Quadratic Maximization and Semidefinite Relaxation

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ABSTRACT

In this paper we study a class of quadratic maximization problems and their semidefinite programming (SDP) relaxation. For a special subclass of the problems we show that the SDP relaxation provides an exact optimal solution. Another subclass, which is $\mathcal{NP}$-hard, guarantees that the SDP relaxation yields an approximate solution with a worst-case performance ratio of 0.87856.... This is a generalization of the well-known result of Goemans and Williamson for the maximum-cut problem. Finally, we discuss extensions of these results in the presence of a certain type of sign restrictions.

Key words: Quadratic programming, semidefinite programming relaxation, polynomial-time algorithm, approximation.

AMS subject classification: 90C20, 90C26.

1 Introduction

Semidefinite programming (SDP) has been an active research area following the seminal work of Nesterov and Nemirovski [9]; see also Alizadeh [1]. We refer to Vandenberghe and Boyd [10] for an overview on SDP. SDP has wide applications in many directions including engineering, economics and combinatorial optimization. In the latter category of applications the recent result of Goemans and Williamson [4] on maximum-cut and satisfiability problems using semidefinite programming relaxation and randomization techniques has generated much research interest. It turns out the method of Goemans and Williamson is a powerful tool to approximately solve certain hard problems in (non-convex) quadratic optimization; see Nesterov [7, 8] and Ye [11]. Recently, Nemirovski, Roos and Terlaky [6] improved some of the results in [8] to allow homogeneous convex quadratic constraints. Ye [12] further extended similar results to certain type of non-homogeneous quadratically constrained problems.

Most of the above mentioned results deal with approximations of non-convex quadratic programming problems which are $\mathcal{NP}$-hard. The goal of this paper is twofold. First, we show that semidefinite programming can also be used as an *exact* solution method for a certain class of non-convex quadratic programs, yielding polynomial-time algorithms. Second, we show that the result of Goemans and Williamson [4] can be generalized to a wider class of quadratic maximization problems, retaining the worst-case performance ratio 0.87856..., which is the case for their original method for the maximum-cut problem.

In this paper the following notation will be used. We represent matrices by capital letters, e.g. X. The notation $X \succeq 0$ means that X is positive semidefinite. If X is an n by n matrix, then diag (X) denotes an n-dimensional vector formed by the diagonal elements of X. The inner-product of two matrices X and Y is $\langle X, Y\rangle = \sum_{i,j} X_{ij} Y_{ij}$. For a given one-dimensional function f, we denote $f(X)$ to be $[f(X_{ij})]_{n\times n}$. For a, $b \in \Re^n$, we write $ab \in \Re^n$ as the component-wise product (or the Hadamard product). Along the same line, we write a^2 to denote the n-dimensional vector which is component-wise square of a. If no confusion is possible then for a given vector d we use the capitalized letter D to denote the diagonal matrix which takes d as its diagonal elements.

2 Quadratic maximization

Consider the following form of quadratic maximization problem:

$$(QP) \quad \begin{array}{ll} \text{maximize} & x^T Q x \\ \text{subject to} & x^2 \in \mathcal{F} \end{array}$$

where $\mathcal{F}$ is a closed convex subset of $\Re^n$, and Q is an arbitrary symmetric matrix.

In this paper we always assume that the optimization problem under consideration has an optimal solution.

As we shall discuss later, this problem is an extension of the optimization model for the maximum-cut problem studied by Goemans and Williamson [4]. This kind of extension was first proposed by Ye [11], and in its general form as formulated in (QP) was considered in Nesterov [8]. We remark that it is not a loss of generality to exclude a linear term in the objective function; see Ye [11]. If, for instance, the objective is $x^TQx + c^Tx$, then one may transform it into $x^TQx + x_{n+1}c^Tx$ with an additional variable x_{n+1} and an additional restriction $x_{n+1}^2 = 1$. It is easy to see that $x_{n+1}x$ is a solution to the original problem.

A related semidefinite programming formulation is given as follows:

$$\begin{array}{lll} (SP) & \text{maximize} & \frac{2}{\pi}\langle Q, D\arcsin(X)D\rangle \\ & \text{subject to} & d \geq 0,\ d^2 \in \mathcal{F} \\ & & X \succeq 0,\ \text{diag}\ (X) = e \end{array}$$

where $\arcsin(X) := [\arcsin(X_{ij})]_{n\times n}$ and e denotes the all-one vector.

Let $v(QP)$ denote the optimal value of (QP) and $v(SP)$ denote the optimal value of (SP). The following result is essentially due to Goemans and Williamson [4]; see also Nesterov [7, 8] and Ye [11]. A proof is provided below for completeness.

Theorem 2.1 *It holds that*

$$v(QP) = v(SP).$$

Proof. We can rewrite (QP) as

$$\begin{array}{ll} \text{maximize} & \langle Q, xx^T\rangle \\ \text{subject to} & d \geq 0,\ d^2 \in \mathcal{F} \\ & x = d\sigma,\ \sigma \in \{-1,+1\}^n. \end{array}$$

Clearly, the optimal value of the above problem can never exceed $v(SP)$ since any feasible solution of it corresponds to a feasible solution of (SP) with $X = \sigma\sigma^T$. Notice that if $\sigma \in \{-1,+1\}^n$ then $\frac{2}{\pi}\arcsin(\sigma\sigma^T) = \sigma\sigma^T$. Hence,

$$v(QP) \leq v(SP). \tag{2.1}$$

To prove the reversed direction of the inequality, we take an arbitrary feasible solution of (SP). Let it be (d, X). Since X is positive semidefinite, let $X = V^TV$ where $V = [v_1, ..., v_n]$. Now, let ξ

be a uniformly generated random unit vector whose dimension equal to the number of rows in V. Having generated such a random direction ξ, let

$$\sigma_i = \text{sign } (v_i^T \xi), \text{ for } i = 1, ..., n$$

where sign $(\cdot)$ is the sign function, i.e. it is $+1$ for non-negative numbers, and -1 for negative numbers.

For any i and j, it follows that

$$\begin{aligned} E[\sigma_i \sigma_j] &= 1 - 2\text{Pr } \{\sigma_i \neq \sigma_j\} \\ &= 1 - \frac{2}{\pi} \angle(v_i, v_j) \\ &= 1 - \frac{2}{\pi} \arccos v_i^T v_j \\ &= 1 - \frac{2}{\pi} \arccos X_{ij} \\ &= \frac{2}{\pi} \arcsin X_{ij}, \end{aligned}$$

where the second equation is based on the fact that Pr $\{\sigma_i \neq \sigma_j\} = \frac{1}{\pi}\angle(v_i, v_j)$. This is a nontrivial observation; cf. Lemma 3.2 of [4].

By using the linearity of the mathematical expectation, we conclude that the expected objective value in (QP) of such solution $d\sigma$ is $E[\langle Q, (d\sigma)(d\sigma)^T \rangle] = \frac{2}{\pi} \langle Q, D(\arcsin X) D \rangle$, which implies that the optimum value of (QP) must be at least as large as the optimum value of (SP). That is, $v(QP) \geq v(SP)$, and combining with (2.1) we have $v(QP) = v(SP)$.

□

Now we consider a relaxed semidefinite maximization problem:

$$\begin{array}{lll} (R) & \text{maximize} & \langle Q, Z \rangle \\ & \text{subject to} & \text{diag } (Z) \in \mathcal{F} \\ & & Z \succeq 0. \end{array}$$

Nesterov [7] has shown that $X \succeq 0$ and diag $(X) = e$ imply $\arcsin(X) \succeq X \succeq 0$. As a consequence of this fact we conclude that (R) is a relaxation of (SP). This is because any feasible solution (d, X) for (SP) also yields a feasible solution for (R) given as

$$Z := \frac{2}{\pi} D \arcsin(X) D.$$

To see this we first note that $Z \succeq 0$ as $\arcsin(X) \succeq X \succeq 0$, and secondly, diag $(Z) = d^2 \in \mathcal{F}$. Since (R) is a relaxation it follows immediately that

$$v(SP) \leq v(R). \tag{2.2}$$

We remark here that if $\mathcal{F}$ is a closed convex set then (R) is a well-formulated convex optimization problem.

Furthermore, if Q is a positive semidefinite matrix, then by noting $\arcsin(X) \succeq X$ again, one has

$$v(QP) \geq \frac{2}{\pi} v(R).$$

The above result was established in Nesterov [7]. This means that the solution of (R) provides a good approximation for (QP) which itself can be $\mathcal{NP}$-hard, with the worst-case performance ratio being $2/\pi \approx 0.63661$. In Section 4 we shall see that this performance ratio can be improved for a more restrictive subclass of problems.

3 A polynomially solvable case

In this section we shall concentrate on the conditions under which a solution for (R) also solves (QP) exactly. The main result in this direction is stated as follows.

Theorem 3.1 *If $Q = [q_{ij}]_{n\times n}$ satisfies $q_{ij} \geq 0$ for all $i \neq j$, then $v(R) = v(SP) = v(QP)$. Moreover, suppose that Z^* is an optimal solution for (R). Then, $\sqrt{diag\ (Z^*)}$ is an optimal solution for (QP).*

In order to prove this result we first note a lemma.

Lemma 3.1 *For any $M \geq 1$ there is $\epsilon_M \geq 0$ such that*

$$\frac{2}{\pi} \arcsin t - Mt \geq 1 - M - \epsilon_M$$

for all $t \in [-1, +1]$. Moreover,

$$\lim_{M\to+\infty} \epsilon_M = 0.$$

Proof. For any $M \geq 1$ and $t \in [-1, 0]$ it holds that

$$\frac{2}{\pi} \arcsin t - Mt \geq \frac{2}{\pi} \arcsin t - t \geq 0 \geq 1 - M.$$

Moreover, the function $h(t) := \frac{2}{\pi} \arcsin t - Mt$ is convex on $[0, +1]$, and

$$h'(t) = \frac{2}{\pi\sqrt{1-t^2}} - M.$$

The function attains its minimum value at $t_M = \sqrt{1 - 4/(\pi M)^2}$. Therefore,

$$\begin{aligned} h(t) &\geq h(t_M) = \frac{2}{\pi} \arcsin \sqrt{1 - 4/(\pi M)^2} - M\sqrt{1 - 4/(\pi M)^2} \\ &\geq \frac{2}{\pi} (\arcsin \sqrt{1 - 4/(\pi M)^2} - \pi/2) + 1 - M \\ &= -\frac{2}{\pi} \arccos \sqrt{1 - 4/(\pi M)^2} + 1 - M \end{aligned}$$

for all $t \in [0, +1]$. Let

$$\epsilon_M = \frac{2}{\pi} \arccos \sqrt{1 - 4/(\pi M)^2}.$$

Clearly,

$$\lim_{M \to +\infty} \epsilon_M = 0.$$

□

Proof of Theorem 3.1:

Let Z^* be an optimal solution of (R). Let

$$d^* = \sqrt{\text{diag } (Z^*)} \text{ and } X^* = (D^*)^+ Z^* (D^*)^+ + \bar{D}$$

where $(D^*)^+$ stands for the pseudo-inverse of D^*, i.e. it is also diagonal and

$$(D^*)^+_{ii} = \begin{cases} (d_i^*)^{-1}, & \text{if } d_i^* > 0; \\ 0, & \text{if } d_i^* = 0, \end{cases}$$

and $\bar{D}$ denotes a binary diagonal matrix where $\bar{D}_{ii} = 1$ if $Z^*_{ii} = 0$ and $\bar{D}_{ii} = 0$ otherwise. It can be easily verified that $Z^*_{ij} = d_i^* d_j^* X^*_{ij}$ for all i and j.

Since $(d^*)^2 \in \mathcal{F}$, diag $(X^*) = e$ and $X^* \succeq 0$, it follows that (d^*, X^*) forms a feasible solution to (SP). Therefore,

$$\begin{aligned} v(SP) &\geq \frac{2}{\pi} \langle Q, D^* \arcsin(X^*) D^* \rangle \\ &= \sum_{i \neq j} q_{ij} d_i^* d_j^* (\frac{2}{\pi} \arcsin X^*_{ij}) + \sum_{i=1}^n q_{ii} (d_i^*)^2 \\ &\geq \sum_{i \neq j} q_{ij} d_i^* d_j^* (1 - M + M X^*_{ij} - \epsilon_M) + \sum_{i=1}^n q_{ii} (d_i^*)^2 \\ &= (1 - M - \epsilon_M) \sum_{i,j} q_{ij} d_i^* d_j^* + M v(R) + \epsilon_M \sum_{i=1}^n q_{ii} (d_i^*)^2 \\ &\geq (1 - M - \epsilon_M) v(R) + M v(R) + \epsilon_M \sum_{i=1}^n q_{ii} (d_i^*)^2 \\ &= (1 - \epsilon_M) v(R) + \epsilon_M \sum_{i=1}^n q_{ii} (d_i^*)^2, \end{aligned} \qquad (3.1)$$

where we let $M \geq 1$. The second inequality of the above derivation follows from Lemma 3.1, and the third inequality follows from the fact that $d^*(d^*)^T$ is also a feasible solution for (R).

By taking $M \to +\infty$ we have

$$v(SP) \geq v(R).$$

Combining the above inequality with (2.2) yields

$$v(SP) = v(R).$$

Moreover, from (3.1) it follows that

$$\sum_{i,j} q_{ij} d_i^* d_j^* = v(R)$$

and this implies that d^* is in fact an optimal solution for (QP). □

Corollary 3.1 *Suppose that $\mathcal{F}$ is a closed convex set and that the off-diagonal elements of Q are nonnegative. Then (QP) can be polynomially approximated in terms of the problem size and the logarithm of the accuracy required.*

Proof. By Theorem 3.1, if the off-diagonal elements of Q are nonnegative then any optimal solution of (R) also solves (QP). Moreover, if $\mathcal{F}$ is a closed convex set, then (R) is a convex program for which polynomial-time approximation algorithms exist in terms of the size of the problem and the logarithm of the accuracy required; see e.g. Grötschel, Lovász and Schrijver [5].

□

We remark that linear programming

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

can be cast as

$$\begin{array}{ll} \text{maximize} & c^T x^2 \\ \text{subject to} & Ax^2 = b \end{array}$$

to which, of course, Corollary 3.1 applies.

4 OD nonpositive quadratic maximization

The well-known algorithm for the maximum-cut problem proposed by Goemans and Williamson [4] is based on the following quadratic maximization formulation:

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^n \sum_{j=1}^n w_{ij} \frac{1-x_i x_j}{2} \\ \text{subject to} & x_i^2 = 1,\ i = 1, ..., n \end{array}$$

where $w_{ij} \geq 0$ for all $i, j = 1, ..., n$. It turns out that this problem can be equivalently rewritten as

$$\begin{array}{ll} \text{maximize} & x^T Q x \\ \text{subject to} & x^2 \in \mathcal{F} \end{array}$$

where $\mathcal{F} = e$ is a single point, and $Q = [q_{ij}]_{n\times n}$ satisfies $q_{ij} = -w_{ij}$ for $i \neq j$ and $q_{ii} = \sum_{j=1}^n w_{ij}$ for $i = 1, ..., n$. (See also [3] for a discussion on the formulation of this problem). Specifically, in this quadratic form we have $q_{ij} \leq 0$ for any $i \neq j$, and $Q \succeq 0$. Goemans and Williamson showed that under this formulation it holds that

$$v(QP) = v(SP) \geq \alpha v(R)$$

with $\alpha = 0.87856....$ For the maximum-cut problem this is the best known worst-case ratio for a polynomial approximation algorithm.

We shall see below that this result can be generalized to any (QP) with $Q \succeq 0$ and $q_{ij} \leq 0$ for all $i \neq j$. First we note the following inequality, which was also used in Goemans and Williamson [4], i.e.

$$\frac{2}{\pi} \arcsin t \leq 1 - \alpha + \alpha t \tag{4.1}$$

for all $t \in [-1, +1]$.

Theorem 4.1 *If $Q = [q_{ij}]_{n\times n}$ satisfies $q_{ij} \leq 0$ for all $i \neq j$ and $Q \succeq 0$, then*

$$v(QP) = v(SP) \geq \alpha v(R)$$

with $\alpha = 0.87856....$

Proof. As in the proof of Theorem 3.1, we consider an optimal solution Z^* for (R). Let

$$d^* = \sqrt{\text{diag }(Z^*)} \text{ and } X^* = (D^*)^+ Z^* (D^*)^+ + \bar{D}.$$

Then, due to the fact that (d^*, X^*) is a feasible solution to (SP), one has

$$\begin{aligned} v(SP) &\geq \sum_{i,j} q_{ij} d_i^* d_j^* (\frac{2}{\pi} \arcsin(X_{ij}^*)) \\ &= \sum_{i \neq j} q_{ij} d_i^* d_j^* (\frac{2}{\pi} \arcsin(X_{ij}^*)) + \sum_{i=1}^{n} q_{ii} (d_i^*)^2 \\ &\geq \sum_{i \neq j} q_{ij} d_i^* d_j^* (1 - \alpha + \alpha X_{ij}^*) + \sum_{i=1}^{n} q_{ii} (d_i^*)^2 \\ &= (1-\alpha) \sum_{i,j} q_{ij} d_i^* d_j^* + \alpha \sum_{i,j} q_{ij} d_i^* d_j^* X_{ij}^* \\ &= (1-\alpha) \sum_{i,j} q_{ij} d_i^* d_j^* + \alpha v(R) \\ &\geq \alpha v(R) \end{aligned}$$

where in the second inequality we used $q_{ij} \leq 0$ and also (4.1), and the last inequality follows from $Q \succeq 0$. □

Observe that the proofs of Theorem 3.1 and Theorem 4.1 depend critically on the fact that the terms $q_{ij} d_i^* d_j^*$, $i \neq j$, are of the same sign. This however, does not necessarily require that q_{ij}, $i \neq j$, are of the same sign per se. In fact, different signs are allowed as long as they will have the same sign under a similar diagonal sign transformation. To make this clearer we introduce the following two definitions.

Definition 4.1 *We call a symmetric matrix $Q = [q_{ij}]_{n \times n}$* almost OD-nonnegative *if there exists a sign vector $\sigma \in \{-1, +1\}^n$ such that*

$$q_{ij} \sigma_i \sigma_j \geq 0$$

for all $i, j = 1, ..., n$ and $i \neq j$.

Definition 4.2 *We call a symmetric matrix $Q = [q_{ij}]_{n \times n}$* almost OD-nonpositive *if there exists a sign vector $\sigma \in \{-1, +1\}^n$ such that*

$$q_{ij} \sigma_i \sigma_j \leq 0$$

for all $i, j = 1, ..., n$ and $i \neq j$.

For example,

$$Q_1 = \begin{bmatrix} 2 & -3 & 5 \\ -3 & 1 & -1 \\ 5 & -1 & -4 \end{bmatrix}$$

is almost OD-nonnegative since $\sigma = [+1, -1, +1]$ satisfies the required condition.

We remark that a matrix can be both almost OD-nonnegative and almost OD-nonpositive, e.g.

$$Q_2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Checking whether or not a given matrix Q is OD-nonnegative (OD-nonpositive) is easy. In fact, if Q does not contain zeros then the sign pattern of σ can be determined by a single column or row of Q. The rest is a simply matter of checking if this pattern is consistent for all columns/rows.

Theorem 4.2 *If Q is almost OD-nonnegative, then*

$$v(QP) = v(SP) = v(R).$$

If $Q \succeq 0$ is almost OD-nonpositive, then

$$v(QP) = v(SP) \geq \alpha v(R)$$

with $\alpha = 0.87856....$

Proof. We follow exactly the same arguments as in the proofs of Theorem 3.1 and Theorem 4.1, except that now we let

$$d^* = \sigma\sqrt{\text{diag } (Z^*)}.$$

It can be easily checked that the rest of the proofs simply go through. □

5 Quadratic maximization with sign restrictions

In this section we shall discuss quadratic maximization (QP) with extra restrictions on the signs of the cross-products $x_i x_j$'s. We shall see that some of the results in the previous sections carry over to this case. Note that in their original paper Goemans and Williamson [4] discussed this type of extensions for the maximum-cut problem. Extensive discussions on this issue can also be found in [2].

Let S_+ and S_- be subsets of double indices. Consider

$$\begin{array}{lll} (QP)' & \text{maximize} & x^T Q x \\ & \text{subject to} & x^2 \in \mathcal{F}, \\ & & x_i x_j \geq 0 \text{ for } (i,j) \in S_+, \\ & & x_k x_l \leq 0 \text{ for } (k,l) \in S_-. \end{array}$$

Correspondingly,

$$\begin{array}{lll} (SP)' & \text{maximize} & \frac{2}{\pi}\langle Q, D\arcsin(X)D\rangle \\ & \text{subject to} & d \geq 0,\ d^2 \in \mathcal{F}, \\ & & X \succeq 0,\ \text{diag}\ (X) = e, \\ & & X_{ij} = 1 \text{ for } (i,j) \in S_+, \\ & & X_{kl} = -1 \text{ for } (k,l) \in S_-. \end{array}$$

Rewriting $(QP)'$ as

$$\begin{array}{ll} \text{maximize} & \langle Q, xx^T\rangle \\ \text{subject to} & d \geq 0,\ d^2 \in \mathcal{F} \\ & x = d\sigma,\ \sigma \in \{-1,+1\}^n, \\ & \sigma_i\sigma_j = +1 \text{ for } (i,j) \in S_+, \\ & \sigma_k\sigma_l = -1 \text{ for } (k,l) \in S_- \end{array}$$

one has

$$v(QP)' \leq v(SP)'.$$

Now we prove that the equality must hold.

Let (d,X) be a feasible solution of $(SP)'$. Let $X = V^TV$ and $V = [v_1, ..., v_n]$. Let ξ be a uniformly generated random vector and let

$$\sigma_i = \text{sign}\ (v_i^T\xi),\ \text{for } i = 1, ..., n.$$

As in the proof of Theorem 2.1 we have

$$E[\sigma_i\sigma_j] = \frac{2}{\pi}\arcsin X_{ij}$$

for any pair (i,j). In particular, if $(i,j) \in S_+$ then $\sigma_i\sigma_j = 1$ and if $(i,j) \in S_-$ then $\sigma_i\sigma_j = -1$. This means that $x = d\sigma$ is always a feasible solution for $(QP)'$. Moreover,

$$E[\langle Q, (d\sigma)(d\sigma)^T\rangle] = \frac{2}{\pi}\langle Q, D\arcsin(X)D\rangle.$$

Hence, $v(QP)' \geq v(SP)'$ and consequently

$$v(QP)' = v(SP)',$$

a relation similar to the one established in Theorem 2.1.

The corresponding relaxation of $(SP)'$ is given as follows:

$$\begin{array}{lll} (R)' & \text{maximize} & \langle Q, DXD\rangle \\ & \text{subject to} & d \geq 0,\ d^2 \in \mathcal{F}, \\ & & X \succeq 0,\ \text{diag}\ (X) = e, \\ & & X_{ij} = 1 \text{ for } (i,j) \in S_+, \\ & & X_{kl} = -1 \text{ for } (k,l) \in S_-. \end{array}$$

For any feasible solution of $(SP)'$, say (d, X), it follows that $(d, \frac{2}{\pi}\arcsin(X))$ is also a feasible solution of $(R)'$. Hence,

$$v(QP)' = v(SP)' \leq v(R)'.$$

Unlike (R), problem $(R)'$ may not be a convex optimization problem. However, in some special cases it can still be easy to solve, e.g. in the application of the maximum-cut problem, as we shall discuss later. In any case, if we fix d, then $(R)'$ reduces to a semidefinite program.

Since $(SP)'$ and $(R)'$ have identical feasible sets, we claim that the proofs for Theorem 3.1 and Theorem 4.1 can be copied to yield the following two analogous results.

Theorem 5.1 *Suppose that $Q = [q_{ij}]_{n\times n}$ satisfies $q_{ij} \geq 0$ for all $i \neq j$ and $(i,j) \notin S_+ \cup S_-$. Moreover, suppose that the matrix $Y = [Y_{ij}]_{n\times n}$ is positive semidefinite, where $Y_{ij} = -1$ for $(i,j) \in S_-$ and $Y_{ij} = 1$ for all $(i,j) \notin S_-$. Then*

$$v(R)' = v(SP)' = v(QP)'.$$

Proof. Let (d^*, X^*) be an optimal solution of $(R)'$. Certainly it is also a feasible solution for $(SP)'$. Therefore,

$$\begin{aligned}
v(SP)' &\geq \frac{2}{\pi}\langle Q, D^* \arcsin(X^*) D^*\rangle \\
&= \sum_{\{i\neq j \text{ and } (i,j)\notin S_+\cup S_-\}} q_{ij} d_i^* d_j^* (\frac{2}{\pi}\arcsin X_{ij}^*) \\
&\quad + \sum_{i=1}^n q_{ii}(d_i^*)^2 + \sum_{(i,j)\in S_+} q_{ij} d_i^* d_j^* - \sum_{(i,j)\in S_-} q_{ij} d_i^* d_j^* \\
&\geq \sum_{\{i\neq j \text{ and } (i,j)\notin S_+\cup S_-\}} q_{ij} d_i^* d_j^* (1 - M + MX_{ij}^* - \epsilon_M) \\
&\quad + \sum_{i=1}^n q_{ii}(d_i^*)^2 + \sum_{(i,j)\in S_+} q_{ij} d_i^* d_j^* - \sum_{(i,j)\in S_-} q_{ij} d_i^* d_j^* \\
&= (1 - M - \epsilon_M)\sum_{i,j} q_{ij} d_i^* d_j^* Y_{ij} + Mv(R)' \\
&\quad + \epsilon_M[\sum_{i=1}^n q_{ii}(d_i^*)^2 + \sum_{(i,j)\in S_+} q_{ij} d_i^* d_j^* - \sum_{(i,j)\in S_-} q_{ij} d_i^* d_j^*] \\
&\geq (1 - M - \epsilon_M)v(R)' + Mv(R)' \\
&\quad + \epsilon_M[\sum_{i=1}^n q_{ii}(d_i^*)^2 + \sum_{(i,j)\in S_+} q_{ij} d_i^* d_j^* - \sum_{(i,j)\in S_-} q_{ij} d_i^* d_j^*] \\
&= (1 - \epsilon_M)v(R)'
\end{aligned}$$

$$+\epsilon_M[\sum_{i=1}^n q_{ii}(d_i^*)^2 + \sum_{(i,j)\in S_+} q_{ij}d_i^*d_j^* - \sum_{(i,j)\in S_-} q_{ij}d_i^*d_j^*]$$

where $M \geq 1$, and last inequality is due to the fact that (d^*, Y) forms a feasible solution for $(R)'$. Letting $M \to \infty$ we obtain

$$v(SP)' \geq v(R)'.$$

Therefore, $v(SP)' = v(R)'$, and the theorem is proven. □

Theorem 5.2 *Suppose that $Q = [q_{ij}]_{n\times n} \succeq 0$ satisfies $q_{ij} \leq 0$ for all $i \neq j$. Then,*

$$v(QP)' = v(SP)' \geq \alpha v(R)'$$

with $\alpha = 0.87856...$.

Proof. Let (d^*, X^*) be an optimal solution of $(R)'$.

Then,

$$\begin{aligned}
v(SP)' &\geq \sum_{i,j} q_{ij}d_i^*d_j^*(\frac{2}{\pi}\arcsin(X_{ij}^*)) \\
&= \sum_{i\neq j} q_{ij}d_i^*d_j^*(\frac{2}{\pi}\arcsin(X_{ij}^*)) + \sum_{i=1}^n q_{ii}(d_i^*)^2 \\
&\geq \sum_{i\neq j} q_{ij}d_i^*d_j^*(1-\alpha+\alpha X_{ij}^*) + \sum_{i=1}^n q_{ii}(d_i^*)^2 \\
&= (1-\alpha)\sum_{i,j} q_{ij}d_i^*d_j^* + \alpha\sum_{i,j} q_{ij}d_i^*d_j^*X_{ij}^* \\
&= (1-\alpha)\sum_{i,j} q_{ij}d_i^*d_j^* + \alpha v(R)' \\
&\geq \alpha v(R)'.
\end{aligned}$$

□

For the maximum-cut problem, $d^* = e$, and so the corresponding problem $(R)'$ reads as follows:

$$\begin{array}{ll}
\text{maximize} & \langle Q, X\rangle \\
\text{subject to} & X \succeq 0,\ \text{diag}\ (X) = e, \\
& X_{ij} = 1 \text{ for } (i,j) \in S_+, \\
& X_{kl} = -1 \text{ for } (k,l) \in S_-.
\end{array}$$

This is an SDP problem. An application of Theorem 5.2 is that, even if one requires in advance that a given set of arcs must be in the cut and another given set of arcs must be out of the cut, as long as the problem remains feasible one can still find a solution in polynomial time with worst-case performance ratio no less than 0.87856.... We remark that such a restricted version of the maximum-cut problem is denoted by *MAX RES CUT* in Goemans and Williamson [4]. The above statement is exactly the 0.87856...-approximation result of Goemans and Williamson for the MAX RES CUT problem.

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