{F}_D$$

for $1 \leq s \leq \nu - 1$ and

$$y^* + \sum_{l=1}^{\nu} \epsilon_l d^{(l)} \in \overset{o}{\mathcal{F}}_D \ .$$

Proof.
Similar to the proof of Theorem 4, and is omitted here.

□

Remark 6 We remark here that the new algorithms can be applied to find "lexicographically feasible" extremal rays starting from any optimal pair of solutions (i.e. not necessary strictly complemementary pairs). The advantage of having a pair of primal- and dual- optimal points in the relative interiors is that a complete set of extremal directions will be found.

Finally, we present the combined algorithm to get a Balinski-Tucker tableau.

The Primal-Dual Algorithm (P&D)

- Input: $\bar{x} \in \overset{o}{\mathcal{F}}_P$ and $\bar{y} \in \overset{o}{\mathcal{F}}_D$.
- Output: A Balinski-Tucker Tableau.

Step 1 Apply Megiddo's algorithm to get the primal basic optimal solution x^* and the dual basic optimal solution y^*.

Step 2 Apply the primal algorithm (PA) to get a partition of variables in $\sigma(\bar{x})$ to either B or N, and to get a set of extremal directions $r^{(l)}$ $(1 \le l \le \tau)$.

Step 3 Apply the dual algorithm (DA) to get a partition of variables in $\sigma(c^T - \bar{y}^T A)$ to either B or N, and to get a set of extremal directions $d^{(l)}$ $(1 \le l \le \nu)$.

Step 4 Construct the tableau based on the basis B obtained at Step 2 and 3, and using the procedure described in the proof of Lemma 6 to arrange and get a Balinski-Tucker type tableau.

Summarizing Theorem 3, Theorem 4, Theorem 5 and Lemma 6, we have the following main theorem of this paper.

Theorem 6 *The algorithm (P&D) yields a Balinski-Tucker tableau in strongly polynomial time.*

Proof.
The theorem follows immediately from Theorem 3, Theorem 4, Theorem 5 and Lemma 6. Notice also that B is indeed a basis, because columns in A_B are independent according to the primal algorithm, and $rank\{A_B\} = m$ according to the dual algorithm. The extremal directions so derived satisfy the conditions required by Lemma 6.

□

5 Conclusions

In this paper we showed that if we have a pair of optimal primal and dual solutions satisfying the complementary slackness relation strictly, then we can construct a certain kind of optimal tableau which contains at least the information about the dimension of the optimal faces and the lexicographically feasible extremal directions. The construction is done in strongly polynomial time. Since linear programming is not yet known to be solvable in strongly polynomial time, this shows that optimal solutions can be further classified according to the information carried.

The algorithms can also be used for other purposes, e.g., to find the dimension of the optimal faces.

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