New complexity results for the Iri-Imai method

Jos F. Sturm Shuzhong Zhang

June 1993

Abstract

In this paper we show that the number of main iterations required by the Iri-Imai algorithm to solve a linear programming problem is $\mathcal{O}(nL)$. Moreover, we show that a modification of this algorithm requires only $\mathcal{O}(\sqrt{n}L)$ main iterations. In this modification we measure progress by means of the primal-dual potential function.

Key words: Linear Programming, Iri-Imai method, primal-dual potential function.

1 Introduction

Since Karmarkar [12] showed in 1984 that his interior point method solves linear programming problems in polynomial time, a lot of research has been devoted to interior point methods. This research can be categorized according to the type of interior point method analyzed. In particular, we can distinguish affine scaling methods, path-following methods and potential reduction methods. The affine scaling method was introduced as early as in 1967 by Dikin [2]. That method searches along a scaled steepest descent direction of the objective function and therefore is of a greedy nature. Currently, no pure affine scaling algorithm has been proved to be polynomial. In fact, the convergence of the large-step affine scaling method (without any non-degeneracy assumptions) has been proved only recently by Tsuchiya and Muramutsu [15]. The path-following method was first studied by

Huard [8] in 1967. The method is less greedy in terms of improving the objective since it only searches for approximate analytical centers of a sequence of shrinking subsets of the feasible region that contain the optimal point. The objective function, however, is monotonically improving along this sequence of analytical centers. Polynomiality of a path-following algorithm has first been proved by Renegar [13] in 1988. Karmarkar's method is called a potential reduction method because it minimizes a (quasi-convex) potential function which is a composition of objective as well as constraint functions. Potential reduction methods are not of greedy type with respect to the objective function, since reducing the potential function locally can even result in a worse objective value. In the long run, however, the objective will be improved. The potential function ϕ is defined by

$$\phi(x, z) := q \ln(z - b^{\mathrm{T}} x) - \sum_{i=1}^{n} \ln(c_i - a_i^{\mathrm{T}} x),$$

where $q \geq n$, and z is an upper bound on the optimal value of (D): $\max\{b^{\mathrm{T}} x : A^{\mathrm{T}} x \leq c\}$.

With respect to the potential function, an important observation was made by Iri and Imai [10], when they showed that the exponential of Karmarkar's potential function, the Iri-Imai function, is strictly convex in x if $q \geq n+1$. The method of Iri and Imai is to minimize the Iri-Imai function by means of Newton's method. A polynomial bound on the number of main iterations required by this method was first proved by Zhang and Shi [17] in 1988. The iteration bound obtained by Zhang and Shi was $\mathcal{O}(n^8 L)$ and was reduced to $\mathcal{O}(n^2 L)$ by Imai [9] in 1991. Iri [11] obtained an $\mathcal{O}(nL)$ iteration bound for the case in which l is a multiple of n, where $l := q - n$. In 1992, Zhang [18] extended Iri's result to the smooth convex programming case.

For the potential reduction approach Ye [16] made an important contribution by showing that his primal-dual potential reduction method requires only $\mathcal{O}(\sqrt{n}L)$ main iterations to solve linear programming problems.

In this paper we show that the Iri-Imai method actually requires $\mathcal{O}(qL)$ iterations for any $q \geq n + 2$. Moreover, we show that when $l = q - n = \sqrt{n + 3}$, and a certain updating scheme of upper-bounds on the optimal objective value is used, the number of main iterations required reduces to $\mathcal{O}(\sqrt{n}L)$ too.

This paper is organized as follows. At the end of this section, we present a glossary of symbols used in this paper for easy reference. In Section 2 we try to get the reader acquainted with the ideas behind potential reduction methods in general and the Iri-Imai method in particular. In Section 3, the behavior of relevant quantities along the Newton direction of the Iri-Imai function is analyzed. In addition, the Iri-Imai algorithm is described and the $\mathcal{O}(nL)$ iteration bound is proven for it. In Section 4, we present our primal-dual version of Iri and Imai's method, and prove the number of main iterations needed by the algorithm thus obtained to be $\mathcal{O}(\sqrt{n}L)$. We conclude the paper in Section 5.

List of symbols:

A, b, c, x, λ	data and variables for (P) and (D)
$\mathcal{F}$,$\mathcal{F}^0$	$\{x \mid A^{\mathrm{T}}x \le c\}, \{x \mid A^{\mathrm{T}}x < c\}$
Φ, ϕ, ψ	Iri-Imai, dual, and primal-dual potential functions
q, z, l	parameters of the potential functions Φ,ϕ and ψ
d	$-\nabla^2\Phi(x)^{-1}\nabla\Phi(x)$
a_i	the i-th column of A, $A = [a_1, a_2, \cdots, a_n]$
$\bar{a}_i$	$\frac{a_i^{\mathrm{T}}d}{c_i - a_i^{\mathrm{T}}x}$
$\bar{a}$	$\sum_{i=1}^n \bar{a}_i/n$
$\bar{b}$	$\frac{b^{\mathrm{T}}d}{z - b^{\mathrm{T}}x}$
h	$\sqrt{-\nabla\phi(x,z)^{\mathrm{T}}d}$
e	$[1, 1, \cdots, 1]^{\mathrm{T}}$
e_i	the i-th column of the identity matrix
I_n	identity matrix of order n, $I_n = [e_1, e_2, \cdots, e_n]$
r_k	$q\bar{b}^k - \sum_{i=1}^n \bar{a}_i^k$
s	$\sqrt{\sum_{i=1}^n (\bar{a}_i - \bar{b})^2}$
z^*	optimal objective value, $z^* = \max_{x\in\mathcal{F}} b^{\mathrm{T}}x$

2 Preliminaries

Consider the linear programming problem in the dual form

$$(D) \quad \begin{array}{ll} \text{maximize} & b^{\mathrm{T}}x \\ \text{subject to} & A^{\mathrm{T}}x \le c \end{array}$$

where A is an $m \times n$ matrix, b and c are m- and n-dimensional vectors respectively, and $x \in \mathcal{R}^m$ is the decision variable.

Let $\mathcal{F}$ be the feasible region, $\mathcal{F} := \{x \mid A^{\mathrm{T}}x \leq c\}$ with interior $\mathcal{F}^0 := \{x \mid A^{\mathrm{T}}x < c\}$ and let z be an upper bound on the optimal value of (D), i.e. $b^{\mathrm{T}}x \leq z$ for all $x \in \mathcal{F}$. We will make the following assumptions.

Assumption 1 (Slater condition)

$$\mathcal{F}^0 \neq \emptyset.$$

Assumption 2 *$\mathcal{F}$ is bounded.*

Further assume that an initial interior point $x^0 \in \mathcal{F}^0$ is available.

A potential reduction method tries to construct the next iterative interior solution x' based on an interior point $x \in \mathcal{F}^0$, in such a way that a certain reduction can be guaranteed in the value of the potential function ϕ, which is defined by

$$\phi(x, z) := q \ln(z - b^{\mathrm{T}}x) - \sum_{i=1}^{n} \ln(c_i - a_i^{\mathrm{T}}x)$$

where $q \geq n$, z is an upper bound on the optimal value of (D). Note that the function $-\sum_{i=1}^{n} \ln(c_i - a_i^{\mathrm{T}}x)$ is Huard's center function [8]. The minimizer of this center function, which can be shown to be unique under Assumptions 1 and 2, is called the analytic center of $\mathcal{F}$ [13, 14]. The concept of potential function was introduced by Karmarkar [12] in 1984 for linear programming in the primal form. As ϕ is only defined for $x \in \mathcal{F}^0$, ϕ is also called a barrier function. By reducing the potential function ϕ by at least a constant amount at each iteration, the gap $z - b^{\mathrm{T}}x$ will decrease in the long run also, because the center function is bounded from below by its value in the analytic center of $\mathcal{F}$.

Lemma 1 *Suppose Assumptions 1 and 2 are satisfied, and an initial interior point x^0, and an initial upper bound z^0 have been generated in such a way that $\phi(x^0, z^0) = \mathcal{O}(qL)$. If an algorithm reduces the potential function $\phi(x, z)$ by at least δ at each iteration, then the algorithm can be used to find an optimal solution of (D) in $\mathcal{O}(qL/\delta)$ iterations.*

Proof:

See Gonzaga [5], Lemma 2.1.

□

This paper does not deal with the question how to generate (x^0,z^0); we refer the interested reader to Anstreicher [1] and Freund [4]. To simplify the notation, we will often use $\phi(x)$ to denote $\phi(x,z)$ in this and the next section when there is no confusion.

Lemma 1 shows that by minimizing the potential function, one would solve (D). Up till now, it is by no means clear, however, how to minimize $\phi(x)$. Especially it does not make much sense to apply Newton's method, as $\phi(x)$ happens to be non-convex. Instead of minimizing ϕ itself, we will minimize in this paper the Iri-Imai potential function Φ, which is a monotone transformation of ϕ. The Iri-Imai potential function is

$$\Phi(x) := e^{\phi(x)} = \frac{(z - b^{\mathrm{T}}x)^q}{\prod_{i=1}^n (c_i - a_i^{\mathrm{T}}x)}$$

where $q = n + l$.

In the remaining of this section, we will prove that Φ is strictly convex on $\mathcal{F}^0$ if $l > 1$.

We first note, since $\Phi > 0$ on $\mathcal{F}^0$ and $\phi(x) = \ln(\Phi(x))$, that

$$\nabla\phi(x) = \frac{\nabla\Phi(x)}{\Phi(x)}$$

$$\nabla^2\phi(x) = -\frac{\nabla\Phi(x)\nabla\Phi(x)^{\mathrm{T}}}{\Phi(x)^2} + \frac{\nabla^2\Phi(x)}{\Phi(x)}$$

i.e.

$$\frac{\nabla^2\Phi(x)}{\Phi(x)} = \nabla\phi(x)\nabla\phi(x)^{\mathrm{T}} + \nabla^2\phi(x). \tag{1}$$

Because $\phi(x) = q\ln(z - b^{\mathrm{T}}x) - \sum_{i=1}^n \ln(c_i - a_i^{\mathrm{T}}x)$ we have

$$\nabla\phi(x) = -q\frac{b}{z - b^{\mathrm{T}}x} + \sum_{i=1}^n \frac{a_i}{c_i - a_i^{\mathrm{T}}x}$$

and

$$\nabla^2\phi(x) = \sum_{i=1}^n \frac{a_i a_i^{\mathrm{T}}}{(c_i - a_i^{\mathrm{T}}x)^2} - q\frac{bb^{\mathrm{T}}}{(z - b^{\mathrm{T}}x)^2}.$$

It is well known that $n\sum_{i=1}^n y_i^2 \geq \left(\sum_{i=1}^n y_i\right)^2$ for any $y \in \mathcal{R}^n$. We now prove a somewhat stronger statement in the following lemma to proceed the analysis.

Lemma 2 *For any $\alpha \geq 0, y \in \mathcal{R}^n$ it holds that*

$$(n+\alpha)\sum_{i=1}^n y_i^2 \geq \left(\sum_{i=1}^n y_i\right)^2 + \frac{\alpha(n+\alpha)}{\alpha+1}\max_{1\leq i\leq n} y_i^2.$$

Proof:

For $\alpha \geq 0$ we will prove the lemma by showing

$$(n+\alpha)I_n - ee^{\mathrm{T}} - \beta e_n e_n^{\mathrm{T}} \succeq 0 \Leftrightarrow \beta \leq \frac{\alpha(n+\alpha)}{\alpha+1} \tag{2}$$

where $A \succeq B$ means that $B - A$ is a positive semi-definite matrix.

Note that a matrix is positive semi-definite iff all of its eigenvalues are nonnegative. Clearly, the matrix $A_j := (n+\alpha)I_j - ee^{\mathrm{T}}$ is semi-positive definite for any $j \leq n$ and $\alpha \geq 0$ since it has $j-1$ positive eigenvalues $n+\alpha$ and one nonnegative eigenvalue $n+\alpha-j$. As the determinant is the product of the eigenvalues, we have

$$\det(A_j) = (n+\alpha-j)(n+\alpha)^{j-1}.$$

Therefore, the semi-positive definiteness of $A_n - \beta e_n e_n^{\mathrm{T}}$ is determined solely by the sign of $\det(A_n - \beta e_n e_n^{\mathrm{T}})$.

Finally, notice that the difference between A_n and $A_n - \beta e_n e_n^{\mathrm{T}}$ exists only in one position and the last column of $A_n - \beta e_n e_n^{\mathrm{T}}$ can be written as $e - \beta e_n$. This shows that $\det(A_n - \beta e_n e_n^{\mathrm{T}}) = \det(A_n) - \beta\det(A_{n-1})$, and so $\det(A_n - \beta e_n e_n^{\mathrm{T}}) \geq 0$ iff

$$\beta \leq \frac{\det(A_n)}{\det(A_{n-1})} = \frac{\alpha(n+\alpha)}{\alpha+1}$$

which proves equation (2). By pre-multiplying by y^{T} and post-multiplying by y in (2) the lemma follows.

□

Now we are ready to give an alternative proof for a result which was obtained by Iri and Imai [10]. The new proof technique will be used in the analysis given in the next section.

Lemma 3 *The Hessian matrix* $\nabla^2\Phi(x)$ *is positive definite for any* $x \in \mathcal{F}^0$ *if* $l > 1$.

Proof:

(See also Iri and Imai [10]).

Let $v \in \mathcal{R}^m$ and $x \in \mathcal{F}^0$ be arbitrary. Then we have, using (1),

$$\begin{aligned} v^{\mathrm{T}}\frac{\nabla^2\Phi(x)}{\Phi(x)}v &= (q^2-q)\frac{(b^{\mathrm{T}}v)^2}{(z-b^{\mathrm{T}}x)^2} \\ &\quad -2q\frac{b^{\mathrm{T}}v}{z-b^{\mathrm{T}}x}\sum_{i=1}^{n}\frac{a_i^{\mathrm{T}}v}{c_i-a_i^{\mathrm{T}}x}+\left(\sum_{i=1}^{n}\frac{a_i^{\mathrm{T}}v}{c_i-a_i^{\mathrm{T}}x}\right)^2 \\ &\quad +\sum_{i=1}^{n}\frac{(a_i^{\mathrm{T}}v)^2}{(c_i-a_i^{\mathrm{T}}x)^2}. \end{aligned}$$

Therefore, we obtain for any $j \in \{1, 2, \cdots, n\}$,

$$\begin{aligned} v^{\mathrm{T}}\frac{\nabla^2\Phi(x)}{\Phi(x)}v &= \sum_{i=1}^{n}\frac{(a_i^{\mathrm{T}}v)^2}{(c_i-a_i^{\mathrm{T}}x)^2}-\frac{1}{q-1}\left(\sum_{i=1}^{n}\frac{a_i^{\mathrm{T}}v}{c_i-a_i^{\mathrm{T}}x}\right)^2 \\ &\quad +\left(\sqrt{q^2-q}\frac{b^{\mathrm{T}}v}{z-b^{\mathrm{T}}x}-\sqrt{\frac{q}{q-1}}\sum_{i=1}^{n}\frac{a_i^{\mathrm{T}}v}{c_i-a_i^{\mathrm{T}}x}\right)^2 \\ &\geq \sum_{i=1}^{n}\frac{(a_i^{\mathrm{T}}v)^2}{(c_i-a_i^{\mathrm{T}}x)^2}-\frac{1}{q-1}(\sum_{i=1}^{n}\frac{a_i^{\mathrm{T}}v}{c_i-a_i^{\mathrm{T}}x})^2 \\ &\geq \frac{l-1}{l}\frac{(a_j^{\mathrm{T}}v)^2}{(c_j-a_j^{\mathrm{T}}x)^2} \geq 0 \qquad (3) \end{aligned}$$

where the last inequality follows using Lemma 2. Since v is arbitrary, this shows that $\nabla^2\Phi(x)$ is positive semi-definite. To prove it is actually positive definite, we assume to the contrary that there exists a $v \neq 0$ such that $v^{\mathrm{T}}\frac{\nabla^2\Phi(x)}{\Phi(x)}v = 0$. Then we have for any $t \in \mathcal{R}$, and for any $j \in \{1, 2, \cdots, n\}$,

$$\frac{(ta_j^{\mathrm{T}}v)^2}{(c_j-a_j^{\mathrm{T}}x)^2} \leq \frac{lt^2}{l-1}v^{\mathrm{T}}\frac{\nabla^2\Phi(x)}{\Phi(x)}v = 0.$$

which contradicts the boundedness of $\mathcal{F}$ (Assumption 2).

□

Knowing that the Hessian matrix $\nabla^2\Phi$ is positive definite on $\mathcal{F}^0$, it clearly makes sense to analyze the Newton direction d given by

$$d := -\nabla^2\Phi(x)^{-1}\nabla\Phi(x).$$

This analysis will be carried out in the next section.

3 Along the Newton direction

To simplify the analysis, we introduce the quantity

$$\bar{b} := \frac{b^{\mathrm{T}}d}{z - b^{\mathrm{T}}x},$$

which is the change in objective value caused by a unit Newton step d relative to the gap $z - b^{\mathrm{T}}x$. In addition, we define $\bar{a}_j$ to be the relative change in the slack of constraint j,

$$\bar{a}_j := \frac{a_j^{\mathrm{T}}d}{c_j - a_j^{\mathrm{T}}x} \text{ for } j = 1, 2, \cdots, n$$

and finally h^2 is defined as a first order approximation of the reduction in potential function value along d,

$$h^2 := -\nabla\phi(x)^{\mathrm{T}}d = q\bar{b} - \sum_{i=1}^{n}\bar{a}_i.$$

Due to equation (1) it follows that

$$h^2 = h^4 - q\bar{b}^2 + \sum_{i=1}^{n}\bar{a}_i^2.$$

The following lemma estimates the change in slacks along d. For technical reasons we assume from now on that $l \geq 2$.

Lemma 4 (Change in slacks) *The quantity h^2 can be bounded from above by*

$$h^2 \leq \frac{l}{l-1}$$

and the relative change in slack values along d can be estimated as

$$\max_{1\leq j\leq n} \bar{a}_j^2 \leq \frac{l+3}{l}\left(h^2 - \frac{2l-3}{2l}h^4\right) \leq \frac{l+3}{2(2l-3)}.$$

Proof:

Let $\frac{1}{l} \leq \lambda \leq 1$.

Because $h^4 = (q\bar{b} - \sum_{i=1}^n \bar{a}_i)^2$ we have

$$
\begin{aligned}
d^{\mathrm{T}} \frac{\nabla^2 \Phi(x)}{\Phi(x)} d &= h^4 - q\bar{b}^2 + \sum_{i=1}^n \bar{a}_i^2 - \lambda(h^4 - (q\bar{b} - \sum_{i=1}^n \bar{a}_i)^2) \\
&= (1-\lambda)h^4 + (\lambda q - 1)q\bar{b}^2 \\
&\quad -2\lambda q\bar{b} \sum_{i=1}^n \bar{a}_i + \lambda(\sum_{i=1}^n \bar{a}_i)^2 + \sum_{i=1}^n \bar{a}_i^2. \qquad (4)
\end{aligned}
$$

Therefore, we obtain for any $j \in \{1, 2, \cdots, n\}$,

$$
\begin{aligned}
d^{\mathrm{T}} \frac{\nabla^2 \Phi(x)}{\Phi(x)} d &= (1-\lambda)h^4 + \sum_{i=1}^n \bar{a}_i^2 - \frac{\lambda}{\lambda q - 1}(\sum_{i=1}^n \bar{a}_i)^2 \\
&\quad + \left(\sqrt{\lambda q^2 - q}\bar{b} - \lambda\sqrt{q/(\lambda q - 1)} \sum_{i=1}^n \bar{a}_i \right)^2 \\
&\geq (1-\lambda)h^4 + \sum_{i=1}^n \bar{a}_i^2 - \frac{\lambda}{\lambda q - 1}(\sum_{i=1}^n \bar{a}_i)^2 \\
&\geq (1-\lambda)h^4 + \frac{l\lambda - 1}{l\lambda + \lambda - 1}\bar{a}_j^2 \qquad (5)
\end{aligned}
$$

where the last inequality follows using Lemma 2.

By choosing $\lambda = 1/l$ in (5) it follows that $h^2 \geq \frac{l-1}{l}h^4$, or equivalently,

$$h^2 \leq \frac{l}{l-1}$$

which proves the first part of the lemma (see also Iri [11], equation (3.24)).

Moreover, from (5) we obtain

$$
\begin{aligned}
\bar{a}_j^2 &\leq \min_{1/l < \lambda \leq 1} \frac{l\lambda + \lambda - 1}{l\lambda - 1}(h^2 - (1-\lambda)h^4) \\
&\leq \frac{l+3}{l}(h^2 - \frac{2l-3}{2l}h^4) \leq \frac{l+3}{2(2l-3)}
\end{aligned}
$$

where the second inequality follows by letting $\lambda = \frac{3}{2l}$.

□

The following lemma gives a bound on the change in objective value along the Newton direction d by means of bounding the quantity $s := \sqrt{\sum_{i=1}^n (\bar{a}_i - \bar{b})^2}$ which will prove useful in the subsequent analysis as well.

Lemma 5 *Let $s := \sqrt{\sum_{i=1}^n (\bar{a}_i - \bar{b})^2}$. Then,*

$$s^2 = (1 + 2\bar{b})h^2 - h^4 - l\bar{b}^2 \le h^2 - \frac{l-1}{l} h^4 \le \frac{1}{4}\frac{l}{l-1}.$$

Moreover, we have

$$\bar{b} = \frac{h^2}{l} \pm \sqrt{\frac{1}{l}(h^2 - \frac{l-1}{l} h^4 - s^2)}$$

and

$$|\, \bar{b} \,| < \frac{1}{l-1} + \sqrt{\frac{1}{2(l-1)}}.$$

Proof:

We first notice that

$$\begin{aligned} s^2 &= \sum_{i=1}^n (\bar{a}_i - \bar{b})^2 \\ &= \sum_{i=1}^n \bar{a}_i^2 + n\bar{b}^2 - 2\bar{b}\sum_{i=1}^n \bar{a}_i \\ &= (h^2 - h^4 + q\bar{b}^2) + n\bar{b}^2 + 2\bar{b}(h^2 - q\bar{b}) \\ &= (1 + 2\bar{b})h^2 - h^4 - l\bar{b}^2 \end{aligned}$$

which proves the first part of the lemma. Now solving for $\bar{b}$ the above derived identity

$$l\bar{b}^2 - 2h^2\bar{b} + s^2 - h^2 + h^4 = 0$$

yields

$$\bar{b} = \frac{h^2}{l} \pm \sqrt{\frac{1}{l}(h^2 - \frac{l-1}{l} h^4 - s^2)}$$

and consequently,

$$|\, \bar{b} \,| \le \frac{h^2}{l} + \sqrt{\frac{1}{l}(h^2 - \frac{l-1}{l} h^4)} < \frac{1}{l-1} + \sqrt{\frac{1}{2(l-1)}}.$$

□

Clearly $(\bar{a}_i - \bar{b})^2 \leq s^2$ for any $1 \leq i \leq n$ and so we have for any $j \geq 2$

$$\sum_{i=1}^{n} \mid \bar{a}_i - \bar{b} \mid^j \leq s^{j-2} \sum_{i=1}^{n} (\bar{a}_i - \bar{b})^2 = s^j \tag{6}$$

where the equality follows from the definition of s.

In view of Lemma 1, we want to estimate the improvement in potential function that can be obtained by performing a line search along the Newton direction d. In order to do so, we will use the power series expansion of the potential function.

It is well known that for any $\mid y \mid < 1$ we have the power series expansion

$$\ln(1-y) = -\sum_{k=1}^{\infty} \frac{y^k}{k}.$$

Let t be such that $t \mid \bar{b} \mid < 1$ and $t \mid \bar{a}_i \mid < 1$ for any $i \in \{1, 2, \cdots, n\}$. Then we have

$$\begin{aligned} \phi(x+td) &= q \ln\left((z - b^{\mathrm{T}}x)(1-t\bar{b})\right) - \sum_{i=1}^{n} \ln((c_i - a_i^{\mathrm{T}}x)(1-t\bar{a}_i)) \\ &= \phi(x) + q\ln(1-t\bar{b}) - \sum_{i=1}^{n} \ln(1-t\bar{a}_i) \\ &= \phi(x) - \sum_{k=1}^{\infty} \frac{q(t\bar{b})^k - \sum_{i=1}^{n}(t\bar{a}_i)^k}{k} \\ &= \phi(x) - \sum_{k=1}^{\infty} \frac{t^k r_k}{k} \end{aligned}$$

where $r_k := q\bar{b}^k - \sum_{i=1}^{n} \bar{a}_i^k$. Therefore, the reduction in potential function value obtained by the step td is

$$\phi(x) - \phi(x+td) = \sum_{k=1}^{\infty} \frac{t^k r_k}{k}. \tag{7}$$

From this equation it is clear that estimating r_k should be interesting. First notice that $r_1 = h^2$ and $r_2 = h^4 - h^2$.

Lemma 6 *Let $r_k := q\bar{b}^k - \sum_{i=1}^{n} \bar{a}_i^k$. Then for any $k \geq 2$ it holds that*

$$\mid r_k \mid \leq (l-1) \mid \bar{b} \mid^k + k(h^2 + l \mid \bar{b} \mid - s) \mid \bar{b} \mid^{k-1} + (s + \mid \bar{b} \mid)^k.$$

Proof:

Using Newton's binomial formula, we have for any $k \geq 2$

$$\begin{aligned} r_k &= q\bar{b}^k - \sum_{i=1}^{n}(\bar{a}_i - \bar{b} + \bar{b})^k \\ &= q\bar{b}^k - \sum_{j=0}^{k}\sum_{i=1}^{n}\begin{pmatrix} k \\ j \end{pmatrix}(\bar{a}_i - \bar{b})^j\bar{b}^{k-j} \\ &= q\bar{b}^k - n\bar{b}^k - k\sum_{i=1}^{n}(\bar{a}_i - \bar{b})\bar{b}^{k-1} - \sum_{j=2}^{k}\sum_{i=1}^{n}\begin{pmatrix} k \\ j \end{pmatrix}(\bar{a}_i - \bar{b})^j\bar{b}^{k-j} \\ &= l\bar{b}^k + k(h^2 - l\bar{b})\bar{b}^{k-1} - \sum_{j=2}^{k}\sum_{i=1}^{n}\begin{pmatrix} k \\ j \end{pmatrix}(\bar{a}_i - \bar{b})^j\bar{b}^{k-j} \end{aligned}$$

which implies

$$\begin{aligned} \mid r_k \mid &\leq \left| l\bar{b}^k + k(h^2 - l\bar{b})\bar{b}^{k-1} \right| + \sum_{j=2}^{k}\sum_{i=1}^{n}\begin{pmatrix} k \\ j \end{pmatrix} \mid \bar{a}_i - \bar{b} \mid^j \mid \bar{b} \mid^{k-j} \\ &\leq \left| l\bar{b}^k + k(h^2 - l\bar{b})\bar{b}^{k-1} \right| + \sum_{j=2}^{k}\sum_{i=1}^{n}\begin{pmatrix} k \\ j \end{pmatrix} s^j \mid \bar{b} \mid^{k-j} \\ &\leq \left| l\bar{b}^k + k(h^2 - l\bar{b})\bar{b}^{k-1} \right| + (s + \mid \bar{b} \mid)^k - \mid \bar{b} \mid^k - ks \mid \bar{b} \mid^{k-1} \\ &\leq (l-1) \mid \bar{b} \mid^k + k(h^2 + l \mid \bar{b} \mid - s) \mid \bar{b} \mid^{k-1} + (s + \mid \bar{b} \mid)^k \end{aligned}$$

where the second inequality follows from (6).

□

The reduction in potential function value by searching along the Newton direction is estimated in the following lemma.

Lemma 7 (reduction by searching along Newton direction) *Let* $t := 0.1$*, then*

$$\phi(x, z) - \phi(x + td, z) > 0.07h^2.$$

Proof:

First note that if $t := 0.1$ then it follows using Lemma 4 and Lemma 5

$$t^2\bar{a}_j^2 \leq (0.1)^2\frac{l+3}{2(2l-3)} < 1 \text{ and } t \mid \bar{b} \mid < 0.1\left(\frac{1}{l-1} + \sqrt{\frac{1}{2(l-1)}}\right) < 1$$

which shows that td is a feasible step. Therefore, we can use (7) to obtain

$$\begin{aligned}
\phi(x,z)-\phi(x+td,z) &= \sum_{k=1}^{\infty}\frac{t^k r_k}{k} \\
&= th^2+\frac{t^2}{2}(h^4-h^2)+\sum_{k=3}^{\infty}\frac{t^k r_k}{k} \\
&\geq \frac{2-t+th^2}{2}th^2 \\
&\quad -\sum_{k=3}^{\infty}\frac{(l-1)\mid\bar b\mid^k+k(h^2+l\mid\bar b\mid-s)\mid\bar b\mid^{k-1}}{k}t^k \\
&\quad -\sum_{k=3}^{\infty}\frac{(s+\mid\bar b\mid)^k}{k}t^k \\
&= \frac{2-t+th^2}{2}th^2-(l-1)t^3\mid\bar b\mid^3\sum_{k=0}^{\infty}\frac{\mid t\bar b\mid^k}{k+3}- \\
&\quad -(h^2+l\mid\bar b\mid-s)t^3\bar b^2\sum_{k=0}^{\infty}\mid t\bar b\mid^k \\
&\quad -t^3(s+\mid\bar b\mid)^3\sum_{k=0}^{\infty}\frac{t^k(s+\mid\bar b\mid)^k}{k+3} \\
&\geq \frac{2-t+th^2}{2}th^2-\frac{(l-1)t^3\mid\bar b\mid^3}{3(1-t\mid\bar b\mid)} \\
&\quad -\frac{(h^2+l\mid\bar b\mid-s)t^3\bar b^2}{1-t\mid\bar b\mid}-\frac{t^3(s+\mid\bar b\mid)^3}{3(1-t(s+\mid\bar b\mid))}. \qquad (8)
\end{aligned}$$

From Lemma 5 we know that $s^2=(1+2\bar b)h^2-h^4-l\bar b^2$. Hence

$$\begin{aligned}
(s+\mid\bar b\mid)^2 &= s^2+2s\mid\bar b\mid+\bar b^2 \\
&= h^2-h^4+2h^2\bar b+2s\mid\bar b\mid-(l-1)\bar b^2 \\
&\leq h^2-h^4+(2(h^2+s)-(l-1)\mid\bar b\mid)\mid\bar b\mid \\
&\leq h^2-h^4+\frac{(h^2+s)^2}{l-1} \\
&\leq h^2-h^4+2\frac{h^4+s^2}{l-1} \\
&\leq \frac{l+1}{l-1}h^2-(1-\frac{2}{l(l-1)})h^4
\end{aligned}$$

$$\leq \frac{l+1}{l-1}h^2,$$

where the fourth inequality follows by noticing $s^2 \leq h^2 - \frac{l-1}{l}h^4$.

Clearly, $t^2(s+ \mid \bar{b} \mid)^2 \leq 0.03h^2 \leq 0.06$. Here notice that we assume $l \geq 2$ and so $h^2 \leq 2$. It follows then $t(s+ \mid \bar{b} \mid) < 0.3$. Therefore,

$$\frac{t^3(s+ \mid \bar{b} \mid)^3}{3(1-t(s+ \mid \bar{b} \mid))} < \frac{0.3 \times 0.03h^2}{3(1-0.3)} < 0.005h^2. \quad (9)$$

From Lemma 5 we know that $\mid \bar{b} \mid < \frac{1}{l-1} + \sqrt{\frac{1}{2(l-1)}}$, which implies

$$t \mid \bar{b} \mid < 0.1(1 + \frac{\sqrt{2}}{2}) < 0.2.$$

Moreover,

$$\bar{b}^2 \leq 2(\frac{h^4}{l^2} + (\frac{h^2}{l} - \frac{(l-1)h^4}{l^2})) \leq \frac{2h^2}{l}.$$

So,

$$\frac{(l-1)t^3 \mid \bar{b} \mid^3}{3(1-t \mid \bar{b} \mid)} < \frac{(l-1) \times 0.2 \times (0.1)^2 \times \frac{2h^2}{l}}{3(1-0.2)} < 0.002h^2 \quad (10)$$

and

$$\begin{aligned}\frac{(h^2 + l \mid \bar{b} \mid -s)t^3\bar{b}^2}{1-t \mid \bar{b} \mid} &< \frac{t^3 \mid \bar{b} \mid^2 h^2}{1-0.2} + \frac{lt^3 \mid \bar{b} \mid^3}{1-0.2} \\ &< \frac{0.004h^2}{0.8} + \frac{0.004h^2}{0.8} \\ &= 0.01h^2. \quad (11)\end{aligned}$$

Combining (9),(10) and (11) with (8), we get

$$\begin{aligned}\phi(x,z) - \phi(x+td,z) &\geq \frac{2-t+th^2}{2}th^2 - 0.017h^2 \\ &> 0.07h^2.\end{aligned}$$

□

From Lemma 7, we know that for a certain step length along the Newton direction, the reduction in potential function value will be at least $0.07h^2$. If the upper bound z is fixed to the optimal value $z^* := \max_{x \in \mathcal{F}} b^{\mathrm{T}}x$, this implies a constant reduction in potential function value, due to the following lemma by Iri [11].

Lemma 8 *If $z = z^*$, then $h^2 \geq 1/2$.*

Proof:

See Iri [11], inequality (3.15). Iri's proof is based on Cauchy-Schwartz' inequality. As an alternative, we will show that the inequality also follows from a duality relation, see Corollary 1.

□

Combining Lemma 7 and Lemma 8 with Lemma 1, it follows that the following Iri-imai algorithm can be used to solve (D) in $\mathcal{O}(nL)$ iterations.

Algorithm 1 (Iri and Imai's algorithm. Input data: A,b,c,l,x^0,z^*)

Step 0 *Set $i = 0$.*

Step 1 *If $z^* - b^{\mathrm{T}} x^i \leq 2^{-2L}$, stop.*

Step 2 *Set $d = -\nabla^2 \Phi(x^i, z^*)^{-1} \nabla \Phi(x^i, z^*)$.*

Step 3 *Compute x^{i+1} by minimizing $\phi(x, z^*)$ along d.*

Step 4 *Set $i = i + 1$ and return to Step 1.*

Theorem 1 *For any $l \geq 2$, Algorithm 1 solves (D) in $\mathcal{O}((n + l)L)$ main iterations.*

Proof:

Combining Lemma 7 and Lemma 8 with Lemma 1, the theorem follows.

□

4 Primal–dual potential reduction

So far the Iri-Imai method, like the original Karmarkar method, does not include both primal and dual variables. In this section we will modify the Iri-Imai method to incorporate a primal–dual potential function and so it becomes a primal–dual potential reduction method.

More specifically, we shall now try to construct the next iterative interior solution x', or update the variable λ, based on the current x and λ, in such a way that a certain reduction can be guaranteed in the value of the primal–dual potential function ψ defined by

$$\psi(x, \lambda) := q \ln(c^{\mathrm{T}}\lambda - b^{\mathrm{T}}x) - \sum_{i=1}^{n} \ln(c_i - a_i^{\mathrm{T}}x) - \sum_{i=1}^{n} \ln(\lambda_i)$$

where $q = n + l > n$ and x and λ are interior solutions of (D) and (P) respectively (i.e. $x \in \mathcal{F}^0$ and $\lambda > 0, A\lambda = b$).

The primal–dual potential function was introduced by Tanabe in 1987. Unlike Karmarkar's potential function, the primal–dual potential function needs to be reduced by an amount of only $\mathcal{O}(lL)$, compared to $\mathcal{O}(qL)$ in Karmarkar's case, in order to obtain x and λ such that $c^{\mathrm{T}}\lambda - b^{\mathrm{T}}x \leq 2^{-2L}$. Here it is assumed that the initial dual interior point x^0 and the initial upper bound z^0 are generated in such a way that there exists a primal feasible $\lambda^0 > 0$ with $z^0 = c^{\mathrm{T}}\lambda^0$ and $\psi(x^0, \lambda^0) = \mathcal{O}(lL)$. This is because the arithmetic mean inequality

$$\left(\prod_{i=1}^{n} y_i\right)^{1/n} \leq \sum_{i=1}^{n} y_i/n$$

implies

$$\begin{aligned} \sum_{i=1}^{n} \ln(\lambda_i(c_i - a_i x)) &\leq n \ln(\frac{\lambda^{\mathrm{T}}(c - A^{\mathrm{T}}x)}{n}) \\ &= n \ln(c^{\mathrm{T}}\lambda - b^{\mathrm{T}}x) - n \ln(n) \end{aligned}$$

i.e.,

$$\sum_{i=1}^{n} \ln(\frac{\lambda_i(c_i - a_i x)}{(c^{\mathrm{T}}\lambda - b^{\mathrm{T}}x)/n}) \leq 0. \tag{12}$$

By rewriting the primal-dual potential function as

$$\begin{aligned} \psi(x, \lambda) &= l \ln(c^{\mathrm{T}}\lambda - b^{\mathrm{T}}x) + n \ln(n) - \sum_{i=1}^{n} \ln(\frac{\lambda_i(c_i - a_i x)}{(c^{\mathrm{T}}\lambda - b^{\mathrm{T}}x)/n}) \\ &\geq l \ln(c^{\mathrm{T}}\lambda - b^{\mathrm{T}}x) + n \ln(n) \end{aligned}$$

it is clear that if $\psi(x, \lambda) \leq -2lL\ln(2)$, then $c^{\mathrm{T}}\lambda - b^{\mathrm{T}}x \leq 2^{-2L}$, in which case x can be purified to an optimal solution of (D) in only $\mathcal{O}(n^3)$ operations. This proves the following lemma.

Lemma 9 *Let $\delta > 0$ be a fixed quantity. Suppose Assumptions 1 and 2 are satisfied, and an initial interior point $x^0 \in \mathcal{F}^0$ and an initial upper bound z^0 are available satisfying the property that there exist $\lambda^0 > 0$ with $A\lambda^0 = b, c^{\mathrm{T}}\lambda^0 = z^0$ and $\psi(x^0, \lambda^0) = \mathcal{O}(lL)$. If an algorithm reduces the primal–dual potential function $\psi(x, \lambda)$ by at least δ at each iteration, then the algorithm can be used to solve (D) in $\mathcal{O}(\frac{l}{\delta} L)$ iterations.*

Proof:
See Ye [16] or Freund [3].
□

By using Lemma 9, Ye [16] was the first to obtain an $\mathcal{O}(\sqrt{n}L)$-iteration bound for a potential reduction algorithm.

It is easy to see that

$$\nabla_x \psi(x, \lambda) = \nabla \phi(x, c^{\mathrm{T}}\lambda)$$

and

$$\nabla_x^2 \psi(x, \lambda) = \nabla^2 \phi(x, c^{\mathrm{T}}\lambda).$$

Therefore, if $h^2 = \nabla\phi(x, c^{\mathrm{T}}\lambda)^{\mathrm{T}} \nabla^2 \Phi(x)^{-1} \nabla \Phi(x, c^{\mathrm{T}}\lambda) \geq \frac{1}{9}$ it follows from Lemma 7 that the primal–dual potential function can be reduced by an amount of at least 0.007 by doing a line-search in $\mathcal{F}^0$ along d where $d := -\nabla^2\Phi(x)^{-1}\nabla\Phi(x, c^{\mathrm{T}}\lambda)$. We proceed by showing that if $h < \frac{1}{3}$ then it is possible to construct new primal variables λ', in such a way that the primal–dual potential function ψ is again reduced by at least a constant amount, for some special choice of l.

In the derivation of the primal updating formula, we will divide by the quantity $1 - h^2 + \bar{b}$, which is valid for small h due to the following lemma.

Lemma 10 *If $h^2 < 1$ then*

$$1 - h^2 + \bar{b} > 0.$$

Proof:
Suppose that we have $h^2 < 1$ and $1 - h^2 + \bar{b} \leq 0$. This implies that $\bar{b} < 0$. We then get the following impossible inequalities (cf. Lemma 5)

$$0 \leq s^2 = -l\bar{b}^2 + (1 + 2\bar{b})h^2 - h^4 = h^2(1 - h^2 + \bar{b}) + \bar{b}h^2 - l\bar{b}^2 < 0.$$

The lemma is proved by contradiction.

□

From

$$\begin{aligned}\nabla\phi(x,z) &= -\frac{\nabla^2\Phi(x)d}{\Phi(x)} \\ &= -(\nabla\phi(x)^{\mathrm{T}}d)\nabla\phi(x) - \sum_{i=1}^{n}\bar{a}_i\frac{a_i}{c_i - a_i^{\mathrm{T}}x} + q\bar{b}\frac{b}{z - b^{\mathrm{T}}x}\end{aligned}$$

where $-\nabla\phi(x)^{\mathrm{T}}d = h^2$ and $\nabla\phi(x) = -q\frac{b}{z-b^{\mathrm{T}}x} + \sum_{i=1}^{n}\frac{a_i}{c_i - a_i^{\mathrm{T}}x}$, it follows that

$$\begin{aligned}b &= \frac{z - b^{\mathrm{T}}x}{q(1-h^2+\bar{b})}\sum_{i=1}^{n}(1-h^2+\bar{a}_i)\frac{a_i}{c_i - a_i^{\mathrm{T}}x} \\ &= A\lambda' \end{aligned} \tag{13}$$

where we define $\lambda'_1, \lambda'_2, \cdots, \lambda'_n$ as

$$\lambda'_i := \frac{z - b^{\mathrm{T}}x}{q(c_i - a_i^{\mathrm{T}}x)}\frac{1-h^2+\bar{a}_i}{1-h^2+\bar{b}}. \tag{14}$$

Therefore

$$\begin{aligned}c^{\mathrm{T}}\lambda' - b^{\mathrm{T}}x &= (c^{\mathrm{T}} - x^{\mathrm{T}}A)\lambda' \\ &= \frac{n(1-h^2) + \sum_{i=1}^{n}\bar{a}_i}{q(1-h^2+\bar{b})}(z - b^{\mathrm{T}}x)\end{aligned}$$

$$= \frac{n(1-h^2+\bar{a})}{q(1-h^2+\bar{b})}(z - b^{\mathrm{T}}x) \tag{15}$$

where we define $\bar{a} := \sum_{i=1}^{n}\bar{a}_i/n$ to simplify notations.

Using relation (15), it follows from (14), for $i = 1, 2, \cdots, n$, that

$$\frac{\lambda'_i(c_i - a_i^{\mathrm{T}}x)}{(c^{\mathrm{T}}\lambda' - b^{\mathrm{T}}x)/n} = \frac{1-h^2+\bar{a}_i}{1-h^2+\bar{a}}, \tag{16}$$

which shows that when h is small, the mean-inequality (12) will be tight after updating λ.

By noting that $q\bar{b} - h^2 = n\bar{a}$, we rewrite (15) as follows

$$\begin{aligned} c^{\mathrm{T}}\lambda' - b^{\mathrm{T}}x &= \frac{n(1-h^2) + q\bar{b} - h^2}{q(1-h^2+\bar{b})}(z - b^{\mathrm{T}}x) \\ &= (1 - \frac{l - (l-1)h^2}{q(1-h^2+\bar{b})})(z - b^{\mathrm{T}}x), \qquad (17) \end{aligned}$$

which shows that $c^{\mathrm{T}}\lambda' < z$, i.e. λ' can be used to reduce the upper bound z if λ' is primal feasible. The following lemma gives a sufficient condition for λ' to be feasible.

Lemma 11 *If $h^2 < \frac{1}{2}$, then $\lambda' > 0$.*

Proof:
Since $\lambda_i' := \frac{z - b^{\mathrm{T}}x}{q(c_i - a_i^{\mathrm{T}}x)}(1 + \frac{\bar{a}_i - \bar{b}}{1-h^2+\bar{b}})$, we have $\lambda' > 0$ iff $(1 + \frac{\bar{a}_i - \bar{b}}{1-h^2+\bar{b}}) > 0$. Moreover,

$$1 + \frac{\bar{a}_i - \bar{b}}{1-h^2+\bar{b}} \geq 1 - \frac{|\ \bar{a}_i - \bar{b}\ |}{1-h^2+\bar{b}} \geq 1 - \frac{s}{1-h^2+\bar{b}}.$$

Because $s^2 = h^2 + 2h^2\bar{b} - h^4 - l\bar{b}^2$ we have

$$\begin{aligned} (1-h^2+\bar{b})^2 - s^2 &= 1 - 3h^2 + 2h^4 + (2-4h^2)\bar{b} + (l+1)\bar{b}^2 \\ &\geq 1 - 3h^2 + 2h^4 - \frac{(1-2h^2)^2}{l+1}. \end{aligned}$$

If $h^2 < 1/2$, it follows from the inequality above that

$$(1-h^2+\bar{b})^2 - s^2 > 0,$$

which implies

$$1 - \frac{s}{1-h^2+\bar{b}} > 0.$$

The lemma is proved.
□

As a corollary, we have a restatement of Lemma 8.

Corollary 1 *If $z = z^* := \max_{x\in\mathcal{F}} b^{\mathrm{T}}x$, then $h^2 \geq \frac{1}{2}$.*

Proof:

By the duality theorem, we have for any primal feasible λ and dual feasible x that $c^{\mathrm{T}}\lambda \geq b^{\mathrm{T}}x$. Lemma 11 however, shows that if $h^2 < \frac{1}{2}$, there exists a primal feasible λ' for which (17) holds, i.e. $c^{\mathrm{T}}\lambda' < z$. This contradicts the duality theorem, and proves the corollary and therefore also Lemma 8.

□

Consider now the case $h^2 < 1/2$. The reduction in potential function value caused by the primal update, is given by

$$\begin{aligned}
\psi(x,\lambda) - \psi(x,\lambda') &= q\ln(\frac{c^{\mathrm{T}}\lambda - b^{\mathrm{T}}x}{c^{\mathrm{T}}\lambda' - b^{\mathrm{T}}x}) - \sum_{i=1}^{n}\ln(\lambda_i(c_i - a_i^{\mathrm{T}}x)) \\
&\quad + \sum_{i=1}^{n}\ln(\lambda'(c_i - a_i^{\mathrm{T}}x)) \\
&= -l\ln(\frac{c^{\mathrm{T}}\lambda' - b^{\mathrm{T}}x}{z - b^{\mathrm{T}}x}) \\
&\quad - \sum_{i=1}^{n}\ln(\frac{\lambda_i(c_i - a_i^{\mathrm{T}}x)}{(z - b^{\mathrm{T}}x)/n}) \\
&\quad + \sum_{i=1}^{n}\ln(\frac{\lambda_i'(c_i - a_i^{\mathrm{T}}x)}{(c^{\mathrm{T}}\lambda' - b^{\mathrm{T}}x)/n}). \qquad (18)
\end{aligned}$$

The components in (18) can be further estimated. Using (17) we obtain

$$l\ln(\frac{c^{\mathrm{T}}\lambda' - b^{\mathrm{T}}x}{z - b^{\mathrm{T}}x}) = l\ln(1 - \frac{l - (l-1)h^2}{q(1 - h^2 + \bar{b})}) \qquad (19)$$

and using (12),

$$\sum_{i=1}^{n}\ln(\frac{\lambda_i(c_i - a_i^{\mathrm{T}}x)}{(z - b^{\mathrm{T}}x)/n}) \leq 0 \qquad (20)$$

and finally using (16),

$$\begin{aligned}
\sum_{i=1}^{n}\ln(\frac{\lambda_i'(c_i - a_i^{\mathrm{T}}x)}{(c^{\mathrm{T}}\lambda' - b^{\mathrm{T}}x)/n}) &= \sum_{i=1}^{n}\ln(\frac{1 - h^2 + \bar{a}_i}{1 - h^2 + \bar{a}}) \\
&= \sum_{i=1}^{n}\ln(1 + \frac{\bar{a}_i - \bar{a}}{1 - h^2 + \bar{a}}). \qquad (21)
\end{aligned}$$

Now using (19), (20) and (21) together with (18) it follows that

$$\psi(x,\lambda)-\psi(x,\lambda') \geq -l\ln(1-\frac{l-(l-1)h^2}{q(1-h^2+\bar{b})})+\sum_{i=1}^{n}\ln(1+\frac{\bar{a}_i-\bar{a}}{1-h^2+\bar{a}}). \tag{22}$$

In order to further estimate the reduction, we need the following lemma (see also Karmarkar [12]).

Lemma 12 *Let* $|\, y \,| < 1$. *Then,*

$$-y \geq \ln(1-y) \geq -y - \frac{y^2}{2(1-|\, y\,|)}$$

If additionally, $-1 < y \leq 0$, *then*

$$-y = |\, y\,| \geq \ln(1-y) \geq -y - \frac{y^2}{2}.$$

Proof:

The first part is proven in Karmarkar [12]. The second part of the lemma follows by observing

$$\begin{aligned}\ln(1-y) &= -y - \frac{y^2}{2} - y^3\sum_{k=0}^{\infty}\frac{y^k}{k+3} \\ &= -y - \frac{y^2}{2} - y^3\sum_{k=0}^{\infty}(\frac{y^{2k}}{2k+3}+\frac{y^{2k+1}}{2k+4})\end{aligned}$$

and, if $-1 < y \leq 0$, we have $\frac{1}{2k+3}+\frac{y}{2k+4} > 0$.
□

Now we will use Lemma 12 to prove that inequality (22) implies a reduction in primal-dual potential function value by at least a constant amount, when $l=\sqrt{n+3}$ and $h < 1/3$.

Lemma 13 (Reduction by the primal step) *Let* $l=\sqrt{n+3}$ *and* $h<1/3$ *and let* λ *be an interior point of (P) such that* $c^{\mathrm{T}}\lambda = z$. *Then there exists an interior point* λ' *of (P) such that*

$$c^{\mathrm{T}}\lambda' = z - \frac{l-(l-1)h^2}{q(1-h^2+\bar{b})}(z-b^{\mathrm{T}}x)$$

and

$$\psi(x,\lambda)-\psi(x,\lambda') > 0.25.$$

Proof:

Let $\lambda' := \frac{z-b^{\mathrm{T}}x}{q(c_i-a_i^{\mathrm{T}}x)}\frac{1-h^2+\bar{a}_i}{1-h^2+\bar{b}}$. From (13) and Lemma 11 it follows that λ' is an interior point of (P), i.e. $A\lambda' = b$ and $\lambda' > 0$. From (17) we know that

$$c^{\mathrm{T}}\lambda' = z - \frac{l-(l-1)h^2}{q(1-h^2+\bar{b})}(z-b^{\mathrm{T}}x),$$

and from (22) we have

$$\psi(x,\lambda) - \psi(x,\lambda') \geq -l\ln(1 - \frac{l-(l-1)h^2}{q(1-h^2+\bar{b})}) + \sum_{i=1}^{n}\ln(1+\frac{\bar{a}_i-\bar{a}}{1-h^2+\bar{a}}).$$

Clearly, we have

$$0 < \frac{c^{\mathrm{T}}\lambda' - b^{\mathrm{T}}x}{z-b^{\mathrm{T}}x} = 1 - \frac{l-(l-1)h^2}{q(1-h^2+\bar{b})} < 1$$

which enables us to apply Lemma 12 and obtain

$$-l\ln(1-\frac{l-(l-1)h^2}{q(1-h^2+\bar{b})}) \geq \frac{l^2}{q}\frac{1-\frac{l-1}{l}h^2}{1-h^2+\bar{b}}.$$

As we know from Lemma 5,

$$\bar{b} = \frac{h^2}{l} \pm \sqrt{\frac{1}{l}(h^2 - \frac{l-1}{l}h^4 - s^2)}$$

and therefore,

$$\begin{aligned}
-l\ln(1-\frac{l-(l-1)h^2}{q(1-h^2+\bar{b})}) &\geq \frac{l^2}{q}\frac{1-\frac{l-1}{l}h^2}{1-\frac{l-1}{l}h^2+\sqrt{\frac{1}{l}(h^2-\frac{l-1}{l}h^4)}} \\
&= \frac{l^2}{q}\frac{1}{1+\sqrt{\frac{h^2}{l-(l-1)h^2}}} \\
&\geq \frac{1}{2}\frac{1}{1+\sqrt{1/16}} = 0.4. \qquad (23)
\end{aligned}$$

We will complete the proof by estimating the second term in (22), namely $\sum_{i=1}^{n}\ln(1+\frac{\bar{a}_i-\bar{a}}{1-h^2+\bar{a}})$, from below.

As we already know from (3),

$$\bar{a}_i \geq -\sqrt{\frac{l}{l-1}}h$$

for $i = 1, 2, \cdots, n$. This inequality holds also for the mean value, i.e. $\bar{a} \geq -\sqrt{\frac{l}{l-1}}h$. Using $l = \sqrt{n+3} \geq 2$ and $h < 1/3$ we obtain

$$(1 - h^2 - \sqrt{l/(l-1)}h)^2 \geq (1 - \frac{1}{9} - \frac{\sqrt{2}}{3})^2 > 0.4. \qquad (24)$$

We know that the arithmetic mean minimizes the sum of squared deviations, therefore

$$\sum_{i=1}^{n} (\bar{a}_i - \bar{a})^2 \leq \sum_{i=1}^{n} (\bar{a}_i - \bar{b})^2 = s^2.$$

Using Lemma 5 we have $s^2 \leq h^2 - \frac{l-1}{l}h^4$, and this gives

$$\sum_{i=1}^{n} (\bar{a}_i - \bar{a})^2 \leq h^2 - \frac{l-1}{l}h^4 \leq \frac{1}{9} - \frac{1}{162} < 0.11. \qquad (25)$$

This shows that

$$\left| \frac{\bar{a}_i - \bar{a}}{1 - h^2 + \bar{a}} \right| < \sqrt{\frac{0.11}{0.4}} < 1,$$

which enables us to write the logarithmic function as a power series. In particular, it follows using Lemma 12 that if $\bar{a} - \bar{a}_i \geq 0$,

$$\ln(1 - \frac{\bar{a} - \bar{a}_i}{1 - h^2 + \bar{a}}) \geq \frac{\bar{a}_i - \bar{a}}{1 - h^2 + \bar{a}} - \frac{1}{2} \frac{(\bar{a}_i - \bar{a})^2}{(1 - h^2 + \bar{a})(1 - h^2 + \bar{a}_i)}$$

and if $\bar{a} - \bar{a}_i < 0$,

$$\ln(1 - \frac{\bar{a} - \bar{a}_i}{1 - h^2 + \bar{a}}) \geq \frac{\bar{a}_i - \bar{a}}{1 - h^2 + \bar{a}} - \frac{1}{2} \frac{(\bar{a}_i - \bar{a})^2}{(1 - h^2 + \bar{a})^2}.$$

Summarizing, we have derived the following inequality

$$\begin{aligned} \sum_{i=1}^{n} \ln(1 + \frac{\bar{a}_i - \bar{a}}{1 - h^2 + \bar{a}}) &\geq \sum_{i=1}^{n} \frac{\bar{a}_i - \bar{a}}{1 - h^2 + \bar{a}} \\ &\quad -\frac{1}{2} \sum_{i=1}^{n} \frac{(\bar{a}_i - \bar{a})^2}{(1 - h^2 - \sqrt{l/(l-1)}h)^2} \\ &= -\frac{1}{2} \frac{\sum_{i=1}^{n} (\bar{a}_i - \bar{a})^2}{(1 - h^2 - \sqrt{l/(l-1)}h)^2}. \end{aligned}$$

Therefore, using the estimations (24) and (25),

$$\sum_{i=1}^{n} \ln(1 + \frac{\bar{a}_i - \bar{a}}{1 - h^2 + \bar{a}}) > -0.5\frac{0.11}{0.4} > -0.15. \tag{26}$$

Combining (23) and (26) with (22) shows that the reduction will be at least $0.4 - 0.15 = 0.25$, which proves the lemma.
□

Combining Lemma 13 and Lemma 7 with Lemma 9, it follows that the following algorithm can be used to solve (D) in $\mathcal{O}(\sqrt{n}L)$ iterations.

Algorithm 2 (Primal–dual Iri and Imai's Algorithm. Input data: A,b,c,x^0,z^0)

Step 0 *Set* $l = \sqrt{n+3}, q = n + l,\ i = 0$.

Step 1 *If* $z^i - b^{\mathrm{T}} x^i \leq 2^{-2L}$, *stop.*

Step 2 *Set* $d = -\nabla^2\Phi(x^i, z^i)^{-1}\nabla\Phi(x^i, z^i)$ *and* $h = \sqrt{-\nabla\phi(x^i, z^i)^{\mathrm{T}} d}$.

Step 3 *If* $h < \frac{1}{3}$ *go to Step 6. Otherwise go to Step 4.*

Step 4 *Compute* x^{i+1} *by minimizing* $\phi(x, z^i)$ *along* d. *Set* $z^{i+1} = z^i$.

Step 5 *Set* $i = i + 1$ *and return to Step 1.*

Step 6 *Set* $z^{i+1} = z^i - \frac{l-(l-1)h^2}{q(1-h^2+\bar{b})} (z^i - b^{\mathrm{T}} x^i)$.

Step 7 *Set* $i = i + 1$ *and return to Step 1.*

Theorem 2 *Algorithm 2 solves (D) in* $\mathcal{O}(\sqrt{n}L)$ *main iterations.*

Proof:
Combining Lemma 13 and Lemma 7 with Lemma 9, the theorem follows.
□

5 Conclusions

To the best knowledge of the authors, the $\mathcal{O}(\sqrt{n}L)$- iteration bound is currently the best one for any interior point algorithm available. The benefit of the algorithm presented in this paper is that it uses Newton directions for minimizing the dual part of the potential function. A fast convergence can therefore be expected.

The quantity h^2 is a good proximity measure for the current dual solution x to the reference point on the central path determined by the current primal objective value z. The choice of the parameter l plays an important role in the complexity analysis of the algorithm. Once h^2 gets small, larger l will result in a better primal updating formula in terms of reducing the duality gap. However, when l is large it is more difficult to get close to the reference point on the central path for the dual iterative points. A compromise yields the theoretically best choice of l: $l = \mathcal{O}(\sqrt{n})$. Since the dual iterates are expected to converge fast due to the Newton method being used, it is advisable in practice to let $l = \mathcal{O}(n)$. The difference in the choice of l resembles the difference between the short-step and the large-step path following methods. For the original Iri-Imai algorithm without incorporating the primal variables, it is then better to use small l (e.g. let l be a constant). Because in this case the reduction of the potential function will be more directly linked to the reduction in the gap between the optimal value and the current objective value, which is of primary interest for solving the problem.

Finally, as a remark we mention that using a primal–dual reduction method so as the one presented in this paper, one can conclude that if an LP problem would have nonempty interior feasible region and an optimal solution then the dual problem must have nonempty interior feasible region as well.

References

[1] Anstreicher, K.M., *A combined phase I-phase II scaled potential algorithm for linear programming*, Mathematical Programming 52, pp. 429-439, 1991.

[2] Dikin, I.I., *Iterative solution of problems of linear and quadratic programming*, Soviet Mathematics Doklady 8, pp. 674-675, 1967.

[3] Freund, R.M., *Polynomial-time algorithms for linear programming based only on primal scaling and projected gradients of a potential function*, Mathematical Programming 51, pp. 203-222, 1991.

[4] Freund, R.M., *A potential-function reduction algorithm for solving a linear program directly from an infeasible "warm start"*, Mathematical Programming 52, pp. 441-466, 1991.

[5] Gonzaga, C.C., *Polynomial Affine Algorithms for Linear Programming*, Mathematical Programming 49, pp. 7-21, 1990.

[6] Gonzaga, C.C., *Interior Point Algorithms for Linear Programming Problems with Inequality Constraints*, Mathematical Programming 52, pp. 209-226, 1991.

[7] Gonzaga, C.C., *Path-following methods for Linear Programming*, SIAM Review, Vol. 34, No. 2, pp. 167-224, 1992.

[8] Huard, P., *Resolution of mathematical programming with nonlinear constraints by the method of centres*, Nonlinear Programming, ed. J. Abadie, North Holland Publishing Company, Amsterdam, pp. 207-219, 1967.

[9] Imai, H., *On the polynomiality of the multiplicative penalty function method for linear programming and related inscribed ellipsoids*, IEICE Transactions, Vol. E74, No. 4, pp. 669-671, 1991.

[10] Iri, M., Imai, H., *A multiplicative Barrier Function Method for Linear Programming*, Algorithmica 1, pp. 455-482, 1986.

[11] Iri, M., *A Proof of the Polynomiality of the Iri-Imai Method*, Presented at the 14^{th} International Symposium on Mathematical Programming, Amsterdam, 1991.

[12] Karmarkar, N., *A New Polynomial-Time Algorithm for Linear Programming*, Combinatorica, Vol. 4, pp. 373-395, 1984.

[13] Renegar, J., *A polynomial-time algorithm, based on Newton's method, for Linear Programming*, Mathematical Programming 40, pp. 59-93, 1988.

[14] Sonnevend, Gy., *An "analytical centre" for polyhedrons and new classes of global algorithms for linear (smooth, convex) programming*, Lecture Notes in Control and Information Sciences, Vol. 84, ed. Prekopa, A., Szelezsan, J. and Strazicky, B., Springer Verlag, pp. 866-875, 1986.

[15] Tsuchiya, T. and Muramutsu, M., *Global convergence of the long-step affine scaling algorithm for degenerate linear programming problems*, Research Memorandum 423, The Institute of Statistical Mathematics, 4-6-7 Minami-Azabu, Minato-ku, Tokyo 106, Japan, 1992.

[16] Ye, Y, *An* $\mathcal{O}(n^3L)$ *potential reduction algorithm for linear programming*, Mathematical Programming 50, pp. 239-258, 1991.

[17] Zhang, S. and Shi, M., *On the polynomiality of Iri and Imai's new algorithm for linear programming*, Journal of Qinhua University 28, pp. 121-126, 1988 (in Chinese).

[18] Zhang, S., *The convergence property of Iri-Imai's Algorithm for some Smooth Convex Programming Problems*, Research Memorandum Nr. 485, Department of Econometrics, University of Groningen, the Netherlands, 1992 (to appear in Journal of Optimization Theory and Applications).