

**MARGINAL AND PARAMETRIC ANALYSIS OF THE
CENTRAL OPTIMAL SOLUTION**

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ABSTRACT

In this paper we investigate the sensitivity analysis of the parameterized central path. First, a complete marginal analysis of the central optimal solution is developed. This analysis explains the differential properties of the central optimal solution with respect to both the cost coefficients and the right-hand side components. We also show that the marginal derivatives are uniformly bounded. Second, we present three conditions for which the parameterized central path converges. Two of these results allow the difficult situation of simultaneous perturbations in the cost coefficients and right-hand side levels.

Key words: Parametric linear programming, sensitivity analysis, analytic central path.

AMS subject classification: 90C05, 90C31.

RÉSUMÉ

Dans cet article on recherche l'analyse de sensibilité du chemin central paramétrisé. D'abord, nous complétons l'analyse marginale de la solution optimale centrale par rapport aux coefficients de la fonction coût et aux composantes du côté droit. On démontre aussi que les dérivées marginales sont bornées uniformément. Ensuite on présente trois conditions chacune assurant que le chemin central paramétrisé converge. Deux des ces résultats permettent des perturbations simultanées des coefficients de la fonction coût et des valeurs du vecteur du côté droit (dont le traitement était autrefois difficile).

Mots clés: Programmation linéaire paramétrique, analyse de sensibilité, chemin central analytique.

AMS classification du sujet: 90C05, 90C31.

1 Introduction

Sensitivity analysis and parametric programming play an important role in linear programming. In some cases, knowing how an optimum changes relative to perturbations in the input data is more important than simply computing an optimal solution. Indeed, the data of a given problem can never be absolutely accurate in real applications. Hence, it is crucial to keep track of how optimal solutions, or the optimal value, change if the data changes. Because of this, many researchers have investigated the area of sensitivity analysis; for recent advances see Gal and Greenberg [13]. There are several software packages, many of them developed in-house by large companies, that provide aid to practitioners when trying to understand a specific solutions characteristics. As an example we mention ANALYZE [9], which has been used by the Energy Information Administration, British Telecommunications, and Amoco Oil.

We make the distinction between the following two questions:

- How does an optimal solution behave with respect to data perturbations?
- How does the optimal objective value behave with respect to data perturbations?

Both questions are completely addressed from a basic solution perspective. However, interior point algorithms for linear programming terminate with a solution, called the *central optimal solution*, that has different characteristics from that of a basic optimal solution, and there are circumstances under which the central optimal solution is more desirable than a basic optimal solution [10, 19]. The parametric analysis of the optimal value is developed using the central optimal solution in [2, 19, 29]. Monteiro and Mehrotra [24] and Roos, Terlaky, and Vial [30] independently developed an algorithm that uses the central optimal solution to completely describe the objective function along any single direction of change in either the cost coefficients or the right-hand side vector (the algorithm in [30] has the stronger property that any description of the optimal face may be used). Greenberg [11] shows that the question of how the objective function responds to simultaneous changes in the cost coefficients and the right-hand side levels may be answered using the central optimal solution (also see the earlier work of Mills [22]). All of the above analysis deals with information that is attainable from asking the second question.

We address the first question by discussing how the central optimal solution behaves when perturbations occur in either the objective coefficients or the right-hand side levels. Specifically, we present an analysis of both the marginal properties of the central optimal solution and the convergence properties of the parameterized central path. The organization of the paper is as follows. Section 2 presents the notation and terminology used throughout. Sections 3 and 4 present the marginal and parametric analysis, respectively. Section 5 concludes with a summary of the results.

2 Notation and Terminology

For the scalars θ_b and θ_c , consider the following standard form linear program:

$$\begin{aligned} (LP)_{(\theta_b, \theta_c)} \quad & \text{minimize} && (c + \theta_c \delta c)^T x \\ & \text{subject to} && Ax = b + \theta_b \delta b \\ & && x \geq 0 \end{aligned}$$

and its dual problem

$$\begin{aligned} (LD)_{(\theta_b, \theta_c)} \quad & \text{maximize} && (b + \theta_b \delta b)^T y \\ & \text{subject to} && A^T y + s = c + \theta_c \delta c \\ & && s \geq 0, \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$, $m \leq n$, $b + \theta_b \delta b \in \mathbb{R}^m$, and $c + \theta_c \delta c \in \mathbb{R}^n$. The vectors δb and δc are called the right-hand side direction of change and the cost coefficient direction of change, respectively. Similarly, the non-negative scalars θ_b and θ_c are referred to as the right-hand side parameter and the cost coefficient parameter. The primal feasible region is denoted by \mathcal{P}_{θ_b} , and the dual feasible region is denoted by \mathcal{D}_{θ_c} . The strict interiors of these sets are $\mathcal{P}_{\theta_b}^o$ and $\mathcal{D}_{\theta_c}^o$, defined by $\{x \in \mathcal{P}_{\theta_b} : x > 0\}$ and $\{(y, s) \in \mathcal{D}_{\theta_c} : s > 0\}$. Furthermore, the primal and dual optimality sets are $\mathcal{P}_{(\theta_b, \theta_c)}^*$ and $\mathcal{D}_{(\theta_b, \theta_c)}^*$.

We say the input data, (b, c) , is *admissible* if both $(LP)_{(0,0)}$ and $(LD)_{(0,0)}$ are *strictly feasible* —i.e. $\mathcal{P}_{(0,0)}^o \neq \emptyset$ and $\mathcal{D}_{(0,0)}^o \neq \emptyset$. Moreover, $(\delta b, \delta c)$ is an *admissible direction* of change if there exists $\theta^* > 0$ such that $(b, c) + \theta(\delta b, \delta c)$ is admissible for all $\theta \in [0, \theta^*]$. For convenience, we define

$$\begin{aligned} \mathcal{H} &\equiv \{(\delta b, \delta c) : (\delta b, \delta c) \text{ is an admissible direction of change}\}, \\ \mathcal{H}_b &\equiv \{\delta b : (\delta b, 0) \text{ is an admissible direction of change}\}, \text{ and} \\ \mathcal{H}_c &\equiv \{\delta c : (0, \delta c) \text{ is an admissible direction of change}\}. \end{aligned}$$

We make the following assumption throughout.

Assumption: Both (b, c) and $(\delta b, \delta c)$ are admissible.

This assumption guarantees that both $(LP)_{(\theta_b, \theta_c)}$ and $(LD)_{(\theta_b, \theta_c)}$ are admissible for sufficiently small θ_b and θ_c . It is easy to show that the strict feasibility assumption implies that $\mathcal{H}_c = \mathbb{R}^n$. However, because we do not assume that A has full row rank, strict feasibility does not imply that $\mathcal{H}_b = \mathbb{R}^m$. As an illustration, consider a standard transportation problem with equality constraints. A right-hand side direction of change that implies an increase in demand without a corresponding increase in supply is not admissible. What is true, and rather easy to show, is that \mathcal{H}_b is the column space of A .

Because we do not assume that A has full row rank, the dual variables y and s are not related in a one-to-one fashion, and from a mathematical perspective we desire y and s to have such a relationship. Denote the rank of A by \bar{m} , and let \bar{A} be an $\bar{m} \times n$ matrix consisting of \bar{m} independent rows of A . Denote the corresponding subvectors of y , b , and \bar{b} by \bar{y} , \bar{b} , and $\bar{\bar{b}}$, respectively. Furthermore, denote the remaining entries of y by \hat{y} , so that $y = \begin{bmatrix} \bar{y}^T & \hat{y}^T \end{bmatrix}^T$, assuming that a suitable reordering has taken place. One may proceed with the (LP) problem

$$\min\{(c + \theta_c \delta c)^T x : \bar{A}x = \bar{b} + \theta_b \bar{\bar{b}}, x \geq 0\},$$

and its dual

$$\max\{(\bar{b} + \theta_b \bar{\bar{b}})^T \bar{y} : \bar{A}^T \bar{y} + s = c + \theta_c \delta c, s \geq 0\}.$$

The full row rank of \bar{A} now means that \bar{y} and s are related in a one-to-one fashion. Given a primal feasible element x and a dual feasible slack s , we always choose the unique dual multiplier y with $\hat{y} = 0$. We note that one can not realistically disregard rows because of the fact that many real world models do not have the property that the coefficient matrix is full row rank and practitioners might want to make sensitivity queries about any row. Because of this, we prefer to establish our results with the original model $(LP)_{(\theta_b, \theta_c)}$, where A may not have full row rank (but always with the assumption that $\hat{y} = 0$).

Each of \mathcal{H} , \mathcal{H}_b , and \mathcal{H}_c have a subset of interest that relies on the concept of the *optimal partition*. Define

$$\begin{aligned} B(\theta_b, \theta_c) &= \{i : \text{there exists } x \in \mathcal{P}_{(\theta_b, \theta_c)}^* \text{ such that } x_i > 0\} \text{ and} \\ N(\theta_b, \theta_c) &= \{1, 2, 3, \dots, n\} \setminus B(\theta_b, \theta_c). \end{aligned}$$

The sets $B(\theta_b, \theta_c)$ and $N(\theta_b, \theta_c)$ form the optimal partition and completely define the optimal sets as follows [30],

$$\mathcal{P}_{(\theta_b, \theta_c)}^* = \{x \in \mathcal{P}_{\theta_b} : x_{N(\theta_b, \theta_c)} = 0\} \quad \text{and} \quad \mathcal{D}_{(\theta_b, \theta_c)}^* = \{(y, s) \in \mathcal{D}_{\theta_c} : s_{B(\theta_b, \theta_c)} = 0\},$$

where a set subscript is used to denote the sub-vector corresponding to the indices contained in the set. Let \mathcal{H}^1 , \mathcal{H}_b^1 , and \mathcal{H}_c^1 be respective subsets of \mathcal{H} , \mathcal{H}_b , and \mathcal{H}_c , where a direction of change being in one of these subsets implies that the optimal partition remains intact for sufficiently small amounts of perturbation. In other words, \mathcal{H}^1 , \mathcal{H}_b^1 , and \mathcal{H}_c^1 are the admissible directions of change for which the optimal sets are invariant with respect to small amounts of data change. Investigations into the dimensions of these subsets are found in [14], and a geometric description is found in Figure 1. Denoting the column and row spaces of a matrix, M , by $\text{coll}(M)$ and $\text{row}(M)$, we have the following equalities [11, 16]:

$$\mathcal{H}_b^1 = \text{coll}(A_B), \quad \mathcal{H}_c^1 = \text{row} \left(\begin{bmatrix} A_B & A_N \\ 0 & I \end{bmatrix} \right), \quad \mathcal{H}^1 = \mathcal{H}_b^1 \times \mathcal{H}_c^1. \quad (2.1)$$

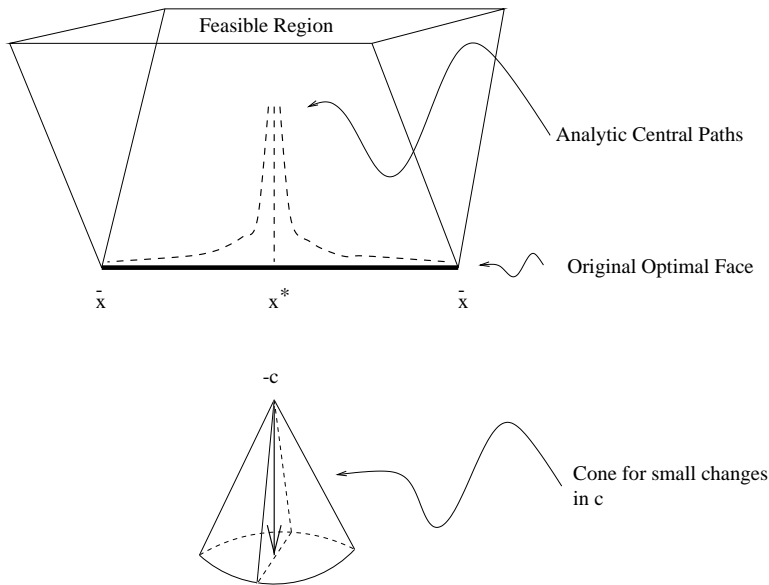


Figure 1: The cut through the cone of cost coefficient changes indicates the perturbations that leave the original optimal partition intact. If c is perturbed so that it is not within this cut, the central optimal solution changes from x^* to one of the \bar{x} 's.

From our assumption that both (b, c) and $(\delta b, \delta c)$ are admissible, there exists $\hat{\theta}_b$ and $\hat{\theta}_c$ such that $(LP)_{(\theta_b, \theta_c)}$ and $(LD)_{(\theta_b, \theta_c)}$ are strictly feasible for (θ_b, θ_c) in $[0, \hat{\theta}_b) \times [0, \hat{\theta}_c)$. It is well known that the analytic central path exists when strict feasibility is guaranteed [30]. Hence, for any fixed $(\theta_b, \theta_c) \in [0, \hat{\theta}_b) \times [0, \hat{\theta}_c)$, there exists a unique $x(\mu) \in \mathcal{P}_{\theta_b}^o$ and a unique $(y(\mu), s(\mu)) \in \{(y, s) \in \mathcal{P}_{\theta_c}^o : \hat{y} = 0\}$, where the uniqueness of y follows because $\hat{y} = 0$, such that

$$x_i(\mu)s_i(\mu) = \mu, \quad i = 1, 2, \dots, n.$$

The above notation indicates a reliance only on μ , which is too restrictive for our purposes. We extend this notation to include a reliance on both θ_b and θ_c . Formally, the *parameterized central path* is defined by

$$x(\theta_b, \theta_c, \mu) \text{ and } (y(\theta_b, \theta_c, \mu), s(\theta_b, \theta_c, \mu)),$$

where these vectors satisfy

$$\left. \begin{aligned} x(\theta_b, \theta_c, \mu) &\in \mathcal{P}_{\theta_b}^o, \\ (y(\theta_b, \theta_c, \mu), s(\theta_b, \theta_c, \mu)) &\in \mathcal{D}_{\theta_c}^o, \\ \hat{y}(\theta_b, \theta_c, \mu) &= 0, \text{ and} \\ x_i(\theta_b, \theta_c, \mu)s_i(\theta_b, \theta_c, \mu) &= \mu, \quad i = 1, 2, \dots, n. \end{aligned} \right\} \quad (2.2)$$

(Note: Technically the parameterized central path is a surface because of its dependence on three parameters. However, we abusively use “path” because it conjures up the correct geometry for

the convergence properties presented later.) Since the work of Sonnevend [32], the elements of the parameterized central path have been known to be analytic functions not only of μ , $\mu > 0$, but also of θ_b and θ_c . Hence, $x(\theta_b, \theta_c, \mu)$ and $s(\theta_b, \theta_c, \mu)$ are analytic functions for $\mu > 0$.

It is useful to notice that the conditions in (2.2) are the necessary and sufficient Lagrange conditions showing that $x(\theta_b, \theta_c, \mu)$ is the unique minimizer of

$$\min \left\{ (c + \theta_c \delta c)^T x - \mu \sum_{i=1}^n \ln(x_i) : x \in \mathcal{P}_{\theta_b}^o \right\}.$$

Here, $\sum_{i=1}^n \ln(x_i)$ is the logarithmic barrier function, and μ is called the *barrier parameter*. One of the well established properties of the analytic central path is that it converges to the central optimal solution as the barrier parameter decreases to zero [20, 30]. We define these limits by,

$$\begin{aligned} x^*(\theta_b, \theta_c) &\equiv \lim_{\mu \downarrow 0} x(\theta_b, \theta_c, \mu) \text{ and} \\ (y^*(\theta_b, \theta_c), s^*(\theta_b, \theta_c)) &\equiv \lim_{\mu \downarrow 0} (y(\theta_b, \theta_c, \mu), s(\theta_b, \theta_c, \mu)). \end{aligned}$$

Furthermore, these limits are the analytic centers, respectively, of $\mathcal{P}_{(\theta_b, \theta_c)}^*$ and $\mathcal{D}_{(\theta_b, \theta_c)}^*$. That is, $x^*(\theta_b, \theta_c)$ is the unique solution to

$$\max \left\{ \sum_{i \in B(\theta_b, \theta_c)} \ln(x_i) : Ax = b + \theta_b \delta b, x_{B(\theta_b, \theta_c)} > 0, x_{N(\theta_b, \theta_c)} = 0 \right\} \quad (2.3)$$

and $(y^*(\theta_b, \theta_c), s^*(\theta_b, \theta_c))$ solves

$$\max \left\{ \sum_{i \in N(\theta_b, \theta_c)} \ln(s_i) : A^T y + s = c + \theta_c \delta c, s_{N(\theta_b, \theta_c)} > 0, s_{B(\theta_b, \theta_c)} = 0 \right\}. \quad (2.4)$$

The standard operator notation, D , is used for differentiation. If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is k times differentiable in a neighborhood of $u \in \mathbb{R}^n$, then $D_{u_i}^k f(u)$ is the vector whose j^{th} component is $\frac{\partial^k f_j}{\partial u_i^k}$, where f_j is the j^{th} component function.

In this paper, we study the existence and value of $D_{\theta_b}^k x^*(\theta_b, 0)$ and $D_{\theta_c}^k x^*(0, \theta_c)$. We extend this notation as follows,

$$\begin{aligned} D_{\theta_b}^1 x^*(0^+, 0) &= \lim_{\theta_b \downarrow 0} D_{\theta_b}^1 x^*(\theta_b, 0), \text{ and} \\ D_{\theta_b}^1 x^*(0, 0) &= \lim_{\theta_b \downarrow 0} \frac{x^*(\theta_b, 0) - x^*(0, 0)}{\theta_b}. \end{aligned}$$

For convenience we make D^0 the identity operator, which allows the following similar extensions for $k = 1, 2, 3, \dots$,

$$D_{\theta_b}^k x^*(0^+, 0) = \lim_{\theta_b \downarrow 0} D_{\theta_b}^k x^*(\theta_b, 0), \text{ and}$$

$$D_{\theta_b^+}^k x^*(0, 0) = \lim_{\theta_b \downarrow 0} \frac{D_{\theta_b}^{k-1} x^*(\theta_b, 0) - D_{\theta_b}^{k-1} x^*(0, 0)}{\theta_b}.$$

In the last two equalities, $D_{\theta_b}^{k-1} x^*(0, 0)$ is the common value of $D_{\theta_b}^{k-1} x^*(0^+, 0)$ and $D_{\theta_b^+}^{k-1} x^*(0, 0)$, provided that these quantities exist and are the same. Analogous definitions hold for

$$D_{\theta_c}^k x^*(0, 0^+), \quad D_{\theta_c^+}^k x^*(0, 0), \quad D_{\mu}^k x(0, 0, 0^+), \quad D_{\mu^+}^k x(0, 0, 0), \quad D_{\mu}^k s(0, 0, 0^+), \quad \text{and} \quad D_{\mu^+}^k s(0, 0, 0).$$

The derivatives of a real valued, single variable function, g , are denoted by $g', g'', \dots, g^{(k)}$. The right sided derivatives and limit of the derivatives are written, $g'_+, g''_+, \dots, g_+^{(k)}$ and $g'(x_0^+), g''(x_0^+), \dots, g^{(k)}(x_0^+)$, respectively. The set of infinitely differentiable functions is denoted by \mathcal{C}^∞ . We say that an analytic property holds *locally* at x if there exists an open neighborhood of x for which the property holds.

Given a vector $x \in \mathbb{R}^n$, we implicitly let the upper case symbol X denote the $n \times n$ diagonal matrix with the elements of x on its diagonal. The standard big- O , small- o , and Θ notation are used [28]. For terminology that is common in the linear programming literature, we refer to the *Math Programming Glossary* [12].

3 Marginal Analysis of the Central Optimal Solution

In this section we investigate the marginal, or differential, properties of the central optimal solution. The marginal analysis of $x^*(0, \theta_c)$ along δc is quite simple because, as shown in Theorem 3.1, $x^*(0, \theta_c)$ behaves like a step function along δc —i.e. it is either constant for θ_c in some interval of the form $[0, \theta_c^*)$ or is discontinuous at $\theta_c = 0$ and then constant for θ_c in an interval of the form $(0, \theta_c^*)$. However, the differential properties of $x^*(\theta_b, 0)$ along δb are more intricate, and our analysis has several objectives. We show that the central optimal solution is always a continuous function along δb . Moreover, so long as the optimal partition does not change, the central optimal solution is analytic along δb . We also provide uniform bounds for the derivatives of the central optimal solution. We begin with the following monotonicity relationship for the optimal partition, which is found in both [2, 3].

Lemma 3.1 (Adler and Monteiro [2]) *For sufficiently small $(\theta_b, \theta_c) > 0$, we have*

$$N(\theta_b, 0) \subseteq N(0, 0) \subseteq N(0, \theta_c) \quad \text{and} \quad B(0, \theta_c) \subseteq B(0, 0) \subseteq B(\theta_b, 0).$$

In addition to Lemma 3.1, Greenberg [11] has shown that if $\theta_c = 0$, there exists $\theta_b^* > 0$ such that the optimal partition is invariant for $\theta_b \in (0, \theta_b^*)$. Similarly, if $\theta_b = 0$, there exists $\theta_c^* > 0$ such that the optimal partition does not change for any $\theta_c \in (0, \theta_c^*)$.

The first result of this section describes the behavior of the central optimal solution with respect to cost coefficient parameterization. Specifically, the following result shows that the central optimal solution is either constant some distance along δc or is immediately discontinuous and then constant some distance along δc . Although our development follows from a primal perspective, we include for future reference the corresponding dual statement in this result. The corollary that follows states the subsequent marginal properties.

Theorem 3.1 *The analytic properties of $x^*(0, \theta_c)$ depend on δc in the following way,*

1. *if $\delta c \in \mathcal{H}_c^1$, there exist $\theta_c^* > 0$ such that $x^*(0, \theta_c)$ is constant for $\theta_c \in [0, \theta_c^*]$,*
2. *if $\delta c \notin \mathcal{H}_c^1$, $x^*(0, \theta_c)$ is discontinuous at $\theta_c = 0$ and there exists $\theta_c^* > 0$ such that $x^*(0, \theta_c)$ is constant for $\theta_c \in (0, \theta_c^*)$.*

The behavior of $s^(\theta_b, 0)$ is analogous.*

Proof. If $\delta c \in \mathcal{H}_c^1$, there exists $\theta_c^* > 0$ such that the optimal partition, $(B(0, \theta_c)|N(0, \theta_c))$, is invariant for any $\theta_c \in [0, \theta_c^*]$. Since $x^*(0, \theta_c)$ is the analytic center of $\{x : Ax = b, x_{B(0,0)} \geq 0, x_{N(0,0)} = 0\}$, we have that $x^*(0, \theta_c) = x^*(0, 0)$ for any $\theta_c \in [0, \theta_c^*]$. If $\delta c \notin \mathcal{H}_c^1$, there exists $\theta_c^* > 0$ such that the optimal partition, $(B(0, \theta_c)|N(0, \theta_c))$, is invariant with respect to $\theta_c \in (0, \theta_c^*)$, and hence the same argument just employed shows that $x^*(0, \theta_c)$ is constant for $\theta_c \in (0, \theta_c^*)$. From Lemma 3.1, $N(0, 0) \subset N(0, \theta_c)$ and $B(0, \theta_c) \subset B(0, 0)$. Let $\mathcal{I} = B(0, 0) \setminus B(0, \theta_c)$. From (2.3), we have that $x_{\mathcal{I}}^*(0, 0) > 0$. However, for all $\theta_c \in (0, \theta_c^*)$, $x_{\mathcal{I}}^*(0, \theta_c) = 0$, which implies a discontinuity at $\theta_c = 0$. The dual statement follows from a similar argument. **Q.E.D.**

Corollary 3.1 *If $\delta c \in \mathcal{H}_c^1$, $D_{\theta_c}^k x^*(0, 0) = 0$. Otherwise, $x^*(0, \theta_c)$ is discontinuous at $\theta_c = 0$.*

Proof. From (2.1), $\delta c \in \mathcal{H}_c^1$ implies $-\delta c \in \mathcal{H}_c^1$. So $(B(0, \theta_c)|N(0, \theta_c)) = (B(0, 0)|N(0, 0))$ for all θ_c in some neighborhood of 0. Furthermore, as indicated in the prior proof, $x^*(0, \theta_c)$ is invariant over this neighborhood. Hence, $D_{\theta_c}^k x^*(0, 0) = 0$, for all k . The case when $\delta c \notin \mathcal{H}_c^1$ is addressed in the proof of Theorem 3.1. **Q.E.D.**

Similar to Corollary 3.1, the marginal analysis with respect to the right-hand side parameter depends on whether or not $\delta b \in \mathcal{H}_b^1$. However, unlike the result stated in Corollary 3.1, the analytic center solution is continuous along any admissible δb . In fact, so long as the linear change in the right-hand side does not alter the optimal partition, the analytic center solution is actually analytic. The next result, which is a direct consequence of the implicit function theorem [7], establishes this fact.

Theorem 3.2 *If $\bar{\delta} \in \mathcal{H}_b^1$, $x^*(\theta_b, 0)$ is locally analytic in θ_b .*

Proof. From (2.1), $\bar{\delta} \in \mathcal{H}_b^1$ implies $-\bar{\delta} \in \mathcal{H}_b^1$. Hence, $(B(0, 0)|N(0, 0)) = (B(\theta_b, 0)|N(\theta_b, 0))$ for θ_b in some sufficiently small neighborhood of 0, and $x_{N(\theta_b, 0)}^*(\theta_b, 0) = x_{N(0, 0)}^*(0, 0) = 0$ over the same neighborhood. Since $\bar{\delta} \in \text{coll}(A_B)$, we have that $\{x : A_B x = b + \theta_b \bar{\delta}\} = \{x : \bar{A}_B x = \bar{b} + \theta_b \bar{\delta}\}$, where \bar{A}_B is a full row rank submatrix of A_B . The analytic center solution of the unperturbed problem is the unique solution to the following system at $\theta_b = 0$,

$$\bar{A}_B x_B = \bar{b} + \theta_b \bar{\delta} \quad (3.5)$$

$$\bar{A}_B^T \bar{y} + s_B = 0 \quad (3.6)$$

$$X_B s_B = e \quad (3.7)$$

$$x_B > 0 \quad (3.8)$$

$$s_B > 0, \quad (3.9)$$

where e the vector of ones, and X_B is the diagonal matrix with the elements of x_B on its diagonal. Denote the solution to (3.5)–(3.9), with $\theta = 0$, by $(x_B^*, \bar{s}_B, \bar{y})$. Define $\phi : \mathbb{R}^{2|B|+\bar{m}+1} \rightarrow \mathbb{R}^{2|B|+\bar{m}}$ as

$$\phi(x_B, \bar{y}, s_B, \theta_b) = \begin{bmatrix} \bar{A}_B x_B - \theta_b \bar{\delta} - \bar{b} \\ \bar{A}_B^T \bar{y} + s_B \\ X_B s_B - e \end{bmatrix}.$$

Then, ϕ is analytic in an open neighborhood of $(x_B^*, \bar{y}, \bar{s}, 0)$ and $\phi(x_B^*, \bar{y}, \bar{s}, 0) = 0$. Since the Jacobian of $\phi((x_B^*, \bar{y}, \bar{s}, 0))$ with respect to (x_B, \bar{y}, s_B) is non-singular, the implicit function theorem implies that x_B is an analytic function of θ_b in some sufficiently small neighborhood of zero. **Q.E.D.**

Notice that if $\bar{\delta}$ had not been in $\text{coll}(A_B)$, the implicit function theorem is not applicable since the needed row reduction to form $(\bar{A}_B, \bar{b}, \bar{\delta})$ is not possible. Theorem 3.2 implies that if we are concerned with a right-hand side direction of change that does not immediately alter the partition, $x^*(\theta_b, 0)$ is of class \mathcal{C}^∞ and consequently has a power series expansion. Differentiating (3.5), (3.6), and (3.7) with respect to θ_b and denoting $x^*(\theta_b, 0)$ as x^* , we have

$$\begin{aligned} \bar{A}_B(D_{\theta_b}^1 x_B^*) &= \bar{\delta} \\ \bar{A}_B^T(D_{\theta_b}^1 \bar{y}) + D_{\theta_b}^1 \bar{s}_B &= 0 \\ \tilde{S}_B(D_{\theta_b}^1 x_B^*) + X_B^*(D_{\theta_b}^1 \bar{s}_B) &= 0. \end{aligned} \quad (3.10)$$

Noticing that this is a non-singular system of linear equations in $D_{\theta_b}^1 x_B^*$, $D_{\theta_b}^1 \bar{s}_B$, and $D_{\theta_b}^1 \bar{y}$, we have the following,

$$\begin{aligned} D_{\theta_b}^1 x_B^* &= (X_B^*)^2 \bar{A}_B^T (\bar{A}_B (X_B^*)^2 \bar{A}_B^T)^{-1} \bar{\delta}, \\ D_{\theta_b}^1 \bar{s}_B &= -\bar{A}_B^T (\bar{A}_B (X_B^*)^2 \bar{A}_B^T)^{-1} \bar{\delta}, \text{ and} \\ D_{\theta_b}^1 \bar{y} &= (\bar{A}_B (X_B^*)^2 \bar{A}_B^T)^{-1} \bar{\delta}. \end{aligned} \quad (3.11)$$

Using (3.10), we are able to recursively establish the higher order derivatives. For $k \geq 2$, define

$$\Omega^k \equiv - \sum_{i=1}^{k-1} \binom{k}{i} (D_{\theta_b}^i \tilde{S}_B)(D_{\theta_b}^{k-i} x_B^*).$$

Then, for $k \geq 2$ we have

$$\begin{aligned} \bar{A}_B(D_{\theta_b}^k x_B^*) &= 0 \\ \bar{A}_B^T(D_{\theta_b}^k \tilde{y}) + D_{\theta_b}^k \tilde{s}_B &= 0 \\ \tilde{S}_B(D_{\theta_b}^k x_B^*) + X_B^*(D_{\theta_b}^k \tilde{s}_B) &= \Omega^k, \end{aligned}$$

which implies

$$\begin{aligned} D_{\theta_b}^k x_B^* &= (X_B^*)^2 \Omega^k - X_B^* \bar{A}_B^T (\bar{A}_B (X_B^*)^2 \bar{A}_B^T)^{-1} \bar{A}_B X_B^* \Omega^k \\ D_{\theta_b}^k \tilde{s}_B &= \bar{A}_B^T (\bar{A}_B (X_B^*)^2 \bar{A}_B^T)^{-1} \bar{A}_B X_B^* \Omega^k \\ D_{\theta_b}^k \tilde{y} &= -(\bar{A}_B (X_B^*)^2 \bar{A}_B^T)^{-1} \bar{A}_B X_B^* \Omega^k. \end{aligned} \tag{3.12}$$

Hence,

$$x^*(\theta_b, 0) = \sum_{k=0}^{\infty} \frac{(D_{\theta_b}^k x^*(0, 0))}{k!} (\theta_b)^k, \tag{3.13}$$

for θ_b sufficiently close to zero. Notice that this power series expansion is easy to calculate to a given order, since it requires only the single matrix factorization of $(\bar{A}_B (X_B^*)^2 \bar{A}_B^T)$.

We now consider the case when $\mathfrak{B} \notin \mathcal{H}_b^1$. In this situation, the only analytic property for $x^*(\theta_b, 0)$ that holds locally at $\theta_b = 0$ is continuity. The following example demonstrates that even if $-\mathfrak{B}$ is admissible, the one-sided derivatives are not guaranteed to be equal.

Example: Consider the linear program $\min\{-x_2 : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, x_1 + x_2 \leq 1 + \theta_b\}$, which is illustrated in Figure 2. After adding a slack vector,

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathfrak{B} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and } c = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Using \mathfrak{B} and $\theta_b \geq 0$, we find that the central optimal solution is

$$(x_1^*(\theta_b, 0), x_2^*(\theta_b, 0)) = \left(1 + \theta_b - \frac{1}{3} \sqrt{\theta_b^2 + 3\theta_b + 9}, 1\right).$$

However, for $-\mathfrak{B}$ and $0 < \theta_b < 1$ the central optimal solution is

$$(x_1^*(\theta_b, 0), x_2^*(\theta_b, 0)) = (0, 1 - \theta_b).$$

When using \mathfrak{B} , we have that the optimal partitions for $\theta_b > 0$ and $\theta_b = 0$ are respectively $(\{1, 2, 3, 5\}, \{4\})$ and $(\{2, 3\}, \{1, 4, 5\})$. When using $-\mathfrak{B}$, we have that the optimal partition for $0 < \theta_b < 1$ is $(\{2, 3, 4\}, \{1, 5\})$.

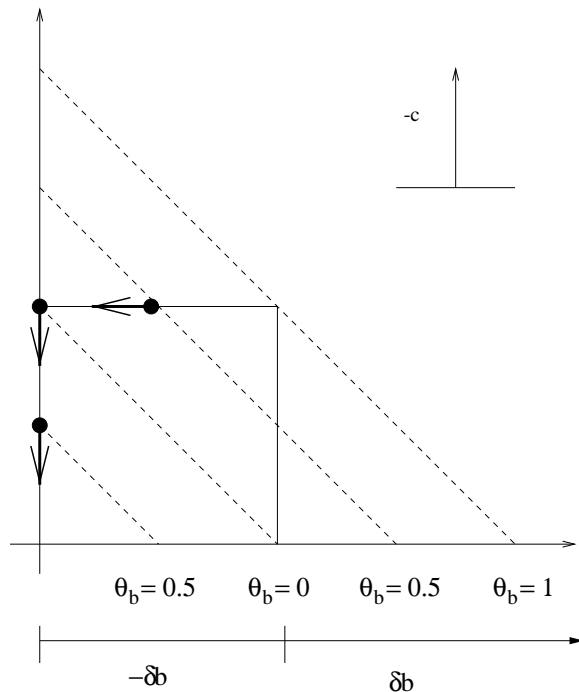


Figure 2: The movement of the central optimal solution as the right-hand side is perturbed. The dark arrows indicate how the central optimal solution changes as the third component of the right-hand side decreases from 2 to 1/2.

If $\theta_b = 0$, $\delta b \notin \mathcal{H}_b^1$ because $\delta b \notin \text{coll}(A_B)$. In this situation, we have that the limit of the derivative is

$$D_{\theta_b}^1 x^*(0^+, 0) = \lim_{\theta_b \downarrow 0} \begin{pmatrix} 1 - \frac{2\theta_b + 3}{6\sqrt{\theta_b^2 + 3\theta_b + 9}} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{5}{6} \\ 0 \end{pmatrix}.$$

However, for $-\delta b$ the limit of the derivative is

$$D_{\theta_b}^1 x^*(0^+, 0) = \lim_{\theta_b \downarrow 0} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Although continuity is the strongest local property allowed when $\delta b \notin \mathcal{H}_b^1$, we do establish that $x^*(\theta_b, 0)$ is one-sided, infinitely differentiable. A key ingredient in our argument is the result that the limit of the derivatives of $x(\theta_b, \theta_c, \mu)$ exist with respect to μ [1, 15, 35]. The formal statement is cited below.

Lemma 3.2 (Güler [15]) For $k = 1, 2, 3, \dots$, both $D_{\mu}^k x(0, 0, 0^+)$ and $D_{\mu}^k s(0, 0, 0^+)$ exist. More-

over,

$$\begin{aligned}
D_{\mu}^k x(0, 0, 0^+) &= D_{\mu^+}^k x(0, 0, 0), \\
D_{\mu}^k s(0, 0, 0^+) &= D_{\mu^+}^k s(0, 0, 0), \\
D_{\mu}^1 x_{N(0,0)}(0, 0, 0^+) &= (S_{N(0,0)}^*(0, 0))^{-1} e_{N(0,0)} > 0, \text{ and} \\
D_{\mu}^1 s_{B(0,0)}(0, 0, 0^+) &= (X_{B(0,0)}^*(0, 0))^{-1} e_{B(0,0)} > 0.
\end{aligned}$$

The above differential properties are used in conjunction with the formula for the k th derivative of a composite function to establish the one-sided marginal analysis of the central optimal solution. Let $h(t) = f(g(t))$, where both f and g are in \mathcal{C}^∞ on some suitable neighborhoods. Setting $y = g(t)$, we have that $h^{(k)}(t)$ is

$$\sum_{m=1}^k \sum \frac{k!}{j_1! j_2! \dots j_k!} \cdot \left[\left(\frac{g'(t)}{1!} \right)^{j_1} \left(\frac{g''(t)}{2!} \right)^{j_2} \dots \left(\frac{g^{(k)}(t)}{k!} \right)^{j_k} \right] \cdot \frac{d^m f}{dy^m}(g(t)), \quad (3.14)$$

where the second sum is taken over all non-negative integer solutions of $\sum_{i=1}^k i j_i = k$ and $\sum_{i=1}^k j_i = m$ [8].

The following lemma provides some needed differential properties.

Lemma 3.3 *Let f map $[0, \mu^*)$ onto $[0, \theta^*)$. Furthermore, let f be in \mathcal{C}^∞ on $(0, \mu^*)$, let $f^{(k)}(0^+)$ exist for all $k \geq 1$, and let $f'(0^+) > 0$. Then, there exists $\bar{\theta} > 0$ such that $g = f^{-1}$ exists on $(0, \bar{\theta})$ and has the property that $g^{(k)}(0^+)$ exists for all $k \geq 1$. Moreover, if f is continuous at zero, f is in \mathcal{C}^∞ on $[0, \mu^*)$ and g is in \mathcal{C}^∞ on $[0, \bar{\theta})$.*

Proof. Let f be as above. The general inverse function theorem [26] establishes that $g = f^{-1}$ exists and is in \mathcal{C}^∞ on $(0, \bar{\theta})$, for some $\bar{\theta} > 0$. For all $\theta \in (0, \bar{\theta})$,

$$g'(\theta) = \frac{1}{f'(g(\theta))}, \quad (3.15)$$

and since $f'(0^+) > 0$,

$$g'(0^+) = \frac{1}{f'(g(0^+))},$$

which completes the first statement when $k = 1$.

Applying (3.14) on (3.15) yields an expression for $g^{(k)}(\theta)$ in terms of $g'(\theta)$, $g''(\theta)$, $g'''(\theta)$, ..., $g^{(k-1)}(\theta)$, and $f'(g(\theta))$, $f''(g(\theta))$, $f'''(g(\theta))$, ..., $f^{(k)}(g(\theta))$. So, a simple induction argument shows that $g^{(k)}(0^+)$ exists for all k .

Now, suppose that f is continuous at zero, and let $t(\mu)$ be from the mean value theorem. Then

$$\frac{f(\mu) - f(0)}{\mu} = f'(t(\mu)) \text{ for all } \mu \in (0, \mu^*).$$

Then using the assumption that $f'(0+)$ exists implies

$$\lim_{\mu \downarrow 0} \left(\frac{f(\mu) - f(0)}{\mu} - f'(\mu) \right) = \lim_{\mu \downarrow 0} (f'(t(\mu)) - f'(\mu)) = 0.$$

So, $f'_+(0) = f'(0+)$ and we have that f' is continuous at zero. Repeated applications of the mean value theorem give that f is in \mathcal{C}^∞ on $[0, \mu^*)$. Since f being continuous at zero implies g is continuous at zero, a similar argument shows g is in \mathcal{C}^∞ on $[0, \bar{\theta}]$. **Q.E.D.**

We now prove that if $\mathfrak{B} \notin \mathcal{H}_b^1$, the primal central optimal solution is not only continuous, but is one-sided, infinitely differentiable with respect to the perturbation parameter θ_b .

Theorem 3.3 $D_{\theta_b}^k x^*(0^+, 0)$ exists for $k = 0, 1, 2, \dots$, and $D_{\theta_b}^k x^*(0^+, 0) = D_{\theta_b^+}^k x^*(0, 0)$.

Proof. The case when $\mathfrak{B} \in \mathcal{H}_b^1$ is already addressed in Theorem 3.2, so assume that $\mathfrak{B} \notin \mathcal{H}_b^1$. Let $\theta_b^* > 0$ be such that $(B(\theta_b, 0), N(\theta_b, 0))$ is invariant for $\theta_b \in (0, \theta_b^*)$, and denote this common partition by $(B'|N')$. From Lemma 3.1, $B(0, 0) \subset B'$ and $N' \subset N(0, 0)$. Therefore, if $i \in N'$, $x_i^*(\theta_b, 0) = 0$ for all $\theta_b \in [0, \theta_b^*)$. Hence, $D_{\theta_b}^k x_{N'}^*(0^+, 0) = D_{\theta_b^+}^k x_{N'}^*(0, 0) = 0$ for all k .

Let $i \in B'$, and consider the following linear program

$$\min\{\theta_b : A_{B(0,0)} z_{B(0,0)} + A_{\mathcal{I}} z_{\mathcal{I}} - \mathfrak{B}\theta_b = b, z_{B(0,0)} \geq 0, z_{\mathcal{I}} \geq 0, \theta_b \geq 0\},$$

where $\mathcal{I} = B' \setminus B(0, 0)$. Furthermore, let $\{(z_{B(0,0)}(\mu), z_{\mathcal{I}}(\mu), \theta_b(\mu)) : \mu \geq 0\}$ be the central path for this linear program. Then,

$$z_{B(0,0)}(\mu) = x_{B(0,0)}^*(\theta_b(\mu), 0) \text{ and } z_{\mathcal{I}}(\mu) = x_{\mathcal{I}}^*(\theta_b(\mu), 0).$$

Since the optimal value of the linear program is zero, $\theta_b(\mu) \rightarrow 0$, as $\mu \downarrow 0$, and from Lemma 3.2, $\theta_b'(0+) > 0$. So, there exists some interval, say $[0, \hat{\mu})$, where $\theta_b(\mu)$ is invertible. Denote the inverse by $\mu(\theta_b)$ and let the corresponding interval be $[0, \hat{\theta}_b)$, where we assume that $\hat{\theta}_b \leq \theta_b^*$. Then for all $\theta_b \in (0, \hat{\theta}_b)$ we have

$$z_{B(0,0)}(\mu(\theta_b)) = x_{B(0,0)}^*(\theta_b, 0) \text{ , } z_{\mathcal{I}}(\mu(\theta_b)) = x_{\mathcal{I}}^*(\theta_b, 0)$$

Applying (3.14) we have the following for $i \in B'$,

$$D_{\theta_b}^k x_i(\theta_b) = \sum_{m=1}^k \sum \frac{k!}{j_1! j_2! \dots j_k!} \cdot \left[\left(\frac{\mu'(\theta_b)}{1!} \right)^{j_1} \left(\frac{\mu''(\theta_b)}{2!} \right)^{j_2} \dots \left(\frac{\mu^{(k)}(\theta_b)}{k!} \right)^{j_k} \right] \cdot z_i^{(m)}(\mu(\theta_b)).$$

Lemmas 3.2 and 3.3 imply that the right-hand side converges as $\theta_b \downarrow 0$. So, $D_{\theta_b}^k x^*(0^+, 0)$ exists. Now, z is continuous at zero because of the convergence properties of the analytic central path,

and μ is continuous at zero because it is the inverse of θ_b , which is in C^∞ and increasing on $[0, \hat{\mu})$. Hence, $x^*(\theta_b, 0)$ is continuous at $\theta_b = 0$, and the mean value theorem is used as in the proof of Lemma 3.3 to conclude the result. **Q.E.D.**

Theorems 3.3 and 3.2 have shown the existence of the one-sided derivatives of $x^*(\theta_b, 0)$. We now conclude this section by showing that the first of these derivatives is uniformly bounded. Let $\mu > 0$ and consider the system (2.2), –i.e. the definition of the parametric central path. Using the full rank matrix \bar{A} , we have that differentiating with respect to θ_b yields

$$\begin{aligned} \bar{A}(D_{\theta_b}^1 x(\theta_b, 0, \mu)) &= \bar{\delta} \\ \bar{A}^T(D_{\theta_b}^1 \bar{y}(\theta_b, 0, \mu)) + D_{\theta_b}^1 s(\theta_b, 0, \mu) &= 0 \\ S(D_{\theta_b}^1 x(\theta_b, 0, \mu)) + X(\theta_b, 0, \mu)(D_{\theta_b}^1 s(\theta_b, 0, \mu)) &= 0. \end{aligned}$$

This system implies

$$D_{\theta_b}^1 x(\theta_b, 0, \mu) = X^2(\theta_b, 0, \mu) \bar{A}^T (\bar{A} X^2(\theta_b, 0, \mu) \bar{A}^T)^{-1} \bar{\delta}. \quad (3.16)$$

The equality in (3.16) is convenient because Dikin [6] showed that for a given full row-rank matrix, M , the following condition number, χ_M , is finite (also see Stewart [33], Todd [34], and for the rank deficient case, Holder and Caron [17]):

$$\chi_M \equiv \sup\{\|(MDM^T)^{-1}MD\| : D \text{ is positive diagonal}\}.$$

This condition number, together with (3.11) and (3.16), provides a uniform bound on the first order derivatives.

Theorem 3.4 *It holds for any positive μ that*

$$\|D_{\theta_b}^1 x(\theta_b, 0, \mu)\| \leq \chi_{\bar{A}} \|\bar{\delta}\|. \quad (3.17)$$

Furthermore, if $\bar{\delta} \in \mathcal{H}_b^1$,

$$\|D_{\theta_b}^1 x^*(0, 0)\| \leq \chi_{\bar{A}} \|\bar{\delta}\|. \quad (3.18)$$

Otherwise $\bar{\delta} \notin \mathcal{H}_b^1$, and

$$\|D_{\theta_b^+}^1 x^*(0, 0)\| \leq \chi_{\bar{A}} \|\bar{\delta}\|. \quad (3.19)$$

Proof. Both (3.17) and (3.18) follow directly from equations 3.11 and 3.16, and from the definition of $\chi_{\bar{A}}$. The bound in (3.19) follows since Theorem 3.3 implies $D_{\theta_b^+}^1 x^*(0, 0) = \lim_{\theta_b \downarrow 0} D_{\theta_b}^1 x^*(\theta_b, 0)$, and from (3.18) the right side of this equality is bounded by $\chi_{\bar{A}} \|\bar{\delta}\|$. **Q.E.D.**

We remark that the bounds in Theorem 3.4 depend only on the A -matrix and $\|\delta\|$; in particular, they are independent of μ , θ_b , and b . Furthermore, $\|\delta\|$ can obviously be normalized to one.

The following corollary lists several order relations that follow from Theorems 3.3 and 3.4. This corollary highlights the fact that the entire primal central path behaves “smoothly” with respect to right-hand side parameterization.

Corollary 3.2 *For all $\mu > 0$ and θ_b sufficiently small,*

$$\|x(\theta_b, 0, \mu) - x(0, 0, \mu)\| = O(\theta_b), \quad (3.20)$$

and from (3.17) the constant in the big- O is independent of μ . Consequently,

$$\|x^*(\theta_b, 0) - x^*(0, 0)\| = O(\theta_b). \quad (3.21)$$

Furthermore, for sufficiently small θ_b , and \mathcal{I} and N' as in the proof Theorem 3.3, we have

$$x_{\mathcal{I}}^*(\theta_b, 0) = \Theta(\theta_b), \quad (3.22)$$

$$x_{N'}(\theta_b, 0, \mu) = \Theta(\mu). \quad (3.23)$$

Proof. Equations (3.20) and (3.21) follow immediately from the uniform bound presented in (3.17). Equation (3.22) follows from the fact that $x_{\mathcal{I}}^*(0, 0) = 0$ and Lemma 3.2 shows $D_{\theta_b^+}^1 x_{\mathcal{I}}(0, 0) > 0$. To show equation (3.23) we first recognize that from Lemma 3.2, $D_{\mu}^1 x_{N'}(\theta_b, 0, 0^+) = (S_{N'}^*(\theta_b, 0))^{-1} e_{N'}$. Theorem 3.1 implies that $s_{N'}^*(\theta_b, 0)$ is constant over some suitably small neighborhood, say $(0, \theta_b^*)$. Furthermore, the definition of N' implies $s_{N'}(\theta_b, 0)$ and $s_{N'}(0, 0)$ are both positive. So over $[0, \theta_b^*)$, $D_{\mu}^1 x_{N'}(\theta_b, 0, 0^+)$ is uniformly bounded, which establishes (3.23). **Q.E.D.**

The bounds indicated in the last corollary are related to those of Theorem 4.1 in [27] by Nunez and Freund. However, their upper bound on $\|x(\theta_b, \theta_c, \mu) - x(0, 0, \mu)\|$ not only depends on μ , but increases to infinity as $\mu \downarrow 0$. The reason why our results are stronger is that we allow only right-hand side changes, whereas the work of Nunez and Freund allows simultaneous changes in A , b , and c .

4 Convergence and Order Relations Under Parameterization

Each of the three results of this section establish conditions under which the parameterized central path converges. The first of these results shows that the parameterized central path converges to the unperturbed central optimal solution, provided that there is no cost perturbation. This result follows directly from Theorem 3.1 in [5]. While this first result does not allow cost coefficient

perturbation, it is quite useful because cost coefficient perturbations may be modeled as right-hand side changes. Hence, we are able to include simultaneous perturbations in the second and third results. Specifically, the second result shows that if the cost parameter and barrier parameter are scalars of each other, the parameterized central path converges to an element of a central path defined on the unperturbed optimal set. The third result shows that the parameterized central path converges to the unperturbed central optimal solution if the barrier parameter converges to zero slower than the cost coefficient parameter. The following example illustrates all three convergence results and demonstrates that convergence is not always assured.

Example: Consider the linear program $\min\{\theta_c(10x_1) + \theta_c x_2 + x_3 : 0 \leq x_1, x_2, x_3 \leq 1 + \theta_b\}$, for which $b = \mathfrak{b} = (1, 1, 1)^T$, $c = (0, 0, 1)^T$, and $\delta c = (10, 1, 0)^T$. The parameterized central path is defined by

$$x_1(\theta_b, \theta_c, \mu) = \begin{cases} \frac{1}{20\theta_c} \left(10(1 + \theta_b)\theta_c + 2\mu - \sqrt{100(1 + \theta_b)^2\theta_c^2 + 4\mu^2} \right), & \theta_c \neq 0 \\ \frac{1 + \theta_b}{2}, & \theta_c = 0 \end{cases}$$

$$x_2(\theta_b, \theta_c, \mu) = \begin{cases} \frac{1}{2\theta_c} \left((1 + \theta_b)\theta_c + 2\mu - \sqrt{(1 + \theta_b)^2\theta_c^2 + 4\mu^2} \right), & \theta_c \neq 0 \\ \frac{1 + \theta_b}{2}, & \theta_c = 0 \text{ and} \end{cases}$$

$$x_3(\theta_b, \theta_c, \mu) = \frac{1}{2} \left((1 + \theta_b) + 2\mu - \sqrt{(1 + \theta_b)^2 + 4\mu^2} \right).$$

To illustrate the next three results, we consider the situations of $(\theta_b, \theta_c) = (\theta_b, 0)$, $(\theta_b, \theta_c) = (\theta_b, \mu/\nu)$, and $(\theta_b, \theta_c) = (\theta_b, \mu^2)$.

Setting θ_c to 0, we see that the limit as $(\theta_b, \mu) \downarrow 0$ is

$$(x_1^*(0, 0), x_2^*(0, 0), x_3^*(0, 0)) = (1/2, 1/2, 0).$$

This is because the entire central path, including the analytic center solution, shifts “smoothly” as the right-hand side is perturbed. Figure 3 depicts this situation, and a formal convergence result is stated in Theorem 4.1

The second, and most difficult situation to explain, makes the substitutions $\theta_c = \mu/\nu$. Under this substitution we have that the limits as $(\theta_b, \theta_c) \downarrow 0$ are

$$x_1 = \frac{1}{20} \left(10 + 2\nu - \sqrt{100 + 4\nu^2} \right),$$

$$x_2 = \frac{1}{2} \left(1 + 2\nu - \sqrt{1 + 4\nu^2} \right), \text{ and}$$

$$x_3 = 0.$$

It so happens that these limits are the ν element of the central path associated with the linear program $\min\{10x_1 + x_2 : 0 \leq x_1, x_2 \leq 1, x_3 = 0\}$, or equivalently $\min\{10x_1 + x_2 : (x_1, x_2, x_3) \in \mathcal{P}_{(0,0)}^*\}$.

So, not only does the linear relationship between μ and θ_c allow convergence, but the limit is an element of the central path contained in the optimal set and defined by the cost coefficient direction of change. This special type of convergence is illustrated by Figure 4.

Making the substitution $\theta_c = \mu^2$ and allowing $(\theta_b, \mu) \downarrow 0$, we find that the limit is the analytic center solution of the unperturbed problem —i.e. $(x_1^*(0,0), x_2^*(0,0), x_3^*(0,0)) = (1/2, 1/2, 0)$. As shown in Theorem 4.3, this follows because θ_c converges to zero faster than μ . An example is found in Figure 5.

Notice that a mixture of the last two cases may easily produce a sequence that does not allow convergence. For example, let $\mu^k = 1/k$, and θ_c^k be $(\mu^k)^2$ or $(1/2)\mu^k$, depending on whether k is even or odd, respectively. Then, the subsequence of even indices converges to $(1/2, 1/2, 0)$ and the subsequence of odd indices converges to $(\frac{14-\sqrt{116}}{20}, \frac{5-\sqrt{17}}{2}, 0)$. The sporadic behavior of this path is shown in Figure 6

Before continuing, we explain the differences between the results of this section and those found in [4] by Bonnans and Potra, in [23] by Mizuno, Todd, and Ye, and in [25] by Monteiro and Tsuchiya. Bonnans and Potra consider a single shifted analytic center in a specific algorithmic framework (see Theorem 4.13 in [4]). Our results are algorithm independent and deal not only with a single shifted analytic center, but rather with a central path defined on the unperturbed optimal set. Furthermore, our results allow changes in b and c separately, which is not allowed in the horizontal linear complementary framework they address —i.e. the single parameter used to parameterize the right-hand side of the horizontal linear complementarity problem does not allow independent parameterizations. Mizuno, Todd, and Ye provide conditions under which the cluster points of the parameterized central path are contained in either the boundary or the relative interior of the unperturbed optimal set (see Section 5 of [23]), whereas our results establish the existence of a unique limit. Mizuno and Tsuchiya show that $x(0, \mu, \mu)$ converges, as $\mu \downarrow 0$ (see Section 4 of [25]). This is a special case of Theorem 4.2, where we establish the convergence of $x(\theta_b, \theta_c, \nu\theta_c)$, for $\nu > 0$ and $(\theta_b, \theta_c) \downarrow 0$.

Our development begins with two lemmas, the first of which shows that the analytic center is a continuous function of the right-hand side data. This result follows directly from Theorem 3.1 in [5], where simultaneous changes in the A matrix are also allowed. As already mentioned, this result is handy because cost coefficient changes may be modeled as right-hand side changes, and hence, subsequent results involving convergence under cost coefficient perturbations may include simultaneous data perturbations without extra effort.

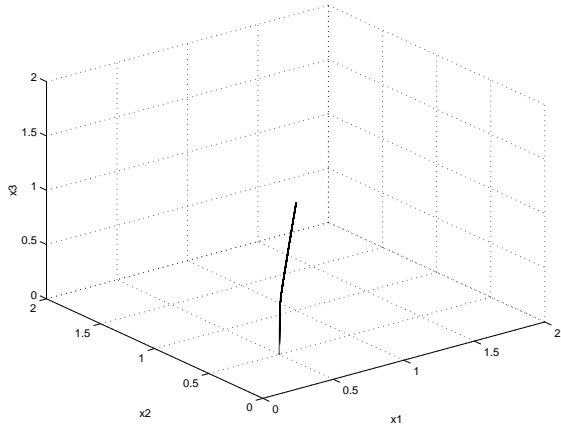


Figure 3: The parameterized central path for $\theta_c = 0$ and θ_b moving from 1 to 0

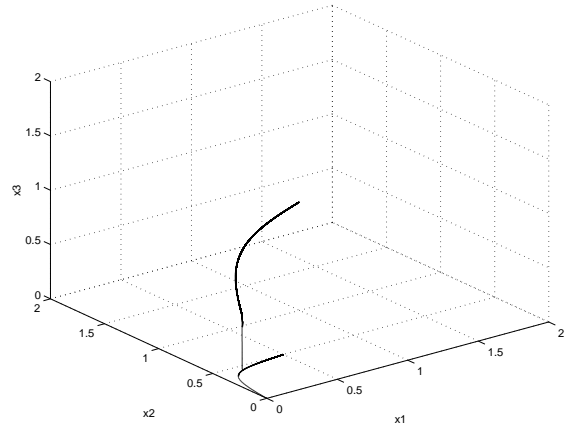


Figure 4: The parameterized central path for θ_b moving from 1 to 0 and θ_c being set to half the barrier parameter. The central path in the plane with $x_3 = 0$ corresponds to minimizing $\delta_{B(0,0)} x_{B(0,0)}$ over the unit square.

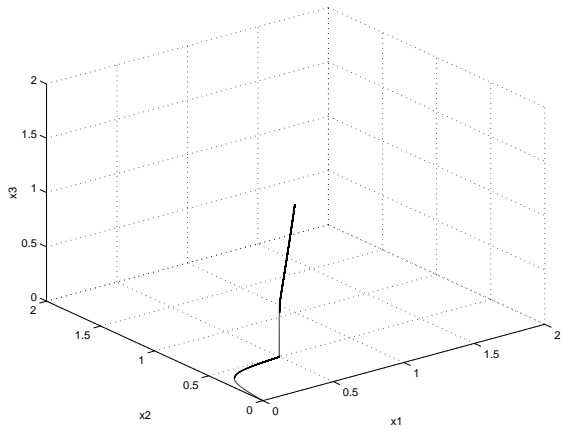


Figure 5: The parameterized central path for θ_b moving from 1 to 0 and θ_c being set to the square of the barrier parameter. The central path in the plane with $x_3 = 0$ is associated with minimizing $\delta_{B(0,0)} x_{B(0,0)}$ over the unit square.

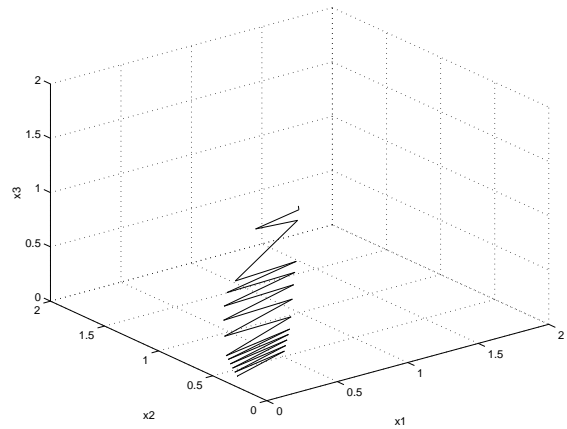


Figure 6: The parameterized central path for θ_b moving from 1 to 0 and θ_c alternating between half the barrier parameter and the square of the barrier parameter.

Lemma 4.1 (Caron, Greenberg, and Holder [5]) *Let $(E|F)$ be a partition of $\{1, 2, \dots, n\}$. Then, the argument maximum of*

$$\left\{ \sum_{i \in E} \ln(x_i) : Ax = b, x_E > 0, x_F = 0 \right\}$$

is continuous with respect to b .

The second lemma extends Lemma I.6 in [29] to include the situation of data perturbations. This result establishes that the union of the level sets of the parameterized central path is bounded. These level sets are defined for each $M \geq 0$ as follows,

$$\mathcal{L}_M^k \equiv \{(x, s) : (x, (y, s)) \in \mathcal{P}_{\theta_b^k} \times \mathcal{D}_{\theta_c^k}, x^T s \leq M\},$$

where (θ_b^k, θ_c^k) is any non-negative sequence of parameters.

Lemma 4.2 *Let $(\theta_b^k, \theta_c^k) \downarrow 0$, as $k \uparrow \infty$. Then, for all $M \geq 0$,*

$$\bigcup_k \mathcal{L}_M^k$$

is bounded.

Proof. For any k , let $(x^k, (y^k, s^k)) \in \mathcal{P}_{\theta_b^k}^o \times \mathcal{D}_{\theta_c^k}^o$. Choose $(x, s) \in \mathcal{L}_M^k$. Then $s^k - s \in \text{row}(A)$ and $x^k - x \in \text{null}(A)$. So

$$0 = (x - x^k)^T (s - s^k) = x^T s - x^T s^k - (x^k)^T s + (x^k)^T s^k.$$

Non-negativity yields

$$x_i s_i^k \leq x^T s^k \leq x^T s + (x^k)^T s^k \leq M + (x^k)^T s^k.$$

Hence,

$$x_i \leq \frac{M + (x^k)^T s^k}{s_i^k}.$$

Similarly,

$$s_i \leq \frac{M + (x^k)^T s^k}{x_i^k}.$$

So, for any fixed k and $M \geq 0$, \mathcal{L}_M^k is bounded.

Let $\bar{\mu} > 0$ and set,

$$s^k = s(0, 0, \bar{\mu}) + \theta_c^k \delta c \quad \text{and} \quad x^k = x(0, 0, \bar{\mu}) + \theta_b^k A^+ \delta b,$$

where A^+ is the Moore-Penrose pseudo inverse of A . Since $x(0, 0, \bar{\mu})$ and $s(0, 0, \bar{\mu})$ are positive, there exists a natural number K^1 such that for all $k \geq K^1$, $s^k > 0$ and $x^k > 0$. Furthermore, $A^T y(0, 0, \bar{\mu}) +$

$s^k = c + \theta_c^k \delta c$ and because $\delta b \in \mathcal{H}_b \Rightarrow \delta b \in \text{coll}(A)$, $Ax^k = b + \theta_b^k \delta b$. So, $(x^k, (y(0, 0, \bar{\mu}), s^k)) \in \mathcal{P}_{\theta_b^k}^o \times \mathcal{D}_{\theta_c^k}^o$, for all $k \geq K^1$. Since $(\theta_b^k, \theta_c^k) \downarrow (0, 0)$ as $k \uparrow \infty$, we have that for $k \geq K^1$, $(x^k, s^k) \geq \alpha > 0$, for some α . Hence, both

$$\frac{M + (x^k)^T s^k}{s_i^k} \quad \text{and} \quad \frac{M + (x^k)^T s^k}{x_i^k}$$

are bounded for all $k \geq K^1$. This implies that

$$\bigcup_{k \geq K^1} \mathcal{L}_M^k$$

is bounded. Since

$$\bigcup_{1 \leq k < K^1} \mathcal{L}_M^k$$

is a finite union of bounded sets, the result follows. **Q.E.D.**

Although the prior example demonstrates that the convergence of the parameterized central path is not guaranteed, Lemma 4.2 assures us that if $\{(\theta_c^k, \theta_b^k, \mu^k)\} \downarrow 0$, both $x(\theta_c^k, \theta_b^k, \mu^k)$ and $s(\theta_c^k, \theta_b^k, \mu^k)$ have at least one cluster point.

We conclude this section with three convergence results that explain situations guaranteeing the convergence of the parameterized central path. The first of these results shows that if there is no change in the cost vector, then the limit as the right-hand side parameter and the barrier parameter decrease to zero is the analytic center solution to the unperturbed linear program.

Theorem 4.1 *It holds that*

$$\lim_{(\theta_b, \mu) \downarrow 0} x(\theta_b, 0, \mu) = x^*(0, 0)$$

Proof. The third equality in (2.2) implies $x^T(\theta_b, 0, \mu)s(\theta_b, 0, \mu) = n\mu$, and hence Lemma 4.2 implies that $\{(x(\theta_b^k, 0, \mu^k), s(\theta_b^k, 0, \mu^k)) : (\theta_b^k, \mu^k) \downarrow 0\}$ is bounded. Without loss in generality, let (θ_b^k, μ^k) be such that $(x(\theta_b^k, 0, \mu^k), s(\theta_b^k, 0, \mu^k)) \rightarrow (\hat{x}, \hat{s})$. Since $\hat{x}^T \hat{s} = 0$, we have that $\hat{x} \in \mathcal{P}^*$, which implies $\hat{x}_{N(0,0)} = 0$.

To complete the proof we must show that $\hat{x}_{B(0,0)} = x_{B(0,0)}^*(0, 0)$. First, note that $x_{B(0,0)}(\theta_b^k, 0, \mu^k)$ is the analytic center of

$$\{x_{B(0,0)} : A_{B(0,0)}x_{B(0,0)} = b + \theta_b \delta b - A_{N(0,0)}x_{N(0,0)}(\theta_b^k, 0, \mu^k), x_{B(0,0)} \geq 0\}.$$

From Lemma 4.1, this analytic center is a continuous function of the right-hand side, $b + \theta_b \delta b - A_{N(0,0)}x_{N(0,0)}(\theta_b^k, 0, \mu^k)$. Now, since $b + \theta_b \delta b - A_{N(0,0)}x_{N(0,0)}(\theta_b^k, 0, \mu^k) \rightarrow b$, we have that

$$x_{B(0,0)}(\theta_b^k, 0, \mu^k) \rightarrow \hat{x}_{B(0,0)} = x_{B(0,0)}^*(0, 0).$$

Q.E.D.

While Theorem 4.1 shows that the parameterized central path is continuous at $(0, 0, 0)$, with respect to both the barrier parameter and the right-hand side parameter, we have already seen in the previous example that a similar continuity result is not possible with respect to the cost coefficient parameter. However, the next result shows that convergence is assured as long as the barrier parameter and the cost coefficient parameter are linearly related. Furthermore, the limit under this special parameterization is an element of the central path contained in the unperturbed optimal set and defined by the cost coefficient direction of change. The elements of this central path are denoted by $z(b, \nu)$ and are the unique solutions to

$$\max \left\{ \delta c_B z - \nu \sum_{i \in B} \ln(z_i) : A_B z = b, z > 0 \right\}.$$

This central path inherits the analytic properties of any other central path. Specifically, we require the use that for any $\nu > 0$,

$$\lim_{b \rightarrow \bar{b}} z(b, \nu) = z(\bar{b}, \nu). \quad (4.24)$$

Theorem 4.2 *For any given parameter $\nu > 0$, it holds that*

$$\lim_{(\theta_b, \theta_c) \downarrow 0} (x_B(\theta_b, \theta_c, \theta_c \nu), x_N(\theta_b, \theta_c, \theta_c \nu)) = (z(b, \nu), 0).$$

Proof. From Lemma 4.2,

$$\{(x(\theta_b^k, \theta_c^k, \theta_c^k \nu), s(\theta_b^k, \theta_c^k, \theta_c^k \nu)) : (\theta_b^k, \theta_c^k) \downarrow 0\}$$

is bounded. Since

$$x^T(\theta_b^k, \theta_c^k, \theta_c^k \nu) s(\theta_b^k, \theta_c^k, \theta_c^k \nu) = n \theta_c^k \nu,$$

we have that any cluster point of $x(\theta_b^k, \theta_c^k, \theta_c^k \nu)$, as $(\theta_b^k, \theta_c^k) \downarrow 0$, is in $\mathcal{P}_{(0,0)}^*$. Hence,

$$x_{N(0,0)}(\theta_b^k, \theta_c^k, \theta_c^k \nu) \rightarrow 0, \text{ as } (\theta_b^k, \theta_c^k) \downarrow 0.$$

Set

$$\begin{aligned} \bar{y}(\theta_b^k, \theta_c^k, \theta_c^k \nu) &= \frac{1}{\theta_c} \left(y(\theta_b^k, \theta_c^k, \theta_c^k \nu) - y^*(0, 0) \right) \text{ and} \\ \bar{s}(\theta_b^k, \theta_c^k, \theta_c^k \nu) &= \frac{1}{\theta_c} \left(s(\theta_b^k, \theta_c^k, \theta_c^k \nu) - s^*(0, 0) \right). \end{aligned}$$

Using that

$$\begin{aligned} s_{B(0,0)}^*(0, 0) &= 0, \\ A_{B(0,0)}^T y(\theta_b^k, \theta_c^k, \theta_c^k \nu) + s_{B(0,0)}(\theta_b^k, \theta_c^k, \theta_c^k \nu) &= c_{B(0,0)} + \theta_c \delta c_{B(0,0)}, \text{ and} \\ A_{B(0,0)}^T y^*(0, 0) &= c_{B(0,0)}, \end{aligned}$$

we have that $(x(\theta_b^k, \theta_c^k, \theta_c^k \nu), \bar{y}(\theta_b^k, \theta_c^k, \theta_c^k \nu), \bar{s}(\theta_b^k, \theta_c^k, \theta_c^k \nu))$ solves

$$\begin{aligned} A_{B(0,0)} x_{B(0,0)} &= b + \theta_b \delta b - A_{N(0,0)} x_{N(0,0)}(\theta_b^k, \theta_c^k, \theta_c^k \nu) \\ A_{B(0,0)}^T \bar{y} + \bar{s}_{B(0,0)} &= \delta c_{B(0,0)} \\ X_{B(0,0)} \bar{S}_{B(0,0)} e &= \nu e. \end{aligned}$$

Since these are the necessary and sufficient Lagrange equations describing

$$z(b + \theta_b \delta b - A_{N(0,0)} x_{N(0,0)}(\theta_b^k, \theta_c^k, \theta_c^k \nu), \nu),$$

we have that $x_{B(0,0)}(\theta_b^k, \theta_c^k, \theta_c^k \nu) = z(b + \theta_b \delta b - A_{N(0,0)} x_{N(0,0)}(\theta_b^k, \theta_c^k, \theta_c^k \nu), \nu)$. From (4.24), we see that

$$\lim_{(\theta_b, \theta_c) \downarrow 0} x_{B(0,0)}(\theta_b^k, \theta_c^k, \theta_c^k \nu) = \lim_{(\theta_b, \theta_c) \downarrow 0} z(b + \theta_b \delta b - A_{N(0,0)} x_{N(0,0)}(\theta_b^k, \theta_c^k, \theta_c^k \nu), \nu) = z(b, \nu).$$

Q.E.D.

The third and final result of this section shows that if the cost coefficient parameter converges to zero faster than the barrier parameter, the parameterized central path converges to the analytic center solution of the unperturbed linear program.

Theorem 4.3 *If $\theta_c = o(\mu)$ and $(\theta_b, \mu) \downarrow (0, 0)$,*

$$\lim_{(\theta_b, \theta_c, \mu) \downarrow 0} x(\theta_b, \theta_c, \mu) = x^*(0, 0).$$

Proof.

Consider a sequence $\{(\theta_b^k, \theta_c^k, \mu^k) : k = 1, 2, \dots\}$ with

$$\lim_{k \uparrow \infty} \theta_b^k = \lim_{k \uparrow \infty} \mu^k = \lim_{k \uparrow \infty} \theta_c^k / \mu^k = 0.$$

Let $F_k(x) = (c^T x / \mu^k) - \sum_{i=1}^n \ln(x_i)$ and notice that $x(\theta_b, 0, \mu^k)$ is the unique solution to

$$\min \left\{ F_k(x) : Ax = b + \theta_b^k \delta b, x > 0 \right\}.$$

Assume, for the sake of attaining a contradiction, that

$$\lim_{k \uparrow \infty} (x(\theta_b, \theta_c, \mu) - x(\theta_b, 0, \mu)) \neq 0.$$

The optimality of $x(\theta_b, \theta_c, \mu)$ and $x(\theta_b, 0, \mu)$ imply that

$$F_k(x(\theta_b^k, 0, \mu^k)) \leq F_k(x(\theta_b^k, \theta_c^k, \mu^k)) \leq F_k(x(\theta_b^k, 0, \mu^k)) - \frac{\theta_c^k}{\mu^k} (\delta c)^T x^k + \frac{\theta_c^k}{\mu^k} (\delta c)^T \hat{x}^k. \quad (4.25)$$

Observe now that $F_k(x)$ is strongly convex and that the eigenvalues of the Hessian matrix of F_k are uniformly bounded from zero. Hence, if $\lim_{k \uparrow \infty} (\hat{x}^k - x^k) \neq 0$, there exists $\epsilon > 0$ such that for each natural number, N , there exists $k \geq N$, such that

$$F_k(x(\theta_b^k, \theta_c^k, \mu^k)) \geq F_k(x(\theta_b^k, 0, \mu^k)) + \epsilon.$$

However, since $\theta_c^k/\mu^k \rightarrow 0$, this is a contradiction to (4.25). So, we must have

$$\lim_{k \uparrow \infty} (x(\theta_b^k, \theta_c^k, \mu^k) - x(\theta_b^k, 0, \mu^k)) = 0.$$

Using Theorem 4.1, we have

$$\begin{aligned} 0 &= \lim_{k \uparrow \infty} (x(\theta_b^k, \theta_c^k, \mu^k) - x(\theta_b^k, 0, \mu^k)) \\ &= \lim_{k \uparrow \infty} (x(\theta_b^k, \theta_c^k, \mu^k) - x^*(0, 0)) + \lim_{k \uparrow \infty} (x^*(0, 0) - x(\theta_b^k, 0, \mu^k)) \\ &= \lim_{k \uparrow \infty} (x(\theta_b^k, \theta_c^k, \mu^k) - x^*(0, 0)). \end{aligned}$$

Q.E.D.

5 Concluding remarks

In this paper we carried out an investigation of how the primal parameterized central path and the primal central optimal solution (analytic center of the optimal set) react to changes in the right-hand side vector, b , and the objective vector, c . These issues are important in the context of sensitivity analysis and parametric programming.

As was shown, the primal parameterized central path behaves differently under parameterization of b and c . In the former case, we proved that the primal central optimal solution has one-sided differentiability with respect to the perturbation parameter. Consequently, the entire central path has a smooth, uniformly bounded shift. Perturbation in the objective vector, however, may cause drastic changes in the central path. In this case we showed that every element of the central path defined on the optimal face is a cluster point of the parametrized central path.

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