POLYNOMIAL PRIMAL–DUAL CONE AFFINE SCALING FOR SEMIDEFINITE PROGRAMMING

Arjan B. Berkelaar[1], Jos F. Sturm[2] and Shuzhong Zhang[3]

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Econometric Institute, Erasmus University Rotterdam
P.O. Box 1738, 3000 DR Rotterdam, The Netherlands

ABSTRACT

Semidefinite programming concerns the problem of optimizing a linear function over a section of the cone of semidefinite matrices. In the cone affine scaling approach, we replace the cone of semidefinite matrices by a certain inscribed cone, in such a way that the resulting optimization problem is analytically solvable. The now easily obtained solution to this modified problem serves as an approximate solution to the semidefinite programming problem. The inscribed cones that we use are affine transformations of second order cones, hence the name 'cone affine scaling'. Compared to other primal–dual affine scaling algorithms for semidefinite programming (see, De Klerk, Roos and Terlaky [14]), our algorithm enjoys the lowest computational complexity.

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[1] E-mail: berkelaa@few.eur.nl.

[2] E-mail: sturm@few.eur.nl.

[3] E-mail: zhang@few.eur.nl.

1 Introduction

There is a fast growing number of applications of semidefinite programming in diverse areas, such as statistics, system and control theory, combinatorial optimization and eigenvalue optimization, see e.g. [1, 31]. Moreover, it turns out that interior point methods can solve semidefinite programming (SDP) problems very efficiently, in a comparable way as they solve linear programming (LP) problems, see [2, 8, 14, 15, 16, 18, 22, 21, 24, 26, 33]. For linear programming (LP), the primal affine scaling algorithm is one of the more popular interior point methods, since it is both simple and efficient. Although proposed by Dikin [4, 5] as early as in 1967, the affine scaling algorithm only received the proper attention when Barnes [3] and Vanderbei et al. [32] rediscovered it as a natural simplification of Karmarkar's algorithm [13], Namely, they derived the affine scaling algorithm by simply replacing the projective transformations in Karmarkar's algorithm by affine transformations. However, a remarkable but disappointing result was recently obtained by Muramatsu [19] who gave an example of an SDP problem which satisfies all usual regularity conditions, nonetheless both the short step and the long step variants of the primal affine scaling algorithm converge to a non-optimal point.

Under Karmarkar's projective scaling, the original linear objective function becomes a linear fraction. The search direction that used in Karmarkar's method (in the transformed space) is obtained by optimizing only the numerator of this linear fraction (thus a simplification) over an inscribed sphere of the solution space. This search direction is in general not a descent direction for the original linear objective, and hence a potential function [13] was introduced in Karmarkar's framework. In contrast to this strategy, Padberg [23] derived a search direction by optimizing the *entire fractional objective* over the sphere, and this direction is now called a cone affine scaling direction. Similar algorithms were independently proposed and analyzed by Goldfarb and Xiao [6] and Jan and Fang [9], see also Gonzaga [7]. In fact, one may obtain the cone affine search direction of Padberg [23] and Goldfarb and Xiao [6] by optimizing the original linear objective over a conic section, using merely an affine transformation. This conic section contains the ellipsoid that is used in the classical affine scaling algorithm of Dikin [4, 5].

Monteiro et al. [17] proposed a variant of the affine scaling algorithm for LP that is symmetric in the duality, henceforth called a *primal–dual* affine scaling algorithm. Other primal–dual affine scaling algorithms were proposed by Jansen et al. [10, 11, 12] and Sturm and Zhang [28]. Although polynomiality of the original method of Dikin is considered unlikely, the primal–dual variants have been shown to have polynomial iteration bounds. The primal–dual affine scaling algorithm of Monteiro, Adler and Resende [17] requires $O(nL^2)$ iterations, whereas the primal–dual Dikin-type affine scaling algorithm of Jansen, Roos and Terlaky [11] and the primal–dual cone affine scaling algorithm of Sturm and Zhang [28] solve linear programs in only $O(nL)$ and $O(\sqrt{n}L)$ main iterations respectively.

In extending primal–dual algorithms from LP to SDP, it is important to specify how to enforce the symmetry of the search directions. For this reason, many different search directions based on the same primal–dual path-following scheme are possible, see [29]. Among these search directions we would like to remark on the one originally proposed by Nesterov and Todd [21, 22]. Independently, Sturm and Zhang [26] proposed a symmetrization framework which yields the same search direction as in [21] for the usual path–following primal–dual methods. However, the so-called V-space interpretation presented in [26] can be used for other non path-following schemes as well. For primal–dual affine scaling method, Muramatsu and Vanderbei [20] investigated the performance of various search directions. For several of the known search directions they showed that the convergence fails, even for a simple example, except for the direction based on Nesterov and Todd [21]. This gives an indication that the symmetrization based on Nesterov and Todd [21], i.e. the V–space framework of Sturm and Zhang [26], is theoretically sound.

Using the V–space notion of Sturm and Zhang [26], De Klerk et al. [14] extended both the primal–dual affine scaling algorithm of Monteiro, Adler and Resende [17] and the primal–dual Dikin-affine scaling algorithm of Jansen, Roos and Terlaky [11] to SDP. They derived iteration bounds that generalize the results known for the respective LP counterparts. In this paper we will generalize the primal–dual cone affine scaling algorithm of Sturm and Zhang [28] to SDP. To attain polynomial complexity we confine the sequence of iterates to a small neigborhood of the central trajectory. We show that $O(\sqrt{n}\log(1/\epsilon))$ main iterations of the cone affine scaling algorithm are sufficient to obtain an ϵ approximate solution to an SDP of order n. Hence, of all affine scaling variants for SDP, the cone affine scaling algorithm has the best worst-case guarantee.

This paper is organized as follows. In Section 2, we will discuss the underlying ideas of the cone affine scaling method. We show in Section 3 how the iterates of the cone affine scaling algorithm for semidefinite programming can be computed. The polynomiality of this algorithm will be established in Section 4.

We will use the following notation. The set $\mathcal{S}^n$ denotes the set of all symmetric matrices in $\mathbb{R}^{n\times n}$. Moreover, we will denote $\mathcal{S}^n_{++}$ ($\mathcal{S}^n_{+}$) the set of symmetric positive (semi)definite matrices. The inner product of two matrices X and Y, denoted as $X \bullet Y$, is defined as $\mathrm{tr}(X^{\mathrm{T}}Y)$. The corresponding Frobenius norm $\|X\|_F$ of a matrix X is defined as $\sqrt{X \bullet X}$. The spectral norm of a matrix X is denoted by $\|X\|$. Given $X \in \mathcal{S}^n$, we let $\lambda_{\min}(X)$ denote its smallest eigenvalue. The identity matrix will be denoted as I. The direct sum of two matrices X and Y is denoted by $X \oplus Y$, i.e.

$$X \oplus Y = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}.$$

2 Cone affine scaling fundamentals

Consider the primal SDP problem (P)

$$(P) \quad \min \left\{ C \bullet X \ : \ A_i \bullet X = b_i,\ i = 1, \ldots, m,\ X \succeq 0 \right\},$$

and its dual

$$(D) \quad \max \left\{ b^T y \ : \ \sum_{i=1}^{m} y_i A_i + Z = C,\ Z \succeq 0 \right\},$$

where $X, Z, C, A_1, \ldots, A_m \in \mathcal{S}^n$ and $b, y \in \mathbb{R}^m$. We make the common assumption that there exist positive definite solutions X and Z which are feasible for (P) and (D) respectively (primal–dual Slater condition). In addition, we assume that $n \geq 2$.

As is well-known, the primal–dual Slater condition implies that both (P) and (D) have optimal solutions. Moreover, if the triple (X, y, Z) satisfies the feasibility requirements

$$A_i \bullet X = b_i \text{ for } i = 1, 2, \ldots, m, \tag{1}$$

$$\sum_{i=1}^{m} y_i A_i + Z = C, \tag{2}$$

then

$$X \bullet Z = C \bullet X - b^{\mathrm{T}} y.$$

The quantity $X \bullet Z$ is known as the *duality gap*. Therefore, solving the primal–dual pair (P) and (D) is equivalent to minimizing the duality gap:

$$\min\{X \bullet Z \ : \ (X, Z) \in \mathcal{M},\ X \oplus Z \in \mathcal{S}^{2n}_{+}\}, \tag{3}$$

where $\mathcal{M}$ is the linear manifold of pairs $(X, Z) \in \mathcal{S}^n \times \mathcal{S}^n$ that satisfy (1) and (2) for some $y \in \Re^m$.

The cone affine scaling algorithm generates a sequence of feasible solution pairs (X^1, Z^1), (X^2, Z^2), ... with

$$(X^{i+1}, Z^{i+1}) = \arg\min\{X \bullet Z \ : \ (X, Z) \in \mathcal{M},\ X \oplus Z \in \mathcal{K}_i\}, \tag{4}$$

where for each iteration $i = 0, 1, \ldots$, the set $\mathcal{K}_i \subseteq \mathcal{S}^{2n}_{+}$ is an inscribed convex cone of the semidefinite cone $\mathcal{S}^{2n}_{+}$. The cones $\mathcal{K}_0, \mathcal{K}_1, \ldots$ will be chosen in such a way that

- the cone program (4) can be solved analytically, and
- the duality gaps $X^1 \bullet Z^1, X^2 \bullet Z^2, \ldots$ converge to zero at least linearly.

In particular, we will consider the case where $\mathcal{K}_1, \mathcal{K}_2, \ldots$ are linearly transformed circular cones. First, we will derive some relations between circular cones and the cone of semidefinite matrices. Then, we will discuss a class of linear transformations that affect the circular cone, but leave the semidefinite cone untouched. In this way, we obtain a class of inscribed cones of the semidefinite cone. The cones $\mathcal{K}_0, \mathcal{K}_1, \ldots$ will be chosen from this class.

2.1 The semidefinite cone and circular cones

Consider the circular cone

$$\mathcal{C}^n_{\mathrm{in}} := \{Y \in \mathcal{S}^n \ : \ \mathrm{tr} Y \geq \sqrt{n-1}\, \|Y\|_F\}.$$

The following lemma states that $\mathcal{C}^n_{\mathrm{in}}$ is an inscribed cone of the semidefinite cone $\mathcal{S}^n_{+}$.

Lemma 2.1 *There holds*

$$\mathcal{C}^n_{in} \subseteq \mathcal{S}^n_{+}.$$

The above result follows immediately from Lemma A.2 in the Appendix of this paper. In fact, $\mathcal{C}^n_{\rm in}$ is the largest inscribed circular cone of the semidefinite cone $\mathcal{S}^n_+$. The following lemma characterizes the symmetric matrices that are both on the boundary of $\mathcal{S}^n_+$ and on the boundary of $\mathcal{C}^n_{\rm in}$.

Lemma 2.2 *Let $Y \in \mathcal{S}^n$ be a symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The following two statements are equivalent:*

1. $Y \in \mathcal{C}^n_{in}$ *and* $\lambda_n \leq 0$,
2. $\lambda_n = 0$ *and* $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} \geq 0$.

Proof: It is straightforward to verify that (2) implies (1). To show that the converse is also true, assume that $Y \in \mathcal{C}^n_{\rm in}$ and $\lambda_n \leq 0$. Since $\mathcal{C}^n_{\rm in} \subseteq \mathcal{S}^n_+$, it follows that $\lambda_n = 0$. Let $u \in \Re^n$ be defined as

$$u_1 = u_2 = \cdots = u_{n-1} = 1, \quad u_n = 0.$$

Then, using the fact that $\lambda_n = 0$,

$$\mathrm{tr} Y = \sum_{i=1}^{n-1} \lambda_i = u^{\rm T}\lambda, \quad \|Y\|_F = \|\lambda\|_2 .$$

By definition, $Y \in \mathcal{C}^n_{\rm in}$ implies that

$$u^{\rm T}\lambda = \mathrm{tr} Y \geq \sqrt{n-1}\, \|Y\|_F = \|u\|_2 \, \|\lambda\|_2 ,$$

which means, by the inequality of Cauchy-Schwarz, that $u^T\lambda = \|u\|_2\|\lambda\|_2$, hence λ is a multiple of u. This completes the proof. □

The smallest circumscribing circular cone of $\mathcal{S}^n_+$ is given as

$$\mathcal{C}^n_{\rm out} := \{Y \in \mathcal{S}^n \;:\; \mathrm{tr} Y \geq \|Y\|_F\}.$$

Indeed, if $Y \in \mathcal{S}^n_+$ then it has n nonnegative eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and

$$\mathrm{tr} Y = \|\lambda\|_1 \geq \|\lambda\|_2 = \|Y\|_F .$$

For $n = 2$, we have $\mathcal{C}^2_{\rm in} = \mathcal{C}^2_{\rm out}$, so that in this case the semidefinite cone is circular itself.

An interesting property of the largest inscribed circular cone $\mathcal{C}^n_{\rm in}$ is that it contains the so-called Dikin-sphere [14]:

$$\{Y \in \mathcal{S}^n \;:\; \|Y - I\|_F \leq 1\}.$$

Lemma 2.3 *Let $Y \in \mathcal{S}^n$. If $\|Y - I\|_F \leq 1$ then $Y \in \mathcal{C}^n_{in}$.*

Proof: Assume that $\|Y-I\|_F \leq 1$. Observe that we have $Y \neq 0$ (since $n \geq 2$) and

$$1 \geq \|Y-I\|_F^2 = \|Y\|_F^2 - 2\mathrm{tr}Y + n.$$

Rearranging terms, we get

$$\mathrm{tr}Y \geq \frac{\|Y\|_F^2 + (n-1)}{2}.$$

However, it follows from the arithmetic-geometric mean inequality that

$$\left(\|Y\|_F + \frac{n-1}{\|Y\|_F}\right) \geq 2\sqrt{n-1},$$

and hence

$$\mathrm{tr}Y \geq \sqrt{n-1}\,\|Y\|_F.$$

□

A well known property (Sylvester's law of inertia) of the semidefinite cone is that for any invertible matrix P of order n, we have

$$Y \in \mathcal{S}_+^n \text{ if and only if } PYP^{\mathrm{T}} \in \mathcal{S}_+^n. \tag{5}$$

Now consider the class of linearly transformed circular cones

$$C^n(\beta, P) := \{Y \in \mathcal{S}^n \;:\; \mathrm{tr}PYP^{\mathrm{T}} \geq \sqrt{(1-\beta^2)n}\left\|PYP^{\mathrm{T}}\right\|_F\}, \tag{6}$$

with $\beta \in [0,1]$ and P an invertible matrix. From Lemma 2.1 we know that

$$Y \in \mathcal{C}^n(\frac{1}{\sqrt{n}}, P) \Rightarrow PYP^{\mathrm{T}} \in \mathcal{C}_{\mathrm{in}}^n \subseteq \mathcal{S}_+^n \Rightarrow Y \in \mathcal{S}_+^n,$$

where the last implication follows from (5). Therefore, we have for all $0 \leq \beta \leq 1/\sqrt{n}$ and invertible P that

$$\mathcal{C}^n(\beta, P) \subseteq \mathcal{C}^n(\frac{1}{\sqrt{n}}, P) \subseteq \mathcal{S}_+^n.$$

Remark also that for any orthogonal matrix Q,

$$\mathcal{C}^n(\frac{1}{\sqrt{n}}, Q) = \mathcal{C}_{\mathrm{in}}^n, \quad \mathcal{C}^n(\sqrt{1-\frac{1}{n}}, Q) = \mathcal{C}_{\mathrm{out}}^n,$$

because circular cones are invariant under orthogonal transformations.

2.2 The symmetric primal–dual transformation

Consider a pair $(X, Z) \in \mathcal{M}$ such that X and Z are both positive definite. Let Λ_{XZ} denote a positive diagonal matrix whose diagonal entries are the eigenvalues of the matrix XZ. Define

$$V := \Lambda_{XZ}^{1/2}.$$

It is shown in [26] that there exists an invertible matrix L_d such that

$$L_d^{-1} X L_d^{-\mathrm{T}} = L_d^{\mathrm{T}} Z L_d = V.$$

We see that the pair (V,V) is feasible for the linearly transformed SDP

$$\min\{\bar{X}\bullet\bar{Z}\ :\ (L_d\bar{X}L_d^{\mathrm{T}}, L_d^{-\mathrm{T}}\bar{Z}L_d^{-1})\in\mathcal{M},\ (\bar{X}\oplus\bar{Z})\in\mathcal{S}^{2n}\}.$$

This elucidates that the above transformation is known as the symmetric primal–dual transformation. Due to the invertability of L_d, the above problem admits a one-to-one correspondence with the untransformed problem (3). Remark that the duality gap of the pair (X,Z) in the original SDP pair is the same as the duality gap of the pair (V,V) for the transformed SDP pair:

$$\mathrm{tr}XZ = \mathrm{tr}\Lambda_{XZ} = \mathrm{tr}V^2.$$

It follows that the optimal solution set will be approached if $V\to 0$.

2.3 Cone affine scaling algorithm

The cone affine scaling algorithm to be introduced is iterative in nature. Suppose that in the i-th iteration of the cone affine scaling algorithm, we have an iterate $(X^i,Z^i)\in\mathcal{M}$ with X^i and Z^i positive definite. We compute L_d^i such that

$$(L_d^i)^{-1}X^i(L_d^i)^{-\mathrm{T}} = (L_d^i)^{\mathrm{T}}Z^iL_d^i.$$

Numerical computation of such L_d^i is straightforward, see [30, 27, 25]. The next iterate is then defined as the solution (X^{i+1},Z^{i+1}) of the cone program (4), with $\mathcal{K}_i := \mathcal{C}^{2n}(\beta/\sqrt{2},(L_d^i)^{-1}\oplus(L_d^i)^{\mathrm{T}})$, for a suitable parameter $\beta\in(0,1/\sqrt{n})$. We will derive an analytic expression for the solution (X^{i+1},Z^{i+1}) of (4) in this section. Obviously, this solution does not exist if $\mathcal{K}_i\cap\mathcal{M}=\emptyset$. However, we will show in Section 4 that this cannot occur if we choose $\beta = 1/(4\sqrt{n})$ in the algorithm.

3 Search directions

In this section we will prove that the above algorithm can be implemented in an explicit way, i.e. we show that at each iteration the search directions can be computed analytically. Since the algorithm is iterative, we will illustrate this fact by amplifying how one particular iteration should proceed. For notational convenience, let $X\succ 0, Z\succ 0$ and $(X,Z)\in\mathcal{M}$ be the current iterates under consideration. We need to compute a new solution as follows

$$\begin{aligned} X^+ &:= X + 2\Delta X,\\ Z^+ &:= Z + 2\Delta Z,\end{aligned}$$

where ΔX and ΔZ are displacements satisfying the feasibility requirements

$$\begin{aligned} A_i\bullet\Delta X &= 0,\ i=1,\ldots,m\\ \textstyle\sum_{i=1}^m \Delta y_i A_i + \Delta Z &= 0 \text{ for some } \Delta y\in\mathbb{R}^m,\end{aligned}\tag{7}$$

Cone Affine Scaling Algorithm

Input:
An initial feasible solution (X^0, Z^0), a parameter $\beta \in (0, 1/\sqrt{n})$ and tolerance $\epsilon > 0$

begin
$i := 0$;
while $X^i \bullet Z^i < \epsilon$ **do**
begin
Compute L_d^i satisfying
$(L_d^i)^{-1} X^i (L_d^i)^{-\mathrm{T}} = (L_d^i)^{\mathrm{T}} Z^i L_d^i$.
Calculate the new iterates
$(X^{i+1}, Z^{i+1}) := \arg\min\{X \bullet Z \; : \; (X, Z) \in \mathcal{M},\, (X \oplus Z) \in \mathcal{K}_i\}$,
with $\mathcal{K}_i := \mathcal{C}^{2n}(\beta/\sqrt{2}, (L_d^i)^{-1} \oplus (L_d^i)^{\mathrm{T}})$,
and let $i := i + 1$.
end
end

Figure 1: Cone Affine Scaling Algorithm

and the constraint

$$(X^+ \oplus Z^+) \in \mathcal{C}^{2n}(\beta/\sqrt{2}, (L_d)^{-1} \oplus (L_d)^{\mathrm{T}}). \tag{8}$$

The displacements in the transformed space are given by

$$\begin{aligned} D_x &= L_d^{-1} \Delta X L_d^{-\mathrm{T}}, \\ D_z &= L_d^{\mathrm{T}} \Delta Z L_d. \end{aligned} \tag{9}$$

Let $D_v := D_x + D_z$ and notice from (7) and (9) that $D_x \perp D_z$. Hence, D_x and D_z form an orthogonal decomposition of D_v. Remark that

$$(L_d^{-1} \oplus L_d^{\mathrm{T}})(X^+ \oplus Z^+)(L_d^{-\mathrm{T}} \oplus L_d) = (V + 2D_x) \oplus (V + 2D_z).$$

Constraint (8) can therefore be rewritten as follows:

$$\mathrm{tr}((V + 2D_x) \oplus (V + 2D_z)) \geq \sqrt{(1 - \frac{\beta^2}{2})2n}\, \|(V + 2D_x) \oplus (V + 2D_z)\|_F . \tag{10}$$

However, using the fact that $D_x \perp D_z$, we have

$$\|D_x\|_F^2 + \|D_z\|_F^2 = \|D_x + D_z\|_F^2 = \|D_v\|_F^2 ,$$

so that

$$\begin{aligned} \|(V + 2D_x) \oplus (V + 2D_z)\|_F^2 &= \|V + 2D_x\|_F^2 + \|V + 2D_z\|_F^2 \\ &= \|V\|_F^2 + \|V + 2D_v\|_F^2 \\ &= \|V \oplus (V + 2D_v)\|_F^2 . \end{aligned} \tag{11}$$

Moreover,

$$\mathrm{tr}((V+2D_x)\oplus(V+2D_z)) = \mathrm{tr}(V\oplus(V+2D_v)). \tag{12}$$

Combining (10)–(12), it follows that (10) is equivalent with

$$\mathrm{tr}(V\oplus(V+2D_v)) \geq \sqrt{(1-\frac{\beta^2}{2})2n}\,\|V\oplus(V+2D_v)\|_F\,,$$

i.e. $V\oplus(V+2D_v)\in\mathcal{C}^{2n}(\beta/\sqrt{2},I)$. In order to solve the cone program (4), we have to minimize the duality gap

$$X^+\bullet Z^+ = \mathrm{tr}((V+2D_x)(V+2D_z)) = \|V\|_F^2 + 2V\bullet D_v.$$

It follows that D_v is the solution of

$$\min\{V\bullet D_v\ :\ \mathrm{tr}(V\oplus(V+2D_v)) \geq \sqrt{(1-\frac{\beta^2}{2})2n}\,\|V\oplus(V+2D_v)\|_F\}. \tag{13}$$

The intuition behind this minimization problem is that we want to minimize the duality gap in the V-space, such that V and $V+D_v$ are confined within a circular cone (which is an approximation of the semidefinite cone). From the above analysis it follows that the minimization problem (13) is equivalent to (4). Denoting the angle between V and the identity matrix I by ϕ, i.e.

$$\phi = \arccos(\frac{\mathrm{tr}V}{\sqrt{n}\,\|V\|_F}),$$

we let

$$\delta := \sin(\phi).$$

In other words

$$\delta = \sqrt{1-\frac{(\mathrm{tr}V)^2}{n\,\|V\|_F^2}}. \tag{14}$$

The crux of the cone affine scaling algorithm is that (13) can be solved analytically as the following lemma shows, yielding a search direction which can easily be computed analytically. We will see later that this direction has other important properties.

Lemma 3.1 *For $\delta<\beta<1$, the solution of (13) is*

$$D_v = \frac{\gamma-1}{2(1-\beta^2)\gamma}\frac{\mathrm{tr}V}{n}I - \frac{\gamma+1}{2\gamma}V \tag{15}$$

where

$$\gamma = \sqrt{\frac{2-\beta^2-\delta^2}{\beta^2-\delta^2}}. \tag{16}$$

Proof: Consider the convex program (13). The Lagrangian is

$$\mathcal{L}_\lambda(D_v) = V\bullet D_v + \lambda(\frac{1}{2}\sqrt{(2-\beta^2)n(\|V\|_F^2+\|V+2D_v\|_F^2)} - \mathrm{tr}(V+D_v))$$

with gradient

$$\nabla \mathcal{L}_\lambda(D_v) = V + \eta(V + 2D_v) - \lambda I,$$

where we let

$$\eta := \lambda \sqrt{\frac{n(2-\beta^2)}{\|V\|_F^2 + \|V + 2D_v\|_F^2}}. \tag{17}$$

The Karush–Kuhn–Tucker optimality conditions are

$$\nabla \mathcal{L}_\lambda(D_v) = V + \eta(V + 2D_v) - \lambda I = 0, \tag{18}$$

$$\lambda \geq 0 \text{ and } \sqrt{\frac{n(1-\beta^2/2)}{2}(\|V\|_F^2 + \|V + 2D_v\|_F^2)} - \mathrm{tr}(V + D_v) \leq 0, \tag{19}$$

$$\lambda(\sqrt{\frac{n(1-\beta^2/2)}{2}(\|V\|_F^2 + \|V + 2D_v\|_F^2)} - \mathrm{tr}(V + D_v)) = 0. \tag{20}$$

Rearranging the terms in (18), we have

$$2\eta(V + D_v) = \lambda I + (\eta - 1)V. \tag{21}$$

From (20) we obtain

$$\begin{aligned} 0 &= 2\eta(\sqrt{\frac{n(1-\beta^2/2)}{2}(\|V\|_F^2 + \|V + 2D_v\|_F^2)} - \mathrm{tr}(V + D_v)) \\ &= 2\lambda n(1-\beta^2/2) - 2\eta \mathrm{tr}(V + D_v) \\ &= 2\lambda n(1-\beta^2/2) - \lambda n - \eta \mathrm{tr}V + \mathrm{tr}V, \end{aligned}$$

where, in the last equation, we used (21). The above relation implies that

$$\lambda = \frac{(\eta - 1)\mathrm{tr}V}{n(1-\beta^2)}. \tag{22}$$

However, the quantity η depends on λ. In particular, we have from the definition of η (17) that

$$\begin{aligned} n(2-\beta^2)\lambda^2 &= \eta^2(\|V\|_F^2 + \|V + 2D_v\|_F^2) \\ &= \eta^2 \|V\|_F^2 + \|\lambda I - V\|_F^2, \end{aligned}$$

which can be rewritten as

$$(\eta^2 + 1) \|V\|_F^2 = n(1-\beta^2)\lambda^2 + 2\lambda \mathrm{tr}V.$$

Substituting λ by (22) in the above relation yields

$$(\eta^2 + 1) \|V\|_F^2 = (\eta - 1)\lambda \mathrm{tr}V + 2\lambda \mathrm{tr}V = (\eta + 1)\lambda \mathrm{tr}V = \frac{(\eta^2 - 1)(\mathrm{tr}V)^2}{n(1-\beta^2)}.$$

Furthermore, using (14) it follows that

$$(\eta^2+1)(1-\beta^2) = (\eta^2-1)(1-\delta^2).$$

As η is nonnegative, we conclude that

$$\eta = \sqrt{\frac{2-\beta^2-\delta^2}{\beta^2-\delta^2}} = \gamma.$$

Together with (18) and (22), the lemma follows. □

From the definition of γ, we have

$$\gamma^2+1 = 2\frac{1-\delta^2}{\beta^2-\delta^2}, \quad \gamma^2-1 = 2\frac{1-\beta^2}{\beta^2-\delta^2}, \tag{23}$$

so that

$$\frac{\gamma-1}{2(1-\beta^2)\gamma} = \frac{\gamma^2-1}{2(1-\beta^2)\gamma(\gamma+1)} = \frac{1}{(\beta^2-\delta^2)\gamma(\gamma+1)},$$

and

$$\frac{\gamma+1}{2\gamma} = \frac{\gamma^2+1}{2\gamma(\gamma+1)} + \frac{1}{\gamma+1} = \frac{1-\delta^2}{(\beta^2-\delta^2)\gamma(\gamma+1)} + \frac{1}{\gamma+1}.$$

Applying the above two relations and (14) to (15) it follows that

$$\begin{aligned} D_v &= \frac{\gamma-1}{2(1-\beta^2)\gamma}\frac{\mathrm{tr}V}{n}I - \frac{\gamma+1}{2\gamma}V \\ &= \frac{1-\delta^2}{(\beta^2-\delta^2)\gamma(\gamma+1)}\left(\frac{\|V\|_F^2}{\mathrm{tr}V}I - V\right) - \frac{1}{\gamma+1}V. \end{aligned} \tag{24}$$

The new value of the duality gap, after taking the cone affine scaling step, is derived in the lemma below. Let us first define Λ_{XZ}^+ as the positive diagonal matrix whose diagonal entries are the eigenvalues of the matrix X^+Z^+. Furthermore, let $V^+ := (\Lambda_{XZ}^+)^{\frac{1}{2}}$.

Lemma 3.2 *We have*

$$\|V^+\|_F^2 = \frac{\gamma-1}{\gamma+1}\|V\|_F^2.$$

Proof: Since $D_x \perp D_z$, we have

$$\|V^+\|_F^2 = \mathrm{tr}(V+2D_x)(V+2D_z) = V \bullet (V+2D_v).$$

Now we derive from (24) that

$$V \bullet D_v = -\frac{1}{\gamma+1}\|V\|_F^2.$$

Combining the above two relations, the result follows. □

4 Polynomiality of the cone affine scaling algorithm

We will show in this section that the cone affine scaling algorithm has a polynomial iteration bound. Observe from Lemma 3.1 that the cone affine scaling step is only defined if $\delta < \beta$. Therefore, it is crucial for the convergence analysis to estimate the next value for δ, viz. the quantity $\delta^+ := \sin(\phi^+)$, where ϕ^+ is the angle between V^+ and the identity matrix.

Lemma 4.1 *For any orthogonal matrix Q, there holds*

$$\delta^+ \le \frac{1}{2}\delta + \frac{\left\| V + D_v - QV^+Q^{\mathrm{T}} \right\|_F}{\|V^+\|_F}.$$

Proof: Since δ^+ is the sine of the angle between V and the identity matrix I, we have

$$\begin{aligned} \delta^+ \|V^+\|_F &= \min_\alpha \|\alpha I - V^+\|_F \\ &= \min_\alpha \left\|\alpha I - QV^+Q^{\mathrm{T}}\right\|_F \\ &\le \min_\alpha \|\alpha I - (V + D_v)\|_F + \left\|V + D_v - QV^+Q^{\mathrm{T}}\right\|_F . \end{aligned}$$

However,

$$V + D_v = \frac{\gamma - 1}{2(1-\beta^2)\gamma}\frac{\mathrm{tr}V}{n}I + \frac{\gamma-1}{2\gamma}V, \tag{25}$$

so that

$$\begin{aligned} \min_\alpha \|\alpha I - (V + D_v)\|_F &= \min_\alpha \left\|\alpha I - \frac{\gamma - 1}{2(1-\beta^2)\gamma}\frac{\mathrm{tr}V}{n}I - \frac{\gamma-1}{2\gamma}V\right\|_F \\ &= \frac{\gamma-1}{2\gamma}\min_\alpha \left\|(\frac{2\gamma\alpha}{\gamma-1} - \frac{\mathrm{tr}V}{(1-\beta^2)n})I - V\right\|_F \\ &= \frac{\gamma-1}{2\gamma}\min_\alpha \|\alpha I - V\|_F = \frac{\gamma-1}{2\gamma}\delta\|V\|_F . \end{aligned}$$

Using Lemma 3.2, we further have

$$\frac{\gamma-1}{\gamma}\|V\|_F = \frac{\gamma-1}{\gamma}\sqrt{\frac{\gamma+1}{\gamma-1}}\|V^+\|_F = \sqrt{1-\frac{1}{\gamma^2}}\|V^+\|_F \le \|V^+\|_F .$$

Now, the lemma follows easily from the above derivations. □

The following result is cited from the thesis [25], Lemma 7.4, which is a modification of Corollary 3.1 in Sturm and Zhang [27].

Lemma 4.2 *Suppose $V + D_v \succ 0$. Let $R = D_x - D_z$ and $\rho = \|(V + D_v)^{-\frac{1}{2}}R(V + D_v)^{-\frac{1}{2}}\|_2$. If $\rho < 4/5$ then there exists an orthogonal matrix Q such that*

$$\left\|V + D_v - QV^+Q^{\mathrm{T}}\right\|_F \le \frac{\rho}{(1-2\rho) + \sqrt{1-\rho^2}}\|D_v\|_F .$$

Based on Lemma 4.1 and Lemma 4.2, a natural way to proceed the estimation of δ^+ is to work out the quantities ρ and $\|D_v\|_F / \|V^+\|_F$. This will be done in the following lemmas.

Lemma 4.3 *There holds*

$$\|D_v\|_F = \sqrt{\frac{\beta^2}{\beta^2-\delta^2} - \frac{(1-\beta^2)\delta^2}{(2-\beta^2-\delta^2)(\beta^2-\delta^2)}}\frac{\|V\|_F}{\gamma+1}.$$

Proof: From (24) and the fact that $((\|V\|_F^2/\mathrm{tr}V)I - V)\perp V$, we have

$$\|D_v\|_F^2 = \left[\frac{1-\delta^2}{(\beta^2-\delta^2)\gamma(\gamma+1)}\right]^2 \tan^2(\phi)\,\|V\|_F^2 + \frac{\|V\|_F^2}{(\gamma+1)^2}.$$

By the definition of δ, we have $\tan^2(\phi) = \delta^2/(1-\delta^2)$. Therefore,

$$\begin{aligned}
\|D_v\|_F^2 &= \frac{\|V\|_F^2}{(\gamma+1)^2}\left[1 + \frac{\delta^2(1-\delta^2)}{(\beta^2-\delta^2)^2\gamma^2}\right] \\
&\overset{(16)}{=} \frac{\|V\|_F^2}{(\gamma+1)^2}\left[1 + \frac{\delta^2(1-\delta^2)}{(\beta^2-\delta^2)(2-\beta^2-\delta^2)}\right] \\
&= \frac{\|V\|_F^2}{(\gamma+1)^2}\left[\frac{\beta^2}{\beta^2-\delta^2} - \frac{(1-\beta^2)\delta^2}{(2-\beta^2-\delta^2)(\beta^2-\delta^2)}\right].
\end{aligned}$$

□

Lemma 4.4 *There holds*

$$\frac{\|D_v\|_F}{\|V^+\|_F} = \sqrt{\frac{\beta^2}{2(1-\beta^2)} - \frac{\delta^2}{2(2-\beta^2-\delta^2)}}$$

Proof: Using Lemmas 3.2 and 4.3 and relation (23), it follows that

$$\begin{aligned}
\frac{\|D_v\|_F^2}{\|V^+\|_F^2} &= \left[\frac{\beta^2}{\beta^2-\delta^2} - \frac{(1-\beta^2)\delta^2}{(2-\beta^2-\delta^2)(\beta^2-\delta^2)}\right]\frac{1}{\gamma^2-1} \\
&= \frac{\beta^2}{2(1-\beta^2)} - \frac{\delta^2}{2(2-\beta^2-\delta^2)}.
\end{aligned}$$

□

Lemma 4.5 *It holds that*

$$\rho \le \sqrt{\frac{2n-1}{2-\beta^2}}\beta.$$

Proof: By definition of the cone affine scaling direction, (X^+, Z^+) satisfies the cone constraint (8). In other words, the sine of the angle between the scaled solution $(V + 2D_x) \oplus (V + 2D_z)$ and the identity matrix is at most $\beta/\sqrt{2}$. Using Lemma A.2, this implies that

$$\begin{cases} (1 + \sqrt{\frac{2n-1}{2-\beta^2}}\beta)I \succeq V + 2D_x \succeq (1 - \sqrt{\frac{2n-1}{2-\beta^2}}\beta)I \\ (1 + \sqrt{\frac{2n-1}{2-\beta^2}}\beta)I \succeq V + 2D_z \succeq (1 - \sqrt{\frac{2n-1}{2-\beta^2}}\beta)I. \end{cases}$$

However, since $D_x + D_z = D_v$ and $R := D_x - D_z$, we have

$$V + 2D_x = (V + D_v) + R, \quad V + 2D_z = (V + D_v) - R.$$

Pre- and postmultiplying with $(V + D_v)^{-\frac{1}{2}}$ yields

$$(1 + \sqrt{\frac{2n-1}{2-\beta^2}}\beta)(V + D_v)^{-1} \succeq I \pm (V + D_v)^{-\frac{1}{2}} R (V + D_v)^{-\frac{1}{2}} \succeq (1 - \sqrt{\frac{2n-1}{2-\beta^2}}\beta)(V + D_v)^{-1},$$

which implies that

$$\begin{cases} \rho \leq (1 + \sqrt{\frac{2n-1}{2-\beta^2}}\beta)\|(V + D_v)^{-1}\| - 1 \\ \rho \leq 1 - (1 - \sqrt{\frac{2n-1}{2-\beta^2}}\beta)\|(V + D_v)^{-1}\| \end{cases}$$

Using the fact that $0 < \beta < 1/\sqrt{n}$, it follows from the above pair of inequalities that

$$\rho \leq \sqrt{\frac{2n-1}{2-\beta^2}}\beta.$$

□

Lemma 4.6 *Suppose that $\beta \leq 1/(2\sqrt{n})$. If $\delta \leq \sqrt{2/3}\beta$ then also $\delta^+ \leq \sqrt{2/3}\beta$.*

Proof: From Lemma 4.5 we know that

$$\frac{\rho}{\sqrt{1-\beta^2}} \leq \sqrt{\frac{2n-1}{2 - 3\beta^2 + \beta^4}}\beta, \tag{26}$$

so that for $\beta \leq 1/(2\sqrt{n})$,

$$\frac{\rho}{\sqrt{1-\beta^2}} \leq \sqrt{n}\beta \leq \frac{1}{2}.$$

Therefore, we certainly also have $\rho \leq 1/2$, and hence

$$\frac{\rho}{(1-2\rho) + \sqrt{1-\rho^2}} \leq \frac{2\rho}{\sqrt{3}}.$$

Combining this with Lemma 4.2 and Lemma 4.4, we have

$$\frac{\left\| V + D_v - QV^+Q^{\mathrm{T}} \right\|_F}{\|V^+\|_F} \leq \frac{2\rho}{\sqrt{3}}\sqrt{\frac{\beta^2}{2(1-\beta^2)} - \frac{\delta^2}{2(2-\beta^2-\delta^2)}},$$

for some orthogonal matrix Q. Applying the bound (26) and the fact that $\delta^2 \geq 0$, we further obtain

$$\frac{\left\|V + D_v - QV^+Q^{\mathrm{T}}\right\|_F}{\|V^+\|_F} \leq \frac{\beta}{\sqrt{6}} = \frac{\sqrt{2/3}\beta}{2}.$$

Combining the above relation with Lemma 4.1, it follows that if $\delta \leq \sqrt{2/3}\beta$ then also $\delta^+ \leq \sqrt{2/3}\beta$. □

Recall that in each main iteration of the cone affine scaling algorithm, we take a full step towards the optimizer of the auxiliary cone program (4), see Figure 1. The radius of the auxiliary, inscribed cone $\mathcal{K}_i$ is determined by the parameter β. Lemma 4.6 shows that if we fix β such that $0 < \beta \leq 1/(2\sqrt{n})$, then the the cone affine scaling algorithm generates iterates in a small neighborhood of the central trajectory, viz. $\delta \leq \sqrt{2/3}\beta$. This means that the cone affine scaling algorithm is well defined, and that the algorithm has a linear reduction rate of $1 - 1/O(\sqrt{n})$, see Lemma 3.2. It is now easy to prove polynomial complexity for the cone affine scaling algorithm.

Theorem 4.7 *Suppose X^0 and Z^0 are feasible interior solutions of (P) and (D) respectively. Let ϵ be an accuracy parameter. Moreover, let $\beta = 1/(2\sqrt{n})$, and $\delta^0 = \sin(V^0, I) \leq \sqrt{2/3}\beta$. Then the cone affine scaling algorithm yields a pair of primal and dual feasible solutions (X, Z) with $X \bullet Z < \epsilon$ in at most $O(\sqrt{n}\log(X^0 \bullet Z^0/\epsilon))$ main iterations.*

Proof: From Lemma 4.6 we have $\delta^i \leq \sqrt{2/3}\beta$ for all i. Therefore, we have using definition (16) that

$$\gamma^i = \sqrt{\frac{2}{\beta^2 - \delta^2} - 1} \leq \sqrt{\frac{6}{\beta^2} - 1} \leq \frac{\sqrt{6}}{\beta}.$$

Now, using Lemma 3.2, we have

$$X^{i+1} \bullet Z^{i+1} = (1 - \frac{2}{\gamma^i + 1})X^i \bullet Z^i \leq (1 - \frac{2}{1 + \sqrt{6}/\beta})X^i \bullet Z^i = (1 - \frac{1}{O(\sqrt{n})})X^i \bullet Z^i,$$

which implies the theorem. □

5 Concluding remarks

Even for the special case of linear programming, there is some novelty in the algorithm that we proposed here. Namely, in each main iteration we take a full step to the optimizer of an auxiliary cone problem, whereas partial steps were used in the original primal–dual cone affine scaling algorithm [28].

Recent studies of primal–dual interior point methods for semidefinite programming have yielded a large number of ways to derive primal–dual search directions. Therefore, we like to conclude with some comments on our choice for the V–space framework [26]. This framework was also used in the study of primal–dual affine scaling type algorithms by De Klerk, Roos and Terlaky [14]. Their choice is easily explained, since they analyze the primal–dual Dikin type algorithm, which is based on the notion of V–space solutions. Similarly, a crucial step in the primal–dual cone affine scaling algorithm is the symmetric primal–dual transformation. This transformation is closely tied to the concept of V–space solutions, since it maps both the primal and the dual solution onto the corresponding V–space solution. This symmetric primal–dual transformation is based on the same symmetrization as proposed by Nesterov and Todd [22]. The difference is that [22] uses a self–scaled barrier and is designed only for a path–following scheme. However, neither the primal–dual Dikin–type direction nor the primal–dual cone affine scaling direction point at solutions on the central path. These directions can therefore not be interpreted as Nesterov–Todd directions.

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A Technical Lemmas

Lemma A.1 *Let $Y \in \mathcal{S}^n$. If* $\mathrm{tr}Y = 0$*, then*

$$\|Y\| \le \sqrt{\frac{n-1}{n}} \|Y\|_F .$$

Proof: Let us denote the eigenvalues of Y by $\lambda_1, \ldots, \lambda_n$, where we assume, without loss of generality, that these eigenvalues are ordered such that

$$|\lambda_1| \le |\lambda_2| \le \cdots \le |\lambda_n| .$$

By definition of the Frobenius norm of Y, and using that the trace of a matrix is the sum of its eigenvalues, we have

$$\|Y\|_F^2 = \sum_{i=1}^{n} \lambda_i^2 = \lambda_n^2 + \sum_{i=1}^{n-1} \lambda_i^2. \tag{27}$$

From $\mathrm{tr}Y = 0$ we have

$$\lambda_n = -\sum_{i=1}^{n-1} \lambda_i,$$

so that

$$\sum_{i=1}^{n-1} \lambda_i^2 - \frac{\lambda_n^2}{n-1} = \sum_{i=1}^{n-1} (\lambda_i + \frac{\lambda_n}{n-1})^2 \ge 0. \tag{28}$$

Combining (27) and (28) yields

$$\|Y\|_F^2 \ge (1 + \frac{1}{n-1})\lambda_n^2 = \frac{n}{n-1} \|Y\|^2 .$$

This completes the proof. □

Lemma A.2 *Let $Y \in \mathcal{S}^n$ with* $\mathrm{tr}Y > 0$*. Let ϕ denote the angle between Y and the identity matrix, i.e.*

$$\phi := \arccos(\frac{\mathrm{tr}Y}{\sqrt{n}\,\|Y\|_F}).$$

It holds that

$$(1 + \sqrt{n-1}\tan(\phi))I \succeq \frac{n}{\mathrm{tr}Y} Y \succeq (1 - \sqrt{n-1}\tan(\phi))I.$$

Proof: For any matrix $A \in \mathcal{S}^n$ we know that $\|A\|\, I \succeq A \succeq -\|A\|\, I$. Applying this property with $A = ((n/\mathrm{tr}Y)Y - I) \in \mathcal{S}^n$ we conclude that

$$(1 - \left\|\frac{n}{\mathrm{tr}Y}Y - I\right\|)I \succeq I + (\frac{n}{\mathrm{tr}Y}Y - I) \succeq (1 - \left\|\frac{n}{\mathrm{tr}Y}Y - I\right\|)I. \tag{29}$$

Since $\mathrm{tr}((n/\mathrm{tr}Y)Y - I) = 0$, we obtain from Lemma A.1 that

$$\left\|\frac{n}{\mathrm{tr}Y}Y - I\right\| \le \sqrt{\frac{n-1}{n}} \left\|\frac{n}{\mathrm{tr}Y}Y - I\right\|_F = \sqrt{n-1}\tan(\phi).$$

Together with (29) this implies the lemma. □