On a wide region of centers and primal-dual interior point algorithms for linear programming

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Abstract

In the adaptive step primal-dual interior point method for linear programming, polynomial algorithms are obtained by computing Newton directions towards targets on the central path, and restricting the iterates to a neighborhood of this central path. In this paper, the adaptive step methodology is extended, by considering targets in a certain central *region*, which contains the usual central path, and subsequently generating iterates in a neighborhood of this region. The size of the central region can vary from the central path to the whole feasible region by choosing a certain parameter.

An $\mathcal{O}(\sqrt{n}L)$ iteration bound is obtained under very mild conditions on the choice of the target points. In particular, we leave plenty of room for experimentation with search directions. The practical performance of the new primal-dual interior point method is measured on the Netlib test set for various sizes of the central region.

Key words: Linear programming, primal-dual interior point method, central path, wide neighborhood.

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1. Introduction

Consider the following standard primal and dual linear programming problems

$$\begin{array}{lll} \text{(P)} & \text{minimize} & c^{\mathrm{T}}x \\ & \text{subject to} & Ax = b, \quad x \geq 0 \end{array}$$

and

$$\begin{array}{lll} \text{(D)} & \text{maximize} & b^{\mathrm{T}}y \\ & \text{subject to} & A^{\mathrm{T}}y + s = c, \quad s \geq 0 \end{array}$$

where A is an m by n matrix.

Further denote the feasible regions of the primal and the dual problems by

$$\mathcal{F}_P := \{x \in \Re^n_+ \mid Ax = b\}$$

and

$$\mathcal{F}_D := \{s \in \Re^n_+ \mid A^{\mathrm{T}}y + s = c, \exists y \in \Re^m\}$$

where $\Re^n_+$ denotes the nonnegative orthant. We assume that both $\mathcal{F}_P$ and $\mathcal{F}_D$ contain positive vectors.

A vast amount of recent research has been contributed to polynomial primal-dual interior point methods that generate iterates in a certain wide neighborhood of the central path, viz. the neighborhood

$$\mathcal{N}^-_\infty(\beta) = \{(x, s) \in \mathcal{F}_p \times \mathcal{F}_d \mid \min_{1 \leq i \leq n} x_i s_i \geq (1 - \beta)\mu e, \mu = \frac{x^{\mathrm{T}}s}{n}\}.$$

A main iteration of this method can be described as follows. First, a search direction is obtained by applying Newton's method for solving a system of equations that describes a target on the central path. Second, the method takes the largest possible step in this Newton direction without leaving the $\mathcal{N}^-_\infty(\beta)$ neighborhood. A new target will be used in the next iteration. These methods are interesting, because they are polynomial and they allow long steps, which is a prerequisite for practical efficiency. We have to remark though, that in these methods, the targets for computing Newton directions are chosen on the central path only. Due to the large size of the neighborhood, the distance between the iterates and the Newton targets on the central path is usually large, which can aversely affect the accuracy of the Newton direction.

In this paper, we propose a generic method where targets are chosen in a wide central *region* that includes the central path, and the iterates are restricted to an — even wider — neighborhood of this region. It will be shown that it is efficient to use targets in the central region that are not necessarily on the central path. In this respect, the method is similar to the target following method of Jansen, Roos, Terlaky and Vial [2]. In other respects however, the two methods differ greatly. In particular, the target following method generates iterates that trace a sequence of targets, whereas the generic method studied in this paper generates a sequence in a neighborhood of the central region, without necessarily tracing the targets.

It has been shown by Kojima, Megiddo, Noma, and Yoshise [4] that there exists a one-to-one correspondance between an arbitrary positive vector v and a pair of positive primal and dual solutions $x \in \mathcal{F}_P$ and $s \in \mathcal{F}_D$ with

$$v_i = \sqrt{x_i s_i} \text{ for } i = 1, 2, \cdots, n.$$

The space of vectors v that is made in this way from primal and dual solutions x and s, is commonly referred to as the v-space. Hence, the v-space coincides with the positive orthant $\Re^n_{++}$, where any vector $v \in \Re^n_{++}$ is uniquely associated to a pair of positive primal and dual solutions. The v-space facilitates a unified and unambiguous treatment of search directions for the interior point method, as will be described in Section 1.1. In Section 1.2, we show how some frequently used neighborhoods of the central path fit in the v-space geometry. The central region and its neighborhood are introduced in Section 2, and the new generic algorithm is described in Section 3. We will proceed in Section 4 by obtaining an $\mathcal{O}(\sqrt{n}L)$ iteration bound for this algorithm under mild conditions on the choice of the target points. In particular, we leave plenty of room for experimentation with search directions. To illustrate this issue, we describe an implementation of the new primal-dual interior point method in Section 5, and we provide results on its performance for various sizes of the neighborhood.

Before proceeding, we mention the notation used in this paper. Let e denote the all-one vector, and e_i the i-th unit vector. For vectors other than e_i, a subscript denotes the corresponding component. For a vector x, $X = \text{diag}(x)$ denotes the diagonal matrix formed by elements of x. The norm $\|\cdot\|$ is Euclidean unless stated otherwise. Two vectors a and b satisfy $a \leq b$ if and only if $a_i \leq b_i$ for all i. We write $\angle(f, w)$ to denote the angle between two vectors f and w. As usual, by the angle $\angle(f, w)$ we mean the smallest nonnegative angle between the two vectors f and w. We use $\sin(f, w), \cos(f, w)$ and $\tan(f, w)$ as a simplified notation for $\sin(\angle(f,w)), \cos(\angle(f,w))$ and $\tan(\angle(f,w))$. Index sets are indicated by a script capital, say $\mathcal{S}$. An n-dimensional vector w subscripted by an index set $\mathcal{S}$, i.e. $w_{\mathcal{S}}$, denotes the n-dimensional vector with components w_i for $i \in \mathcal{S}$ and 0 in the other components. We write $\neg\mathcal{S}$ to denote the complement of $\mathcal{S}$, so that $w_{\neg\mathcal{S}} = w - w_{\mathcal{S}}$. The cardinality of a set $\mathcal{S}$ is denoted by $|\ \mathcal{S}\ |$.

1.1. The v-space and search directions

In this section, we review the v-space approach to the primal-dual interior point method. For a more thorough discussion of this subject, we refer the reader to Kojima, Megiddo, Noma, and Yoshise [4] and Jansen, Roos, Terlaky and Vial [2]. Consider $x, s \in \Re^n_{++}$. Let

$$d_i := \sqrt{x_i/s_i} \text{ for } i = 1, 2, \cdots, n.$$

It is easily seen that for fixed $p \in \Re^n$, $p \neq 0$, and given d, the relation

$$p_x + p_s = p,$$

$$ADp_x = 0 \text{ and } DA^{\mathrm{T}}\Delta y + p_s = 0$$

$$\Delta x = Dp_x \text{ and } \Delta s = D^{-1}p_s$$

uniquely describes the primal and dual directions Δx and Δs. Namely, p_x and p_s form an orthogonal decomposition of p. In the sequel, by 'the search direction' we will mean the vector p which combines the primal and the dual search directions.

Once the direction p is given, if we move along the directions Δx and Δs simultaneously in the primal and the dual space, then the maximum possible step length retaining the feasibility is:

$$t^* := \max\{t \mid v + tp_x \geq 0, v + tp_s \geq 0\},$$

or equivalently,

$$t^* = -\frac{1}{\min_{1 \leq i \leq n} \min(\Delta x_i/x_i, \Delta s_i/s_i)}.$$

For $t \leq t^*$ let

$$v_i(t) := \sqrt{(x_i + t\Delta x_i)(s_i + t\Delta s_i)} \text{ for } i = 1, 2, \cdots, n.$$

Remark that

$$v_i(t) = \sqrt{(v_i + t(p_x)_i)(v_i + t(p_s)_i)} \text{ for } i = 1, 2, \cdots, n, \tag{1}$$

and $v(0) = v$.

From the arithmetic-geometric mean inequality, it follows that

$$v(t) \le \frac{1}{2}(v + tp_x + v + tp_s) = v + tp/2. \tag{2}$$

Because $p_x \perp p_s$, we also have

$$\|v(t)\|^2 = \|v\|^2 + tv^{\mathrm{T}}p. \tag{3}$$

As $\|v\|^2 = x^{\mathrm{T}}s$ is the duality gap, (3) shows that p is a descent direction if and only if $v^{\mathrm{T}}p < 0$.

The trajectory $v(t)$ in the v-space satisfies

$$v(t) = v + \frac{t}{2}p + o(t).$$

The point $v + \frac{t}{2}p$ can thus be interpreted as the target of the next iterate in the v-space if primal and dual steps $t\Delta x$ and $t\Delta s$ are taken, cf. Jansen et al. [2].

From now on, for given v we will scale descent directions p such that

$$v^{\mathrm{T}}p = -\|v\|^2$$

and so

$$\|v(t)\|^2 = (1-t)\|v\|^2. \tag{4}$$

1.2. Neighborhoods of the central path

In this section, we review some neighborhoods of the central path in terms of the v-space geometry. In particular, we will describe the $\mathcal{N}_\infty^-$ - neighborhood [5, 4, 10], the $\mathcal{N}_2$-neighborhood [6, 4, 10] and the circular cone neighborhood [11]. Note that the terminology $\mathcal{N}_\infty^-$ and $\mathcal{N}_2$ is from Mizuno, Todd and Ye [10].

We first mention that the primal-dual central path is a half-line in the v-space, viz.

$$\{v \in \Re^n \mid v = \frac{\|v\|}{\sqrt{n}}e\},$$

see e.g. [4, 2].

The primal-dual interior point method of Kojima, Mizuno and Yoshise [5] originally generated a sequence of primal- dual pairs in an $\mathcal{N}_\infty^-$ neighborhood of the central path. Remark that for any $\beta \in [0,1]$,

$$(x, s) \in \mathcal{N}_\infty^-(\beta)$$

if and only if

$$X^{1/2}S^{1/2}e \in \{v \in \Re^n_+ \mid \min_{1\le i\le n} v_i^2 \ge (1-\beta)\mu e \text{ with } \mu = \frac{\|v\|^2}{n}\}. \quad (5)$$

It follows that the v-space representation of $\mathcal{N}_\infty^-$ is a cone. This cone will be further analyzed in Section 2. Kojima et al. [5] showed that their $\mathcal{N}_\infty^-$ algorithm converges in $\mathcal{O}(nL)$ main iterations. Subsequently, Kojima, Mizuno and Yoshise [6] modified their algorithm by restricting the iterates to an $\mathcal{N}_2(\beta)$ neighborhood, where

$$\mathcal{N}_2(\beta) = \{v \in \Re^n_+ \mid \left\|V^2e - \mu e\right\| \le \beta\mu, \mu = \frac{\|v\|^2}{n}\}.$$

They obtained an $\mathcal{O}(\sqrt{n}L)$ iteration bound for this modification. In [11] it is shown that

$$\mathcal{N}_2(\beta) = \{v \in \Re^n_+ \mid \sqrt{n}\tan(e, V^2e) \le \beta\}.$$

We proposed in [11] to use instead of $\mathcal{N}_2(\beta)$ the circular cone neigborhood $\mathcal{N}(1,\beta)$,

$$\mathcal{N}(1,\beta) := \{v \in \Re^n_+ \mid \sqrt{n-1}\tan(e,v) \le \beta\} \quad (6)$$

and we showed that $\mathcal{N}_2(\sqrt{1+\frac{1}{n-1}}\beta) \subset \mathcal{N}(1,\beta)$. The axis of the circular cone neighborhood, the all-one vector e, corresponds to the primal-dual central path. Figure 1 shows the intersection of $\mathcal{N}_2(\beta)$, $\mathcal{N}_2(\sqrt{1+\frac{1}{n-1}}\beta)$ and $\mathcal{N}(1,\beta)$ with the unit simplex

$$\{v \in \Re^n_+ \mid e^{\mathrm{T}}v = 1\},$$

for the case $n = 3$ and $\beta = 0.9$.

To summarize, in the v-space geometry, the central path is the cone of positive multiples of the all-one vector e, and the neighborhoods are larger cones in $\Re^n_+$ that contain the central path. In the next section, we will introduce a central region, and we will propose a neighborhood of this region.

2. The central region and its neighborhood

We shall first introduce a region of centers in the v-space. The targets that are used in determining a Newton search direction will be chosen in this region. Next, we will introduce a neighborhood of this central region. Based on these two new concepts we will then propose a generic primal-dual algorithm that generates an iterative sequence in this neighborhood.

2.1. Definitions and motivation

In v-space, the central region is defined as the cone

$$\mathcal{C}(\theta) := \{v \mid \min_{1\le i\le n} v_i \ge \theta \|v\| / \sqrt{n}\}$$

for given $\theta \in [0,1]$. It is easily seen from (5) that $\mathcal{C}(\theta)$ is the v-space representation of the $\mathcal{N}_\infty^-(1-\theta^2)$ neighborhood.

As special cases, $\mathcal{C}(1) = \{v \mid v = \frac{\|v\|}{\sqrt{n}} e\}$ is the well known central path, and $\mathcal{C}(0) = \Re^n_+$, the nonnegative orthant. In general, $\mathcal{C}(\theta)$ is the intersection of n circular cones,

$$\mathcal{C}(\theta) = \{v \mid \cos(e_i, v) \ge \theta/\sqrt{n}, 1 \le i \le n\} \cup \{0\}. \tag{7}$$

We will restrict to $\theta > 0$ in the sequel. For this case, we have

$$\mathcal{C}(\theta) = \{v \mid 0 \le \tan(e_i, v) \le r(\theta), 1 \le i \le n\} \cup \{0\},$$

where

$$r(\theta) := \frac{\sqrt{n-\theta^2}}{\theta}. \tag{8}$$

Because $\mathcal{C}(\theta)$ is a closed convex set, there exists for given v a unique vector

$$v^\theta = \arg\min_{f\in\mathcal{C}(\theta)} \|f - v\|.$$

This vector v^θ is known as the *minimal norm projection* of v onto the convex set $\mathcal{C}(\theta)$. It is well known that $(v - v^\theta)$ is the normal direction of a hyperplane that seperates v from $\mathcal{C}(\theta)$, i.e.

$$(v^\theta - v)^{\mathrm{T}}(w - v^\theta) \ge 0 \text{ for all } w \in \mathcal{C}(\theta).$$

The inequality implies

$$(v^\theta - v)^{\mathrm{T}}(w - v) \ge \left\|v^\theta - v\right\|^2 \text{ for all } w \in \mathcal{C}(\theta). \tag{9}$$

The above relation is known as the strong seperating hyperplane theorem. Moreover, we have:

Lemma 2.1. *Let $v \in \Re^n_+, v \ne 0$ and let v^θ be the projection of v onto the closed convex cone $\mathcal{C}(\theta)$, i.e.*

$$v^\theta = \arg\min_{f\in\mathcal{C}(\theta)} \|f - v\|,$$

then

$$\tan(v^\theta, v) = \min_{f\in\mathcal{C}(\theta)} \tan(f, v).$$

Proof: We need only to prove

$$\sin(v^\theta, v) = \min_{f \in \mathcal{C}(\theta)} \sin(f, v).$$

Since $\frac{f^T v}{f^T f} f \in \mathcal{C}(\theta)$ and

$$\left\| v^\theta - v \right\| = \min_{f \in \mathcal{C}(\theta)} \| f - v \|,$$

we obtain

$$\sin(f, v) = \frac{\left\| \frac{f^T v}{f^T f} f - v \right\|}{\|v\|} \geq \frac{\left\| v^\theta - v \right\|}{\|v\|} = \sin(v^\theta, v).$$

□

As a matter of notation, we now define the angle between the closed convex cone $\mathcal{C}(\theta)$ and a vector $v \in \Re^n$ as follows:

$$\tan(\mathcal{C}(\theta), v) := \min_{f \in \mathcal{C}(\theta)} \tan(f, v).$$

The above lemma shows that

$$\tan(\mathcal{C}(\theta), v) = \tan(v^\theta, v).$$

For fixed $\theta \in (0, 1]$ and $\beta \in (0, 1)$, we define a neighborhood of the central region $\mathcal{C}(\theta)$ as:

$$\mathcal{N}(\theta, \beta) := \{ v \in \Re^n_+ \mid r(\theta) \tan(\mathcal{C}(\theta), v) \leq \beta \},$$

where by definition, $r(\theta) = \frac{\sqrt{n - \theta^2}}{\theta}$, see (8). Observe that the above definition is consistent with our earlier definition (6) of $\mathcal{N}(1, \beta)$.

Figure 2 provides an illustration of the intersection of the unit simplex with $\mathcal{C}(0.6)$ and $\mathcal{N}(0.6, 0.7)$ for the case $n = 3$. Notice that the case $\theta = 1$ was shown in Figure 1.

2.2. A basic property of the central region neighborhood

In the previous section, we have extended the notion of *central path* to *central region*, and we have generalized the *circular cone neighborhood* of the central path to a neighborhood of the central region, viz. the *central region neighborhood* $\mathcal{N}(\theta, \beta)$. In this section, we will obtain a useful property of $\mathcal{N}(\theta, \beta)$.

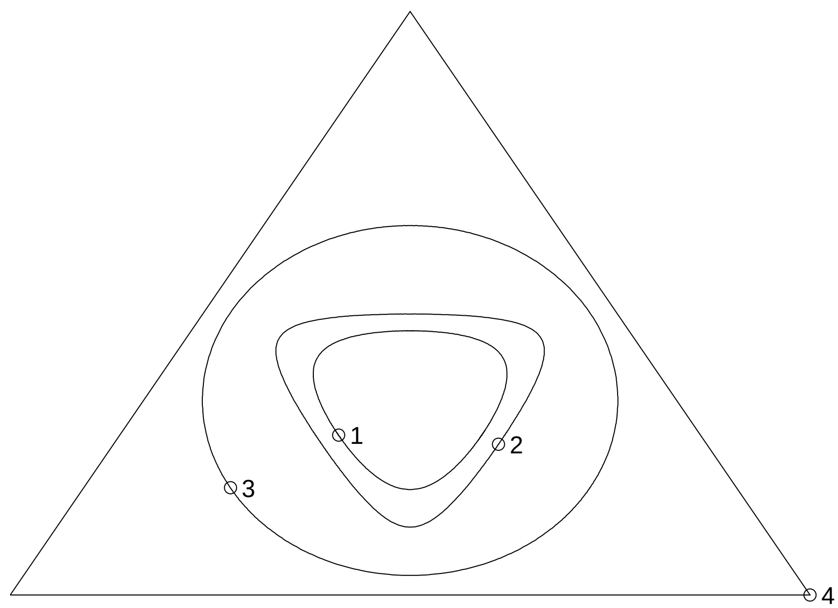

Figure 1: The intersection of (4) the unit simplex with (1) $\mathcal{N}_2(0.9)$, (2) $\mathcal{N}_2(1.102)$ and (3) $\mathcal{N}(1, 0.9)$ for $n = 3$.

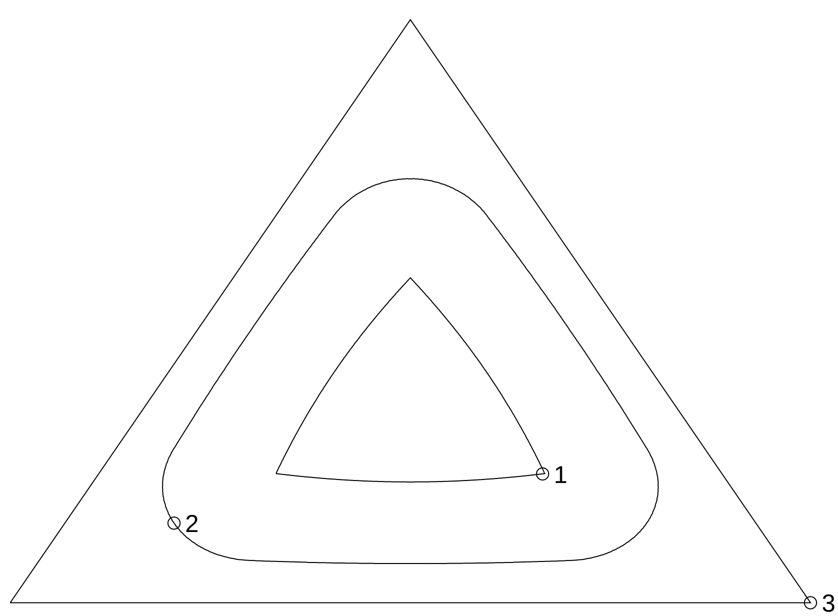

Figure 2: The intersection of (3) the unit simplex with (1) $\mathcal{C}(0.6)$ and (2) $\mathcal{N}(0.6, 0.7)$ for $n = 3$.

Lemma 2.2. *Let $w \in \Re^n$, $f \in \mathcal{C}(\theta)$, $f \neq 0$ for some $\theta \in (0,1]$. Then*

$$w \geq (1 - r(\theta)\tan(f,w))\frac{f^T w}{\|f\|^2} f.$$

Proof:

We will prove the lemma by showing

$$(e_i - \frac{f_i}{\|f\|^2} f)(e_i - \frac{f_i}{\|f\|^2} f)^{\mathrm{T}} \preceq (1 - \frac{f_i^2}{\|f\|^2})(I - \frac{ff^{\mathrm{T}}}{\|f\|^2}) \qquad (10)$$

for arbitrary $i \in \{1, 2, \cdots, n\}$. As a matter of notation, by $A \preceq B$ for symmetric matrices A and B we mean that $B - A$ is positive semi-definite.

The only nonzero eigenvalue of the rank-one matrix

$$(e_i - \frac{f_i}{\|f\|^2} f)(e_i - \frac{f_i}{\|f\|^2} f)^{\mathrm{T}}$$

is $1 - \frac{f_i^2}{\|f\|^2}$ with corresponding eigenvector $(e_i - \frac{f_i}{\|f\|^2} f)$. The positive semi-definite matrix

$$(1 - \frac{f_i^2}{\|f\|^2})(I - \frac{ff^{\mathrm{T}}}{\|f\|^2})$$

has also an eigenvalue $1 - \frac{f_i^2}{\|f\|^2}$ corresponding to the eigenvector $(e_i - \frac{f_i}{\|f\|^2} f)$. This proves (10). By pre-multiplying by w^{T} and post-multiplying by w in (10), it follows that

$$\begin{aligned} \left| w_i - \frac{f^{\mathrm{T}} w}{\|f\|^2} f_i \right| &\leq \sqrt{1 - \frac{f_i^2}{\|f\|^2}} \sqrt{\|w\|^2 - \frac{(f^{\mathrm{T}} w)^2}{\|f\|^2}} \\ &= \sqrt{1 - \frac{f_i^2}{\|f\|^2}} \frac{| f^{\mathrm{T}} w \tan(f,w) |}{\|f\|}. \end{aligned}$$

Using the fact that $\tan(f,w) \geq 0$ if and only if $f^{\mathrm{T}} w \geq 0$ and using $f \geq \theta \|f\| e/\sqrt{n}$, we obtain

$$\begin{aligned} \left| w_i - \frac{f^{\mathrm{T}} w}{\|f\|^2} f_i \right| &\leq \sqrt{1 - \frac{\theta^2}{n}} \frac{f^{\mathrm{T}} w}{\|f\|^2} \tan(f,w) \frac{\sqrt{n}}{\theta} f_i \\ &= r(\theta)\tan(f,w) \frac{f^{\mathrm{T}} w}{\|f\|^2} f_i. \end{aligned}$$

The lemma is proved. □

From Lemma 2.2, it is obvious that

$$\mathcal{N}(\theta, \beta) \subset \Re^n_+ = \mathcal{C}(0).$$

The following theorem provides a stronger result.

Theorem 2.1. *Let $\theta \in (0,1], \beta \in (0,1)$. Define*

$$\theta_\beta := \frac{1}{\sqrt{1+\beta^2/r(\theta)^2}}(1-\beta)\theta.$$

There holds

$$\mathcal{C}(\theta) \subseteq \mathcal{N}(\theta,\beta) \subseteq \mathcal{C}(\theta_\beta).$$

Proof:

The inclusion $\mathcal{C}(\theta) \subseteq \mathcal{N}(\theta,\beta)$ follows immediately from the definition of $\mathcal{N}(\theta,\beta)$. In order to prove the second relation, we use that for any $v \in \mathcal{N}(\theta,\beta)$ there holds

$$r(\theta)\tan(v^\theta, v) \le \beta,$$

so that by applying Lemma 2.2 we obtain

$$v \ge (1-\beta)\frac{(v^\theta)^{\mathrm{T}} v}{\|v^\theta\|^2} v^\theta = (1-\beta)\cos(v^\theta, v)\,\|v\|\,\frac{v^\theta}{\|v^\theta\|}.$$

Using

$$\cos(v^\theta, v) = \frac{1}{\sqrt{1+\tan(v^\theta,v)^2}} \ge \frac{1}{\sqrt{1+\beta^2/r(\theta)^2}},$$

it follows that

$$v \ge \frac{1-\beta}{\sqrt{1+\beta^2/r(\theta)^2}}\theta\,\|v\|\,e/\sqrt{n} = \theta_\beta\,\|v\|\,e/\sqrt{n},$$

i.e. $v \in \mathcal{C}(\theta_\beta)$. □

So far, we have seen some nice properties of the central region and its neighborhood $\mathcal{N}(\theta,\beta)$. In the next section, it will be shown that it is in fact easy to check whether a given vector $v \in \Re^n_+$ belongs to the new neighborhood, which is important for practical implementations.

2.3. Computing the projection on the central region

In order to check the membership $v \in \mathcal{N}(\theta,\beta)$ for some vector $v \in \Re^n_+$, one has to compute the projection v^θ. We will show in this section that this projection can be computed very efficiently.

Let $\mathcal{T} \subset \{1, 2, \cdots, n\}$ be the index set of maximal cardinality such that

$$v_{\mathcal{T}} < \frac{\theta}{\sqrt{n - \theta^2 \mid \mathcal{T} \mid}} \|v_{\neg\mathcal{T}}\| \, e, \tag{11}$$

and let

$$h := \frac{\theta}{\sqrt{n - \theta^2 \mid \mathcal{T} \mid}} \|v_{\neg\mathcal{T}}\| \, .$$

The following lemma states that $i \in \mathcal{T}$ if and only if $v_i < h$.

Lemma 2.3. *There holds*

$$v_{\neg\mathcal{T}} \geq h e_{\neg\mathcal{T}}.$$

Proof:

Suppose to the contrary that there exists some $i \in \neg\mathcal{T}$ with $v_i < h$. Let $\mathcal{T}' := \mathcal{T} \cup \{i\}$. Then

$$\begin{aligned} \|v_{\neg\mathcal{T}'}\|^2 &= \|v_{\neg\mathcal{T}}\|^2 - v_i^2 \\ &> \|v_{\neg\mathcal{T}}\|^2 - h^2 \\ &= \frac{n - \theta^2 \mid \mathcal{T}' \mid}{n - \theta^2 \mid \mathcal{T} \mid} \|v_{\neg\mathcal{T}}\|^2 \, . \end{aligned}$$

Hence

$$v_{\mathcal{T}'} < he < \frac{\theta}{\sqrt{n - \theta^2 \mid \mathcal{T}' \mid}} \|v_{\neg\mathcal{T}'}\| \, e,$$

contradicting the maximal cardinality property of $\mathcal{T}$.

□

The above lemma shows that $v_i < v_j$ for all $i \in \mathcal{T}$ and $j \in \neg\mathcal{T}$. The set $\mathcal{T}$ can thus be computed by sorting the components of v.

Notice that

$$\|v_{\neg\mathcal{T}} + he_{\mathcal{T}}\|^2 = \|v_{\neg\mathcal{T}}\|^2 + h^2 \mid \mathcal{T} \mid = \frac{nh^2}{\theta^2}$$

so that using $(v_{\neg\mathcal{T}} + he_{\mathcal{T}}) \geq he$, we have

$$(v_{\neg\mathcal{T}} + he_{\mathcal{T}}) / \|v_{\neg\mathcal{T}} + he_{\mathcal{T}}\| \geq \theta e / \sqrt{n}.$$

It follows that

$$(v_{\neg\mathcal{T}} + he_{\mathcal{T}}) \in \mathcal{C}(\theta).$$

The following lemma shows that using the set $\mathcal{T}$, one can easily compute the projection v^θ of v onto the closed convex cone $\mathcal{C}(\theta)$.

Lemma 2.4. *Let* $v^\theta = \arg\min_{f\in\mathcal{C}(\theta)} \|f - v\|$. *It holds that*

$$v^\theta = \frac{v^T(v_{\neg\mathcal{T}} + he_{\mathcal{T}})}{\|v_{\neg\mathcal{T}} + he_{\mathcal{T}}\|^2}(v_{\neg\mathcal{T}} + he_{\mathcal{T}}). \tag{12}$$

Proof:

We use the characterization (7) of $\mathcal{C}(\theta)$.

Consider the following problem for finding the projection of v on $\mathcal{C}(\theta)$:

$$\min_{f\in\Re^n}\{\frac{1}{2}\|f-v\|^2 \mid \cos(e_i, f) \geq \theta/\sqrt{n}, i = 1, 2, \cdots, n\}. \tag{13}$$

The above problem will have a unique solution which is the projection v^θ. Note also that the function $\cos(e_i, f)$ is pseudo concave in f on $\Re^n_+$. It is clear that to prove the theorem, we need only to check that the solution given by (12) is a Karush-Kuhn-Tucker point of the problem given in (13).

The Lagrangian function for (13) is constructed as

$$\mathcal{L}(f;\lambda) = \frac{1}{2}\|f-v\|^2 - \sum_{i=1}^n \lambda_i(\cos(e_i, f) - \theta/\sqrt{n}).$$

Its gradient is given by

$$\begin{aligned}\nabla_f\mathcal{L}(f;\lambda) &= f - v - \sum_{i=1}^n \lambda_i \frac{\|f\|^2 e_i - f_i f}{\|f\|^3} \\ &= f - v - \frac{\lambda}{\|f\|} + \frac{f^{\mathrm{T}}\lambda}{\|f\|^3} f.\end{aligned}$$

Now consider the solution

$$f^* = \frac{v^{\mathrm{T}}(v_{\neg\mathcal{T}} + he_{\mathcal{T}})}{\|v_{\neg\mathcal{T}} + he_{\mathcal{T}}\|^2}(v_{\neg\mathcal{T}} + he_{\mathcal{T}})$$

with multiplier

$$\lambda^* = \|f^*\|(he_{\mathcal{T}} - v_{\mathcal{T}}) = \|f^*\|(v_{\neg\mathcal{T}} + he_{\mathcal{T}} - v).$$

Remark that

$$(f^*)^{\mathrm{T}} v = \|f^*\|^2 = \frac{(v^{\mathrm{T}}(v_{\neg\mathcal{T}} + h e_{\mathcal{T}}))^2}{\|v_{\neg\mathcal{T}} + h e_{\mathcal{T}}\|^2},$$

which implies $f^* \perp (f^* - v)$.

Moreover,

$$\begin{aligned}(f^*)^{\mathrm{T}} \lambda^* &= \|f^*\| \frac{v^{\mathrm{T}}(v_{\neg\mathcal{T}} + h e_{\mathcal{T}})}{\|v_{\neg\mathcal{T}} + h e_{\mathcal{T}}\|^2} (v_{\neg\mathcal{T}} + h e_{\mathcal{T}} - v)^{\mathrm{T}} (v_{\neg\mathcal{T}} + h e_{\mathcal{T}}) \\ &= \|f^*\| \, v^{\mathrm{T}}(v_{\neg\mathcal{T}} + h e_{\mathcal{T}}) - \|f^*\|^3 .\end{aligned}$$

Therefore,

$$\frac{(f^*)^{\mathrm{T}} \lambda^*}{\|f^*\|^3} f^* = v_{\neg\mathcal{T}} + h e_{\mathcal{T}} - f^*.$$

Hence, we have

$$\nabla_f \mathcal{L}(f^*; \lambda^*) = 0.$$

Observe that λ^* is nonnegative and $f^* \in \mathcal{C}(\theta)$, i.e. $\cos(e_i, f^*) - \theta/\sqrt{n} \geq 0$ for $i = 1, \cdots, n$. Moreover, there holds

$$\sum_{i=1}^{n} \lambda_i^* (\cos(e_i, f^*) - \theta/\sqrt{n}) = \frac{(\lambda^*)^{\mathrm{T}} f^*}{\|f^*\|} - \frac{\theta}{\sqrt{n}} e^{\mathrm{T}} \lambda^* = 0,$$

where in the last equality we notice

$$\frac{h}{\|v_{\neg\mathcal{T}} + h e_{\mathcal{T}}\|} = \frac{\theta}{\sqrt{n}}.$$

We have thus verified that f^* is a Karush-Kuhn-Tucker point, and therefore $f^* = v^{\theta}$. The theorem is proved. □

We want to remark here that in order to calculate h, $\mathcal{T}$ and v^{θ}, it suffices to sort only a (usually small) subset of the components of v. This can be seen from the following relation:

$$\frac{\theta}{\sqrt{n}} \|v\| \leq h = \frac{\theta}{\sqrt{n - \theta^2 \mid \mathcal{T} \mid}} \|v_{\neg\mathcal{T}}\| \leq \frac{\theta}{\sqrt{n - \theta^2 (n-1)}} \|v\| .$$

Hence, if

$$v_i < \frac{\theta}{\sqrt{n}} \|v\| ,$$

then $i \in \mathcal{T}$, and if

$$v_i \geq \frac{\theta}{\sqrt{n-\theta^2(n-1)}} \|v\|,$$

then $i \in \neg\mathcal{T}$. It follows that we only need to sort those v_i's for which

$$\frac{1}{\sqrt{n}} \leq \frac{v_i}{\theta \|v\|} < \frac{1}{\sqrt{n-\theta^2(n-1)}}.$$

3. A generic central region algorithm

The selection of search directions p in the v-space, or equivalently Δx and Δs in $\mathcal{F}_P$ and $\mathcal{F}_D$, is an important issue in the design of primal-dual interior point algorithms. In our generic central region algorithm, we make sure that the v-space search direction p points towards the central region $\mathcal{C}(\theta)$. Certainly, we are concerned only with descent directions. In mathematical terms, for a given iterate $v \in \mathcal{N}(\theta, \beta)$, we are interested in a descent direction p, with $v^{\mathrm{T}} p = -\|v\|^2$, such that

$$\{t \in [0,1) \mid v + tp \in \mathcal{C}(\theta)\} \neq \emptyset.$$

Let

$$t_L := \min_{t \geq 0} \{t \mid v + tp \in \mathcal{C}(\theta)\}.$$

Clearly, $0 \leq t_L < 1$.

Since $\frac{\mathrm{d}}{\mathrm{d}t} v(t) \mid_{t=0} = \frac{1}{2} p$, the step length $\bar{t}$ with $\bar{t} = 2t_L$ can be interpreted as a Newton step towards the target $(v + t_L p) \in \mathcal{C}(\theta)$.

Let $t(\theta, \beta)$ be the largest step length towards the boundary of $\mathcal{N}(\theta, \beta)$, i.e.

$$t(\theta, \beta) := \max\{t \mid v(\bar{t}) \in \mathcal{N}(\theta, \beta)\ \forall 0 \leq \bar{t} \leq t\}.$$

We propose to use such a step length rule that the resulting sequence of iterates generated by the generic central region algorithm is contained in $\mathcal{N}(\theta, \beta)$.

A generic central region algorithm now follows:

Algorithm 1.

Input data: (A, b, c), *parameters* $0 < \beta < 1$ *and* $0 < \theta \leq 1$ *and initial feasible solution* $(x^{(0)}, s^{(0)})$ *such that* $v^{(0)} = (X^{(0)}S^{(0)})^{1/2}e \in \mathcal{N}(\theta, \beta)$.

Step 0 Initialization. *Set* $k = 0$.

Step 1 Optimality test. *Stop if based on* $(x^{(k)}, s^{(k)})$ *a pair of optimal solutions* (x^*, s^*) *can be found.*

Step 2 Choose direction. *Choose* $p^{(k)}$ *such that* $(v^{(k)})^T p^{(k)} = -\left\|v^{(k)}\right\|^2$ *and*

$$\{t \in [0, 1) \mid v^{(k)} + tp^{(k)} \in \mathcal{C}(\theta)\} \neq \emptyset.$$

Compute $\Delta x^{(k)}$ *and* $\Delta s^{(k)}$.

Step 3 Compute step length. *Compute* $t(\theta, \beta)$ *and let* $t \geq \frac{1}{2}t(\theta, \beta)$ *such that*

$$(X^{(k)} + t\Delta X^{(k)})^{1/2}(S^{(k)} + t\Delta S^{(k)})^{1/2}e \in \mathcal{N}(\theta, \beta).$$

Step 4 Take step. *Set* $x^{(k+1)} = x^{(k)} + t\Delta x^{(k)}$ *and* $s^{(k+1)} = s^{(k)} + t\Delta s^{(k)}$.

Step 5 *Set* $k = k + 1$ *and return to Step 1.*

An initial solution can be obtained by means of a self-dual formulation [13]. For a good optimality test (Step 1), we refer the reader to [9]. In Step 3 one can use a bisection method to compute the step length. The condition $t \geq \frac{1}{2}t(\theta, \beta)$ allows us to accept the step length already after the first time that the lower limit of the bisection interval is updated. Hence, only $\mathcal{O}(\log \frac{1}{t(\theta,\beta)})$ trials are needed by a bisection procedure for determining the step length. In Section 5, we will describe a specific implementation of Algorithm 1.

4. Convergence of the generic algorithm

In this section, we shall analyze the convergence of the generic primal-dual algorithm proposed in the previous section. The analysis is based on a lower bound on the step length $t(\theta, \beta)$. Because $\|v(t)\|^2 = (1 - t)\|v\|^2$, see (4), a linear convergence in duality gap will be achieved once such a lower bound is known.

Throughout this section, we consider only step lengths t satisfying

$$0 \le t \le \max\{\bar{t} \mid v + \bar{t}p \ge 0\}.$$

Letting

$$g_i(t) := \frac{\sqrt{v_i(v_i + tp_i)}}{\sqrt{1-t}} \text{ for } i = 1, 2, \cdots, n,$$

it follows that

$$v_i(t)^2 = (1-t)g_i(t)^2 + t^2 \Delta x_i \Delta s_i \text{ for } i = 1, 2, \cdots, n.$$

Obviously, we have

$$t(\theta, \beta) \ge \max\{t \mid r(\theta)\tan(v^\theta, v(\bar{t})) \le \beta \text{ for all } \bar{t} \in [0, t]\}. \tag{14}$$

We can therefore obtain a lower bound on $t(\theta, \beta)$ by finding an expression for $\tan(v^\theta, v(t))$ as a function of t. In Section 4.1, the angle between v^θ and $g(t)$ will be estimated, and these results are used in Section 4.2 to derive an upper bound on the angle between v^θ and $v(t)$. Using (14) we shall then obtain a lower bound on $t(\theta, \beta)$.

4.1. Analysis concerning $g(t)$

In this section we shall estimate $\sin(v^\theta, g(t))$. Notice that for any $i \in \{1, 2, \cdots, n\}$,

$$\begin{aligned} g_i(t) &= \sqrt{v_i(v_i + tp_i)/(1-t)} \\ &= v_i \sqrt{1 + \frac{t}{1-t} \frac{v_i + p_i}{v_i}}. \end{aligned} \tag{15}$$

This expression can be rewritten by applying the following lemma.

Lemma 4.1. *For $\rho \ge -1$ there holds*

$$\sqrt{1+\rho} = 1 + \frac{\rho}{2} - \frac{1}{2} \frac{\rho^2}{(1 + \sqrt{1+\rho})^2}.$$

The proof of Lemma 4.1 is straightforward, and therefore omitted. Using Lemma 4.1, we obtain from (15) for all i that

$$g_i(t) = v_i + \frac{1}{2} \frac{t}{1-t}(v_i + p_i) - \frac{1}{2} \frac{t^2}{(1-t)^2} \frac{v_i(v_i + p_i)^2}{(v_i + g_i(t))^2}.$$

Consequently,

$$(v^\theta)^{\rm T} g(t) = (v^\theta)^{\rm T} v + \frac{1}{2}\frac{t}{1-t}(v^\theta)^{\rm T}(v+p) - \frac{1}{2}\frac{t^2}{(1-t)^2}(v^\theta)^{\rm T}(V+G(t))^{-2}(V+P)^2 v. \qquad (16)$$

From Lemma 2.2 and the fact that $v \in \mathcal{N}(\theta,\beta)$, it follows that

$$v \geq (1-\beta)\frac{(v^\theta)^{\rm T} v}{\|v^\theta\|^2} v^\theta = (1-\beta)v^\theta,$$

where we used $v^\theta \perp (v^\theta - v)$. Therefore, using the obvious inequality $g(t) \geq 0$, we obtain

$$(v^\theta)^{\rm T}(V+G(t))^{-2}(V+P)^2 v \leq \frac{\|v+p\|^2}{1-\beta}. \qquad (17)$$

First consider the case $v \in \mathcal{C}(\theta)$. In that case, $v^\theta = v$ so that

$$(v^\theta)^{\rm T}(v+p) = 0,$$

and we obtain from (16) and (17) that

$$(v^\theta)^{\rm T} g(t) \geq (v^\theta)^{\rm T} v - \frac{1}{2}\frac{t^2}{(1-t)^2}\frac{\|v+p\|^2}{1-\beta} \text{ if } v \in \mathcal{C}(\theta). \qquad (18)$$

Now suppose $v \notin \mathcal{C}(\theta)$. Then $t_L > 0$, and using $v \perp v + p$, we have

$$(v^\theta)^{\rm T}(v+p) = (v^\theta - v)^{\rm T}(v+p) = \frac{1-t_L}{t_L}(v^\theta - v)^{\rm T}(\frac{v+t_L p}{1-t_L} - v).$$

By definition of t_L and the fact that $\mathcal{C}(\theta)$ is a cone, there holds $\frac{v+t_L p}{1-t_L} \in \mathcal{C}(\theta)$. Applying the inequality (9), yields

$$(v^\theta)^{\rm T}(v+p) = \frac{1-t_L}{t_L}(v^\theta - v)^{\rm T}(\frac{v+t_L p}{1-t_L} - v) \geq \frac{1-t_L}{t_L}\left\|v^\theta - v\right\|^2. \qquad (19)$$

Combining (19) and (17) with (16) we obtain

$$(v^\theta)^{\rm T} g(t) \geq (v^\theta)^{\rm T} v + \frac{1}{2}\frac{t}{1-t}\frac{1-t_L}{t_L}\left\|v^\theta - v\right\|^2 - \frac{1}{2}\frac{t^2}{(1-t)^2}\frac{\|v+p\|^2}{1-\beta}, \qquad (20)$$

for the case $v \notin \mathcal{C}(\theta)$.

Now we easily arrive at the following lemma.

Lemma 4.2. *Let* $0 \le t \le \max\{\bar{t} \mid v + \bar{t}p \ge 0\}$*. If* $v \in \mathcal{C}(\theta)$ *then*

$$\sin(v^\theta, g(t))^2 \le \frac{t^2}{(1-t)^2} \frac{\|v+p\|^2}{(1-\beta)\|v\|^2}.$$

If $v \notin \mathcal{C}(\theta)$ *then*

$$\begin{aligned}\sin(v^\theta, g(t))^2 &\le (1 - \frac{t}{1-t}\frac{1-t_L}{t_L})\sin(\mathcal{C}(\theta), v)^2 \\ &\quad + \frac{t^2}{(1-t)^2}\frac{\|v+p\|^2}{(1-\beta)\|v\|^2}.\end{aligned}$$

Proof:

Using (18) we obtain for $v \in \mathcal{C}(\theta)$ that

$$((v^\theta)^{\mathrm{T}} g(t))^2 \ge ((v^\theta)^{\mathrm{T}} v)^2 - \frac{t^2}{(1-t)^2} \frac{((v^\theta)^{\mathrm{T}} v)\|v+p\|^2}{1-\beta}. \tag{21}$$

where we applied the obvious inequality

$$(1 + \frac{1}{2}\rho)^2 \ge 1 + \rho \text{ for any } \rho \in \Re. \tag{22}$$

We remark here that

$$\|g(t)\|^2 = \frac{v^{\mathrm{T}}(v+tp)}{1-t} = \|v\|^2.$$

For $v \in \mathcal{C}(\theta)$, we thus obtain by using (21) and $v = v^\theta$ that

$$\begin{aligned}\sin(v^\theta, g(t))^2 &= 1 - \frac{((v^\theta)^{\mathrm{T}} g(t))^2}{\|v^\theta\|^2 \|v\|^2} \\ &\le \frac{t^2}{(1-t)^2}\frac{\|v+p\|^2}{(1-\beta)\|v\|^2}.\end{aligned} \tag{23}$$

This concludes the case $v \in \mathcal{C}(\theta)$.

From $v^\theta \perp (v^\theta - v)$ it follows that

$$(v^\theta)^{\mathrm{T}} v = \left\|v^\theta\right\|^2.$$

Applying (22) to (20), we thus obtain for $v \notin \mathcal{C}(\theta)$ that

$$\begin{aligned}\sin(v^\theta, g(t))^2 &= 1 - \frac{((v^\theta)^{\mathrm{T}} g(t))^2}{\|v^\theta\|^2 \|v\|^2} \\ &\le \sin(\mathcal{C}(\theta), v)^2 - \frac{t}{1-t}\frac{1-t_L}{t_L}\frac{\left\|v^\theta - v\right\|^2}{\|v\|^2} \\ &\quad + \frac{t^2}{(1-t)^2}\frac{\|v+p\|^2}{(1-\beta)\|v\|^2}.\end{aligned} \tag{24}$$

Now we notice that $v^\theta \perp (v^\theta - v)$ implies

$$\sin(\mathcal{C}(\theta), v) = \sin(v^\theta, v) = \left\| v^\theta - v \right\| / \|v\| . \tag{25}$$

Using (24) and (25), it follows that

$$\begin{aligned} \sin(v^\theta, g(t))^2 &\leq (1 - \frac{t}{1-t}\frac{1-t_L}{t_L}) \sin(\mathcal{C}(\theta), v)^2 \\ &\quad + \frac{t^2}{(1-t)^2}\frac{\|v+p\|^2}{(1-\beta)\|v\|^2}. \end{aligned}$$

□

For notational simplicity, let

$$\gamma_1 := \beta(1-\beta)\frac{\sqrt{n}r(\theta)}{r(\theta)^2 + \beta^2}$$

and

$$\gamma_2 := \max(1, \frac{t_L}{1-t_L}\frac{r(\theta)\|v+p\|}{\beta\|v\|}).$$

Lemma 4.3. *There holds*

$$\beta\theta_\beta < \gamma_1 < \theta_\beta.$$

Proof:

By definition of γ_1 and θ_β, we have

$$\begin{aligned} \gamma_1 &= \beta(1-\beta)\frac{\sqrt{n}r(\theta)}{r(\theta)^2 + \beta^2} \\ &= \frac{(1-\beta)\theta}{\sqrt{1+\beta^2/r(\theta)^2}}\frac{\beta\sqrt{n}}{\sqrt{\theta^2 r(\theta)^2 + \beta^2\theta^2}} \\ &= \theta_\beta\frac{\beta\sqrt{n}}{\sqrt{n - \theta^2 + \beta^2\theta^2}}. \end{aligned}$$

Using $0 < \beta < 1$ and $0 < \theta \leq 1 < n$, the lemma follows from this relation.

□

A useful consequence of Lemma 4.2 is:

Lemma 4.4. *If* $p \neq -v$ *then*

$$0 \leq r(\theta)\tan(v^\theta, g(t)) \leq \beta$$

for

$$0 \leq \frac{t}{1-t} \leq \frac{\gamma_1 \|v\|}{\sqrt{n}\gamma_2 \|v+p\|}.$$

If $p = -v$ *then*

$$\tan(v^\theta, g(t)) = 0 \text{ for } 0 \leq t < 1.$$

Proof:

A requirement of the direction p in Algorithm 1 is that $t_L < 1$, and so $p = -v$ implies $v \in \mathcal{C}(\theta)$. Hence, $g(t) = v = v^\theta$ if $p = -v$ and the last part of the lemma is proved.

For $p \neq -v$, we first notice that $\gamma_2 \geq 1$ by definition. Therefore, using also Lemma 4.3,

$$\frac{t}{1-t} \leq \frac{\gamma_1}{\sqrt{n}} \frac{\|v\|}{\|v+p\|} \leq \frac{\theta_\beta \|v\|}{\sqrt{n}} \frac{1}{\|v+p\|} \leq \frac{\min_{1\leq i\leq n} v_i}{\|v+p\|},$$

where the last inequality follows from Theorem 2.1 and the fact that $v \in \mathcal{N}(\theta,\beta) \subseteq \mathcal{C}(\theta_\beta)$. This implies that

$$0 \leq t < \max\{\bar{t} \mid v + \bar{t}p \geq 0\}.$$

We notice that $v^\theta > 0$ and $g(t) > 0$ so that $\tan(v^\theta, g(t)) \geq 0$. Remarking here that

$$\sin(v^\theta, g(t))^2 = \frac{\tan(v^\theta, g(t))^2}{1+\tan(v^\theta, g(t))^2},$$

it thus follows that

$$0 \leq \tan(v^\theta, g(t)) \leq \frac{\beta}{r(\theta)} \text{ if and only if } \sin(v^\theta, g(t))^2 \leq \frac{\beta^2}{r(\theta)^2+\beta^2}.$$

The case $\frac{t}{1-t} \geq \frac{t_L}{1-t_L}$ now follows easily from Lemma 4.2 and using $\gamma_2 \geq 1$. In the sequel of the proof we consider $0 \leq \frac{t}{1-t} < \frac{t_L}{1-t_L}$. Because $v \in \mathcal{N}(\theta,\beta)$ there holds

$$\sin(\mathcal{C}(\theta), v)^2 = \frac{\tan(\mathcal{C}(\theta), v)^2}{1+\tan(\mathcal{C}(\theta), v)^2} \leq \frac{\beta^2}{r(\theta)^2+\beta^2}.$$

Using Lemma 4.2 this implies that

$$\begin{aligned}\sin(v^\theta, g(t))^2 \quad &\leq \quad \left(1 - \frac{t}{1-t}\frac{1-t_L}{t_L}\right)\frac{\beta^2}{r(\theta)^2+\beta^2} \\ &\quad + \frac{t^2}{(1-t)^2}\frac{\|v+p\|^2}{(1-\beta)\|v\|^2}.\end{aligned}$$

Therefore, $\sin(v^\theta, g(t))^2 \leq \frac{\beta^2}{r(\theta)^2+\beta^2}$ if

$$0 \leq \frac{t}{1-t} \leq \frac{\gamma_1 \|v\|}{\sqrt{n}\gamma_2 \|v+p\|} \leq \frac{1-t_L}{t_L} \frac{\beta^2}{r(\theta)^2+\beta^2} \frac{(1-\beta)\|v\|^2}{\|v+p\|^2},$$

concluding the proof.

□

4.2. Analysis concerning $v(t)$

We shall now consider step lengths t with

$$0 \leq \frac{t}{1-t} \leq \frac{\gamma_1 \|v\|}{\sqrt{n}\gamma_2 \|p\|}.$$

Because $v \perp (v+p)$, we have $\|p\| > \|v+p\|$, so that Lemma 4.4 yields

$$0 \leq r(\theta) \tan(v^\theta, g(t)) \leq \beta. \tag{26}$$

In this section we shall use this relation to estimate $\sin(v^\theta, v(t))$.

Remark that, using $\gamma_1 < \theta_\beta$ and $\gamma_2 \geq 1$,

$$t \leq \frac{t}{1-t} < \frac{\theta_\beta \|v\|}{\sqrt{n}} \frac{1}{\|p\|} \leq \frac{\min_{1\leq i \leq n} v_i}{\|p\|} \leq t^*, \tag{27}$$

where we applied Theorem 2.1, together with the obvious relations

$$\|p_x\|_\infty \leq \|p_x\| \leq \|p\| \text{ and } \|p_s\|_\infty \leq \|p_s\| \leq \|p\|.$$

Using Lemma 4.1, we obtain

$$\begin{aligned}
\frac{v_i(t)}{\sqrt{1-t}} &= \sqrt{(v_i^2 + t v_i p_i + t^2 \Delta x_i \Delta s_i)/(1-t)} \\
&= g_i(t) \sqrt{1 + \frac{t^2}{1-t} \frac{\Delta x_i \Delta s_i}{g_i(t)^2}} \\
&= g_i(t) + \frac{1}{2} \frac{t^2}{1-t} \frac{\Delta x_i \Delta s_i}{g_i(t)} \\
&\quad - \frac{1}{2} \frac{t^4}{(1-t)^2} \frac{(\Delta x_i \Delta s_i)^2}{g_i(t)(g_i(t) + v_i(t)/\sqrt{1-t})^2},
\end{aligned}$$

for $i \in \{1, 2, \cdots, n\}$, so that

$$\begin{aligned}
\frac{(v^\theta)^{\mathrm{T}} v(t)}{\sqrt{1-t}} &= (v^\theta)^{\mathrm{T}} g(t) + \frac{1}{2} \frac{t^2}{1-t} (v^\theta)^{\mathrm{T}} G(t)^{-1} \Delta X \Delta s \\
&\quad - \frac{1}{2} \frac{t^4}{(1-t)^2} (v^\theta)^{\mathrm{T}} G(t)^{-1} (G(t) + V(t)/\sqrt{1-t})^{-2} \Delta X^2 \Delta S^2 e.
\end{aligned} \tag{28}$$

Using $\Delta x \perp \Delta s$ and $\|p_x\|^2 + \|p_s\|^2 = \|p\|^2$, one easily obtains the following result, see e.g. Jansen et al. [2].

Lemma 4.5. *There holds*

$$\|\Delta X \Delta s\|^2 \leq \|\Delta X \Delta s\|_\infty \|\Delta X \Delta s\|_1 \leq \frac{1}{8}\|p\|^4.$$

Based on (26), (28) and Lemma 4.5 we obtain the following estimation of $\sin(v^\theta, v(t))$.

Lemma 4.6. *If* $0 \leq \frac{t}{1-t} \leq \frac{\gamma_1}{\gamma_2}\frac{\|v\|}{\sqrt{n}\|p\|}$ *then*

$$\begin{aligned}\sin(v^\theta, v(t))^2 \;&\leq\; \sin(v^\theta, g(t))^2 \\ &\quad + \frac{t^2}{1-t}\frac{\sqrt{n}\beta}{\sqrt{8}r(\theta)\theta_\beta}\frac{\|p\|^2}{\|v\|^2} + \frac{t^4}{(1-t)^2}\frac{n}{8(1-\beta)\theta_\beta^2}\frac{\|p\|^4}{\|v\|^4}.\end{aligned}$$

Proof:

As $\Delta x \perp \Delta s$, we have

$$(v^\theta)^{\mathrm{T}} G(t)^{-1}\Delta X \Delta s = (v^\theta - \frac{(v^\theta)^{\mathrm{T}} g(t)}{g(t)^{\mathrm{T}} g(t)} g(t))^{\mathrm{T}} G(t)^{-1} \Delta X \Delta s. \tag{29}$$

Applying here the Cauchy-Schwartz inequality yields

$$\mid (v^\theta)^{\mathrm{T}} G(t)^{-1} \Delta X \Delta s \mid \leq \left\| v^\theta - \frac{(v^\theta)^{\mathrm{T}} g(t)}{g(t)^{\mathrm{T}} g(t)} g(t) \right\| \left\| G(t)^{-1} e \right\|_\infty \|\Delta X \Delta s\|. \tag{30}$$

By construction, $g(t) \perp \left[\; v^\theta - \frac{(v^\theta)^{\mathrm{T}} g(t)}{g(t)^{\mathrm{T}} g(t)} g(t) \;\right]$, and therefore

$$\sin(v^\theta, g(t)) = \left\| v^\theta - \frac{(v^\theta)^{\mathrm{T}} g(t)}{g(t)^{\mathrm{T}} g(t)} g(t) \right\| / \left\| v^\theta \right\|. \tag{31}$$

In order to estimate $G(t)^{-1}$, we apply (26) together with Lemma 2.2 and obtain

$$g(t) \geq (1-\beta)\frac{(v^\theta)^{\mathrm{T}} g(t)}{\|v^\theta\|^2} v^\theta = (1-\beta)\cos(v^\theta, g(t))\frac{\|v\|}{\|v^\theta\|} v^\theta. \tag{32}$$

Now using (26) and $v^\theta / \left\| v^\theta \right\| \geq \theta e/\sqrt{n}$, we have

$$g(t)/\|v\| \geq \frac{1-\beta}{\sqrt{1+\beta^2/r(\theta)^2}} \theta e/\sqrt{n} = \theta_\beta e/\sqrt{n}. \tag{33}$$

Stated differently, $\|G(t)^{-1}e\|_\infty \leq \sqrt{n}/(\theta_\beta \|v\|)$, so that (30) implies together with (31) and (26) that

$$\begin{aligned} |\,(v^\theta)^{\mathrm{T}} G(t)^{-1}\Delta X \Delta s\,| &\leq \frac{\sqrt{n}}{\theta_\beta}\sin(v^\theta, g(t))\,\|\Delta X\Delta s\|\,\frac{\|v^\theta\|}{\|v\|} \\ &\leq \frac{\sqrt{n}}{\theta_\beta}\frac{\beta}{r(\theta)}\,\|\Delta X\Delta s\|\,\frac{\|v^\theta\|}{\|v\|}. \end{aligned} \tag{34}$$

Using (32) and the obvious inequality $v(t) \geq 0$, we obtain

$$\begin{aligned} (v^\theta)^{\mathrm{T}} G(t)^{-1}(G(t) + \frac{V(t)}{\sqrt{1-t}})^{-2}\Delta X^2 \Delta S^2 e &\leq \frac{\|G(t)^{-1}e\|_\infty^2 \,\|\Delta X \Delta s\|^2}{(1-\beta)\cos(v^\theta, g(t))}\frac{\|v^\theta\|}{\|v\|} \\ &\leq \frac{n\,\|\Delta X\Delta s\|^2}{\theta_\beta^2(1-\beta)\cos(v^\theta, g(t))}\frac{\|v^\theta\|}{\|v\|^3}, \end{aligned} \tag{35}$$

where the last inequality follows from (33). Combining (34) and (35) with (28) yields

$$\begin{aligned} \frac{(v^\theta)^{\mathrm{T}} v(t)}{\sqrt{1-t}} \;\geq\; & (v^\theta)^{\mathrm{T}} g(t) - \frac{1}{2}\frac{t^2}{1-t}\frac{\sqrt{n}}{\theta_\beta}\frac{\beta}{r(\theta)}\,\|\Delta X \Delta s\|\,\frac{\|v^\theta\|}{\|v\|} \\ & -\frac{1}{2}\frac{t^4}{(1-t)^2}\frac{n\,\|\Delta X\Delta s\|^2}{\theta_\beta^2(1-\beta)\cos(v^\theta, g(t))}\frac{\|v^\theta\|}{\|v\|^3}. \end{aligned} \tag{36}$$

Using (36) and (22) we obtain

$$\begin{aligned} \cos(v^\theta, v(t))^2 &= \frac{((v^\theta)^{\mathrm{T}} v(t))^2}{(1-t)\,\|v^\theta\|^2\,\|v\|^2} \\ &\geq \cos(v^\theta, g(t))^2 \\ &\quad -\frac{t^2}{1-t}\cos(v^\theta, g(t))\frac{\sqrt{n}}{\theta_\beta}\frac{\beta}{r(\theta)}\frac{\|\Delta X \Delta s\|}{\|v\|^2} \\ &\quad -\frac{t^4}{(1-t)^2}\frac{n}{\theta_\beta^2(1-\beta)}\frac{\|\Delta X\Delta s\|^2}{\|v\|^4}. \end{aligned}$$

Applying Lemma 4.5 to the above relation yields the desired result.

□

Let

$$\gamma_2' := \max(1, \frac{t_L}{1-t_L}\frac{r(\theta)\,\|p\|}{\beta\,\|v\|}).$$

We have already seen that because $v \perp (v+p)$, we have $\|p\| > \|v+p\|$, so that

$$\gamma_2' \geq \max(1, \frac{t_L}{1-t_L}\frac{r(\theta)\,\|v+p\|}{\beta\,\|v\|}) = \gamma_2.$$

Lemma 4.7. *There holds*

$$0 \leq r(\theta)\tan(v^\theta, v(t)) \leq \beta$$

for

$$0 \leq \frac{t}{1-t} \leq \frac{\gamma_1 \|v\|}{2\sqrt{n}\gamma_2' \|p\|}.$$

Proof:

Recall from (27) that $t < t^*$. Therefore, $v(t) > 0$ which implies $\tan(v^\theta, v(t)) \geq 0$. Hence,

$$0 \leq \tan(v^\theta, v(t)) \leq \frac{\beta}{r(\theta)} \text{ if and only if } \sin(v^\theta, v(t))^2 \leq \frac{\beta^2}{r(\theta)^2+\beta^2}.$$

Consider first the case $\frac{t}{1-t} \geq \frac{t_L}{1-t_L}$. It follows easily from Lemma 4.2 and Lemma 4.6, and using $\gamma_1 \leq \theta_\beta, \gamma_2' \geq 1$, that

$$\begin{aligned}\sin(v^\theta, v(t))^2 \;\; &\leq \;\; \frac{t^2}{(1-t)^2}\frac{\|v+p\|^2}{(1-\beta)\|v\|^2} \\ &\quad + \frac{t^2}{1-t}\frac{\sqrt{n}\beta}{\sqrt{8}r(\theta)\theta_\beta}\frac{\|p\|^2}{\|v\|^2} \\ &\quad + \frac{t^4}{(1-t)^2}\frac{n}{8(1-\beta)\theta_\beta^2}\frac{\|p\|^4}{\|v\|^4}.\end{aligned}$$

Now using $0 \leq \frac{t}{1-t} \leq \frac{\gamma_1\|v\|}{2\sqrt{n}\gamma_2'\|p\|}$ and the definition of γ_1, i.e.

$$\gamma_1 = \beta(1-\beta)\frac{\sqrt{n}r(\theta)}{r(\theta)^2+\beta^2},$$

we obtain

$$\begin{aligned}\sin(v^\theta, v(t))^2 \;\; &\leq \;\; \frac{1}{4}\frac{(1-\beta)r(\theta)^2}{r(\theta)^2+\beta^2}\frac{\beta^2}{r(\theta)^2+\beta^2} + \frac{1-\beta}{4\sqrt{8}}\frac{\beta^2}{r(\theta)^2+\beta^2} \\ &\quad + \frac{1}{128}\frac{(1-\beta)r(\theta)^2}{r(\theta)^2+\beta^2}\frac{\beta^2}{r(\theta)^2+\beta^2} \\ &< \;\; \frac{\beta^2}{r(\theta)^2+\beta^2},\end{aligned}$$

concluding the case $\frac{t}{1-t} \geq \frac{t_L}{1-t_L}$.

Now suppose $\frac{t}{1-t} < \frac{t_L}{1-t_L}$. We notice that $t \leq \frac{t}{1-t}$ so that

$$t\frac{\sqrt{n}\beta}{\sqrt{8}r(\theta)\theta_\beta}\frac{\|p\|^2}{\|v\|^2} \leq \frac{\beta\|p\|}{2\sqrt{8}r(\theta)\gamma_2'\|v\|}.$$

This implies, by using the definition of γ_2', that

$$t\frac{\sqrt{n}\beta}{\sqrt{8}r(\theta)\theta_\beta}\frac{\|p\|^2}{\|v\|^2} \leq \frac{1}{4\sqrt{2}}\frac{1-t_L}{t_L}\frac{\beta^2}{r(\theta)^2} \leq \frac{1}{2\sqrt{2}}\frac{1-t_L}{t_L}\frac{\beta^2}{r(\theta)^2+\beta^2}, \tag{37}$$

where in the last inequality we used

$$1+\beta^2/r(\theta)^2 < 2. \tag{38}$$

From $\gamma_1 \leq \theta_\beta$ and $\gamma_2' \geq 1$, it follows that

$$\frac{nt^2}{8(1-\beta)\theta_\beta^2}\frac{\|p\|^4}{\|v\|^4} \leq \frac{\|p\|^2}{32(1-\beta)\|v\|^2}. \tag{39}$$

Now applying Lemma 4.2 and Lemma 4.6 together with (37) and (39) and using (26) yields

$$\begin{aligned}\sin(v^\theta, v(t))^2 &\leq \frac{\beta^2}{r(\theta)^2+\beta^2} - \frac{t}{1-t}\frac{1-t_L}{t_L}\frac{\beta^2}{r(\theta)^2+\beta^2} \\ &\quad + \frac{t^2}{(1-t)^2}\frac{\|v+p\|^2}{(1-\beta)\|v\|^2} \\ &\quad + \frac{t}{1-t}\frac{1}{2\sqrt{2}}\frac{1-t_L}{t_L}\frac{\beta^2}{r(\theta)^2+\beta^2} \\ &\quad + \frac{t^2}{(1-t)^2}\frac{\|p\|^2}{32(1-\beta)\|v\|^2} \\ &\leq \frac{\beta^2}{r(\theta)^2+\beta^2}\left(1-\frac{t}{1-t}\frac{1-t_L}{t_L}\left(1-\frac{1}{2}-\frac{1}{2\sqrt{2}}-\frac{1}{64}\right)\right) \\ &\leq \frac{\beta^2}{r(\theta)^2+\beta^2}.\end{aligned}$$

The lemma is proved.

□

4.3. The main result

In the previous section it has been shown that if

$$0 \leq \frac{t}{1-t} \leq \frac{\gamma_1\|v\|}{2\sqrt{n}\gamma_2'\|p\|},$$

then $0 \leq \tan(v^\theta, v(t)) \leq \beta/r(\theta)$, and therefore

$$v(t) \in \mathcal{N}(\theta,\beta).$$

By definition of $t(\theta,\beta)$, this implies that either $t(\theta,\beta)=1$ or

$$\frac{t(\theta,\beta)}{1-t(\theta,\beta)} \geq \frac{\gamma_1 \|v\|}{2\sqrt{n}\gamma_2' \|p\|}. \tag{40}$$

Based on this relation, we arrive at the following theorem.

Theorem 4.1. *For Algorithm 1, choose parameters θ and β such that $\frac{1}{\theta}=\mathcal{O}(1)$, $\frac{1}{\beta}=\mathcal{O}(1)$ and $\frac{1}{1-\beta}=\mathcal{O}(1)$. In every iteration $k=1,2,\cdots$, choose a direction $p^{(k)}$ such that*

$$t_L^{(k)}=\mathcal{O}(\frac{1}{\sqrt{n}}) \text{ and } \left\|p^{(k)}\right\|=\mathcal{O}(\left\|v^{(k)}\right\|).$$

Suppose that for the initial solution pair $(x^{(0)},s^{(0)})$ there holds $\log((x^0)^T s^0)=\mathcal{O}(L)$. Then Algorithm 1 terminates with an optimal solution in $\mathcal{O}(\sqrt{n}L)$ main iterations.

Proof:

From Lemma 4.3 and using (38), we know that

$$\gamma_1 > \beta\theta_\beta = \frac{\beta(1-\beta)\theta}{\sqrt{1+\beta^2/r(\theta)^2}} > \frac{\beta(1-\beta)\theta}{\sqrt{2}}.$$

Hence, γ_1 is bounded from below by a positive constant, independent of the problem size. As $t_L^{(k)}=\mathcal{O}(\frac{1}{\sqrt{n}})$ and $r(\theta)=\mathcal{O}(\sqrt{n})$, we have

$$(\gamma_2')^{(k)}=\max(1,\frac{t_L^{(k)}}{1-t_L^{(k)}}\frac{r(\theta)\left\|p^{(k)}\right\|}{\beta\left\|v^{(k)}\right\|})=\mathcal{O}(1),$$

for $k=1,2,\cdots$. Using (40) we now obtain that

$$t^{(k)} \geq \frac{1}{2}t(\theta,\beta)=\frac{1}{\mathcal{O}(\sqrt{n})}.$$

From (4) we thus have for $k=0,1,2,\cdots$ that

$$(x^{(k+1)})^{\mathrm{T}}s^{(k+1)}=(1-t^{(k)})(x^{(k)})^{\mathrm{T}}s^{(k)}=(1-\frac{1}{\mathcal{O}(\sqrt{n})})(x^{(k)})^{\mathrm{T}}s^{(k)}.$$

The theorem follows immediately from the above inequality.

□

Theorem 4.1 involves conditions on p and t_L. The following lemma shows how these conditions can be satisfied.

Lemma 4.8. *Let* $f \in \mathcal{C}(\theta)$, $f \neq 0$ *and* $\alpha \in \Re_{++}$. *If*

$$p = -v + \alpha r(\theta) \left[\frac{\|v\|^2}{f^T v} f - v \right]$$

then

$$\frac{t_L}{1 - t_L} \leq \frac{1}{\alpha r(\theta)}$$

and

$$\|p\| = (1 + \alpha r(\theta) \tan(f, v)) \|v\|.$$

Proof:

For $t \in (0, 1)$ we have

$$v + tp = (1 - t)(v + \frac{t}{1 - t} \alpha r(\theta) \left[\frac{\|v\|^2}{f^T v} f - v \right]),$$

so that

$$\frac{t_L}{1 - t_L} \leq \frac{1}{\alpha r(\theta)}.$$

Moreover, using $v \perp (\frac{\|v\|^2}{f^T v} f - v)$,

$$\|p\|^2 = \|v\|^2 + \alpha^2 r(\theta)^2 \tan(f, v)^2 \|v\|^2,$$

concluding the proof.

□

If we choose $f \in \mathcal{C}(\theta)$ with $r(\theta) \tan(f, v) = \mathcal{O}(1)$, e.g. $f = v^\theta$, and if we choose α independent of n, then Lemma 4.8 implies that the conditions of Theorem 4.1 are fulfilled. With respect to the conditions on p and t_L in Theorem 4.1, we also remark here that, because $v \perp (v + p)$ we have

$$\cos(-v, p) = \|v\| / \|p\|, \tan(-v, p) = \|v + p\| / \|v\|.$$

Moreover, it can be shown that if

$$0 \leq t < \max\{\bar{t} \mid v + \bar{t}p \geq 0\},$$

then

$$\tan(v, v + tp) = \frac{t}{1 - t} \tan(-v, p).$$

Therefore,

$$\gamma_2 = \max(1, \frac{t_L}{1 - t_L} \frac{r(\theta) \|v + p\|}{\beta \|v\|}) = \max(1, \frac{r(\theta) \tan(v + t_L p, v)}{\beta}),$$

where $\beta \geq r(\theta) \tan(v^\theta, v)$. Hence, if $v + t_L p$ is a multiple of v^θ, then $\gamma_2 = 1$.

5. An implementation

In the previous sections, we have described a generic wide neighborhood method. We have deliberately stated abstract conditions for polynomiality, in order to leave enough room for experimentation. In this section however, we shall describe a specific implementation of the central region method.

In our implementation, we require three additional fixed parameters, viz. α_L, α_U and β_2, with $\alpha_L \leq \alpha_U$ and $\beta_2 \geq \beta$.

For given $f \in \mathcal{C}(\theta)$ with

$$r(\theta)\tan(f, v) \leq \beta_2, \tag{41}$$

and for some scalar α,

$$\alpha \in [\alpha_L, \alpha_U], \tag{42}$$

we set

$$p = -v + \alpha r(\theta) \left[\ \tfrac{\|v\|^2}{f^{\mathrm{T}} v} f - v \ \right].$$

Using Lemma 4.8, it follows that the conditions of Theorem 4.1 are satisfied. At every iteration, we choose the ray $f \in \mathcal{C}(\theta)$ on the cone generated by v^θ and e, i.e. $f = f(\lambda)$ for some $\lambda \in [0, 1]$, where

$$f(\lambda) := (1-\lambda)\frac{\|v\|^2}{(v^\theta)^{\mathrm{T}} v} v^\theta + \lambda \frac{\|v\|^2}{e^{\mathrm{T}} v} e.$$

In order to satify also (41), we have to add a restriction on λ. Let

$$\lambda^* = \max\{\lambda \in [0,1] \mid r(\theta)\tan(f(\lambda), v) \leq \beta_2\}.$$

In fact, λ^* is the root of a quadratic equation. It is easily seen that

$$f(\lambda) \in \mathcal{C}(\theta) \text{ and } r(\theta)\tan(f(\lambda), v) \leq \beta_2 \text{ for } 0 \leq \lambda \leq \lambda^*.$$

In our implementation, the specific choice of $\lambda \in [0, \lambda^*]$ and $\alpha \in [\alpha_L, \alpha_U]$ is made by maximizing the step length t^* towards the boundary of $\Re^n_+$. In other words, we would like to solve the problem

$$\begin{array}{ll} \max_{t^*,\alpha,\lambda} & t^* \\ s.t. & v + t^* P_{AD}(-v + \alpha r(\theta)((1-\lambda) f(0) + \lambda f(1) - v)) \geq 0 \\ & v + t^*(I - P_{AD})(-v + \alpha r(\theta)((1-\lambda) f(0) + \lambda f(1) - v)) \geq 0 \\ & \alpha_L \leq \alpha \leq a_U, 0 \leq \lambda \leq \lambda^*, t^* > 0. \end{array}$$

Numerical experience has shown that $t(\theta,\beta)$ is usually a large proportion of t^*, typically more than 0.99 if $\theta \le 0.1$, so that the maximizer of t^* approximately also maximizes $t(\theta,\beta)$.

Now we transform the variables of this optimization problem as follows:

$$\psi_1 := 1/t^*$$

$$\psi_2 := \alpha\lambda/\lambda^*$$

$$\psi_3 := \alpha(1-\lambda/\lambda^*),$$

with inverse transformation

$$t^* = 1/\psi_1, \alpha = \psi_2 + \psi_3, \lambda = \lambda^*\psi_2/(\psi_2+\psi_3).$$

We introduce vectors q^0 and q^1,

$$q^0 := r(\theta)(f(0)-v)$$

$$q^1 := r(\theta)((1-\lambda^*)f(0)+\lambda^* f(1)-v).$$

In terms of the new variables, the problem becomes

$$\begin{array}{ll} \min_\psi & \psi_1 \\ s.t. & \psi_1 v + P_{AD}(-v) + \psi_3 P_{AD} q^0 + \psi_2 P_{AD} q^1 \ge 0 \\ & \psi_1 v + (I-P_{AD})(-v) + \psi_3 (I-P_{AD}) q^0 + \psi_2 (I-P_{AD}) q^1 \ge 0 \\ & \alpha_L \le \psi_2 + \psi_3 \le \alpha_U \\ & \psi_1 \ge 0, \psi_2 \ge 0, \psi_3 \ge 0, \end{array}$$

a linear program in 3 variables. Remark that the constraint $\psi_1 \ge 0$ is redundant. We apply the dual simplex method to this problem, starting from the initial feasible solution

$$\psi_2 = \alpha_L, \psi_3 = 0,$$

$$\psi_1 = -\min_{1\le i\le n} \min(e_i^{\mathrm{T}} V^{-1} P_{AD}(-v+\psi_2 q^1), e_i^{\mathrm{T}} V^{-1}(I-P_{AD})(-v+\psi_2 q^1)).$$

We stop after 20 simplex iterations or when optimal solution to this auxiliary problem is found. In this way, the procedure takes only $\mathcal{O}(n)$ operations. In our experiments, an optimal solution to the auxiliary problem was always found before reaching the limit of 20 simplex iterations.

The above paragraphs fully describe our implementation of Step 2 in Algorithm 1. For the initialization, Step 0, we use the self-dual reformulation of Ye, Todd and Mizuno [13]. In particular, we use the all-one vector as an initial solution. Because we use standard double-precision floating point arithmatic, we cannot find an exact optimal solution. Therefore, we alleviate our stopping criterion to eight digits of precision, as described in Xu, Hung and Ye [12]. This criterion is the self-dual analogue of the infeasible interior-point stopping criterion of Lustig, Marsten and Shanno [7, 8].

We have applied the above procedure to those feasible Netlib problems [1] for which no BOUNDS section is specified in the MPS input file. We have used the parameters

$$\beta = 0.7, \beta_2 = 5, \alpha_L = 0.05, \alpha_U = 10,$$

and we have tested three different values for θ, viz.

$$\theta \in \{1, 0.1, 0.01\}.$$

The results are listed in Table 1.

It appears that the wide central region choice $\theta < 1$ performs better than the path-following choice $\theta = 1$. The choices $\theta = 0.1$ and $\theta = 0.01$ appear to be more or less equally efficient.

We remark that our results are comparable to [7], but are not yet competitive to the state-of-the-art interior point codes [8, 12]. However, we have implemented an $\mathcal{O}(\sqrt{n}L)$ central region algorithm exactly as described in the previous pages, whereas the implementations of [8] and [12] do not have any theoretical convergence guarantee. Unlike other implementations we are aware of, we do not use any preprocessing, we take equal step lengths in the primal and the dual space, directions are not corrected and we start simply from the all-one solution.

6. Concluding remarks

Most $\mathcal{O}(\sqrt{n}L)$ iteration interior point algorithms closely trace the central path, or more generally a target sequence [2]. In this paper however, we provided a generic algorithm in which the iterates follow the central path only in a very loose sense. Yet, it achieves the $\mathcal{O}(\sqrt{n}L)$ iteration bound under mild conditions on the target points. Interestingly, the targets do not need to be traced. In fact, our method fits in the adaptive step methodology, where the step lengths are only restricted by a neighborhood of a region of centers. We believe that our approach can help to further reduce the gap between theory and practice of interior point methods.

Name	$\theta = 1.00$	$\theta = 0.10$	$\theta = 0.01$	Name	$\theta = 1.00$	$\theta = 0.10$	$\theta = 0.01$
25FV47	65	44	45	SCAGR7	30	20	21
ADLITTLE	34	19	19	SCFXM1	59	32	31
AFIRO	24	13	12	SCFXM2	66	38	37
AGG	44	31	32	SCFXM3	75	37	37
AGG2	45	28	28	SCORPION	32	19	19
AGG3	51	29	27	SCRS8	55	36	33
BANDM	45	27	27	SCSD1	26	17	15
BEACONFD	30	15	16	SCSD6	33	19	19
BLEND	21	14	14	SCSD8	37	17	16
BNL1	87	51	53	SCTAP1	42	23	23
BNL2	91	53	53	SCTAP2	36	20	17
BRANDY	48	27	28	SCTAP3	44	21	19
D2Q06C	*	70	67	SHARE1B	61	38	37
DEGEN2	40	19	19	SHARE2B	32	18	18
DEGEN3	*	24	25	SHIP04L	41	25	24
E226	43	28	27	SHIP04S	42	26	27
FFFFF800	69	45	45	SHIP08L	52	27	27
ISRAEL	48	28	28	SHIP08S	42	23	25
LOTFI	55	25	25	SHIP12L	73	36	37
SC105	33	16	16	SHIP12S	62	31	29
SC205	32	18	17	STOCFOR1	36	18	19
SC50A	27	16	15	STOCFOR2	81	44	44
SC50B	25	14	14	WOOD1P	*	23	24
SCAGR25	38	23	23				

Table 1: The table lists the number of main iterations. An asterisk (*) means that the program did not attain 8 digits of precision within 100 iterations.

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