A Primal-Dual Semidefinite Programming Approach to Linear Quadratic Control

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Abstract

We study a deterministic linear-quadratic (LQ) control problem over an infinite horizon, without the restriction that the control cost matrix R or the state cost matrix Q be positive definite. We develop a general approach to the problem based on semi-definite programming (SDP) and related duality analysis. We show that the complementary duality condition of the SDP is necessary and sufficient for the existence of an optimal LQ control under certain stability condition (which is satisfied automatically when Q is positive definite). When the complementary duality does hold, an optimal state feedback control is constructed explicitly in terms of the solution to the primal SDP.

Keywords: LQ control, semidefinite programming, complementary duality, generalized Riccati equation.

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1 Introduction

Consider the following deterministic linear-quadratic (LQ) control problem:

$$\text{(LQ)} \quad \min \quad J(x_0, u(\cdot)) := \int_0^\infty [x(t)^T Qx(t) + u(t)^T Ru(t)]dt \tag{1}$$

$$\text{s.t.} \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ x(0) = x_0 \in \Re^n. \end{cases} \tag{2}$$

Here and throughout the paper, A, B, and Q, R are constant matrices, with Q and R both being symmetric matrices; ^{T} denotes the transpose of matrices and vectors; and the control $u(\cdot)$ is an element of $L^2(\Re^m)$, the set of all $\Re^m$-valued, measurable functions satisfying $\int_0^{+\infty} \|u(t)\|^2 dt < +\infty$, where $\|u(t)\| := [\sum_i u_i(t)^2]^{1/2}$. To account for the infinite-horizon nature of the problem, we further require the control to be stabilizing such that the corresponding state trajectory converges to zero.

The LQ control problem, initiated by Kalman [8], is one of the most important classes of optimal control problems, in both theory and applications. It is well known that when $R \succ 0$ (positive definite) and $Q \succeq 0$ (non-negative definite), the problem (LQ) can be solved elegantly via the (algebraic) Riccati equation:

$$Q + A^T P + PA - PBR^{-1}B^T P = 0. \tag{3}$$

Furthermore, the optimal control is explicitly in a feedback form: $u^*(t) = -R^{-1}B^T Px^*(t)$, provided that the control is stabilizing.

The Riccati equation has been a primary, if not predominant, tool in studying LQ control in the literature. This approach, however, requires the control cost matrix R to be non-singular, which often excludes meaningful applications. In the general setting of $R \succeq 0$, the LQ theory itself becomes quite lacking. For instance, we do not even know if and when the LQ problem presented above possesses an optimal solution in the sense of a conventional control – one that is defined above and, in particular, one that does not involve impulsive distributions.

The objective of this paper is to present a unified theory to the general LQ problem — in particular, with $Q \succeq 0$ and $R \succeq 0$ — based on *semi-definite programming* (SDP) and associated duality theory. Our main result (Theorems 3.1 and 3.6) will establish that the existence of optimal controls to (LQ) is equivalent to the *complementary duality* of the corresponding SDP, provided $Q \succ 0$, and the optimal feedback control, when it exists, is directly generated by solving the primal SDP. In the more general case of $Q \succeq 0$, this equivalence condition still holds, provided the feedback control based on the primal SDP is stabilizing. It is worth noting that the classical, Riccati-based LQ theory, with $Q \succ 0$ and $R \succ 0$, reduces to a special case in our primal-dual SDP framework: indeed in this case, the SDP complementary duality is automatically satisfied.

A review of related literature is in order. One may argue that with a possibly singular R, the LQ problem has already been extensively studied in the literature under the heading of singular LQ control; refer to, e.g., [7, 4, 5, 6, 13, 14, 18]. In [6, 18] the existence of optimal controls for singular LQ problems is studied; but the class of controls is defined in the sense of impulsive distributions, which allow controls that can *instantaneously* steer the system dynamics from one state to another state so as to reduce the singular problem to a regular one. In addition to technical complications (such as Dirac measure and its derivatives), this kind of controls is quite impossible to implement in practice.

Another approach to singular LQ control is based on the connection between the Riccati equation and a certain linear matrix inequality (LMI). The LMI approach dates back to Yakubovich [19] and Willems [17]. It works for the singular case because LMI, in contrast to the Riccati equation, does not involve any matrix inverse; hence, in particular, it is not restricted to a non-singular R. Based on this idea, in [5] a necessary condition and a (different) sufficient condition are derived for the well-posedness of a singular LQ problem over a finite-time horizon; while in [13] it is shown that the maximal solution to the LMI provides the optimal cost value of the singular LQ problem via a transfer matrix technique.

In comparison, our results via primal-dual SDP are at once stronger (well-posedness is a considerably weaker condition than the existence of optimal solutions), more complete (ours is an "if and only if" condition), and computationally more viable (standard SDP algorithms are primal-dual based). Moreover, our results apply to both regular and singular LQ control problems in a unified framework.

Briefly, the rest of the paper is organized as follows. In §2, we present the generalized Riccati equation (i.e., the counterpart of (3) allowing $R \succeq 0$), the SDP primal-dual problems that correspond to (LQ), and related preliminary materials. Our main results are presented in §3, where we demonstrate that complementary duality is the key linkage between the SDP and (LQ), and construct an optimal feedback control based on the primal SDP solution. Concluding remarks are summarized in §4.

2 Generalized Riccati Equation and SDP

We start with presenting some regularity conditions concerning the problem (LQ).

(i) An open-loop control $u(\cdot)$ is called *admissible* (w.r.t. x_0), if it is (asymptotically) *stabilizing* (w.r.t. x_0), namely, if the state process under the control, $x(\cdot)$ of (2), with initial state x_0, satisfies $\lim_{t\to+\infty} x(t) = 0$. The set of all admissible controls w.r.t. x_0 is denoted as U^{x_0}.

(ii) A feedback control $u(t) = Kx(t)$, where K is a constant matrix, is called (asymptotically) *stabilizing*, if for every initial state x_0, we have $\lim_{t\to+\infty} x(t) = 0$, where $x(\cdot)$ is the solution to (2), with $u(t) = Kx(t)$.

(iii) Accordingly, the system in (2) is called (asymptotically) *stabilizable*, if there exists a stabilizing feedback control of the form $u(t) = Kx(t)$.

(iv) (LQ) is called *well-posed* (w.r.t. x_0), if its cost objective has a finite infimum:

$$-\infty < \inf_{u(\cdot)\in U^{x_0}} J(x_0, u(\cdot)) < +\infty.$$

(v) (LQ) is called *attainable* (w.r.t. x_0), if it is well-posed and if there exists a control that attains the infimum $\inf_{u(\cdot)\in U^{x_0}} J(x_0, u(\cdot))$, in which case the control is called *optimal.*

Throughout the paper we shall assume that $R \succeq 0$. Note that this condition is *necessary* for the LQ problem (LQ) to be well-posed (cf. [20, Chapter 6, Proposition 2.4]). Since we allow R to be singular, the classical Riccati equation is no longer defined. A natural extension is to consider the following *generalized Riccati equation*:

$$F(P) := A^T P + PA + Q - PBR^+B^T P = 0 \tag{4}$$

where R^+ stands for the pseudo-inverse of R. Note that R^+ satisfies the following properties (refer to [11], and note the symmetry of R):

$$R^+ \succeq 0, \quad (R^+)^T = R^+, \quad R^+R = RR^+, \quad RR^+R = R, \qquad R^+RR^+ = R^+.$$

Next, we introduce an affine transformation of the matrix P,

$$\mathcal{L}(P) := \begin{bmatrix} R, & B^T P \\ PB, & Q + A^T P + PA \end{bmatrix}. \tag{5}$$

The following lemma ([2]) shows that $F(P)$ and $\mathcal{L}(P)$ are closely related.

Lemma 2.1 *$\mathcal{L}(P) \succeq 0$ if and only if $F(P) \succeq 0$ and $(I - RR^+)B^T P = 0$.*

Consider the following SDP:

$$\begin{array}{lll} \text{(P)} & \max & \langle I, P\rangle \\ & \text{s.t.} & \mathcal{L}(P) \succeq 0, \quad P \in \mathcal{S}^{n\times n}. \end{array}$$

Here and below, $\mathcal{S}^{n\times n}$ denotes the set of $n\times n$ symmetric matrices, and $\langle X, Y\rangle := \sum_{i,j} X_{ij}Y_{ij}$ denotes the matrix inner-product. In particular, $\langle I, P\rangle$ (with I being the identity matrix) is equal to the trace of the matrix P.

Note that in the general setting here (allowing $R \succeq 0$), the SDP is still a well defined problem; in particular, it does not impose any restrictions on the definiteness of R. Hence, a viable approach to (LQ) is to solve the SDP first, and then study the relationship between the SDP solution and the solution to (LQ). To this end, consider the dual of (P), which is also an SDP. Let

$$Z := \begin{bmatrix} Z_b, & Z_u \\ Z_u^T, & Z_n \end{bmatrix}$$

denote the dual variable associated with the primal constraint $\mathcal{L}(P) \succeq 0$, with Z_b, Z_u and Z_n being a block partitioning of Z with appropriate dimensions.

The dual of (P) is

$$\begin{array}{lll} \text{(D)} & \min & \langle R, Z_b \rangle + \langle Q, Z_n \rangle \\ & \text{s.t.} & I + Z_u^T B^T + B Z_u + Z_n A^T + A Z_n = 0 \\ & & Z \succeq 0. \end{array}$$

The semidefinite programs are known to be special forms of *conic optimization problems*, for which there exists a well-developed duality theory; see, e.g. [10, 16, 9] and the references therein. Key points of the theory can be highlighted as follows:

- The weak duality always holds.In contrast, the strong duality needs not always hold (unlike the case of linear programming).
- A sufficient condition for the strong duality is that there exists a pair of *complementary* optimal solutions.For (P) and (D) above, this means that the primal optimal solution P^* and the dual optimal solution Z^* both exist and satisfy $\mathcal{L}(P^*)Z^* = 0$.
- If both (P) and (D) satisfy the *strict feasibility* (also known as *Slater's condition*), then the complementary solutions exist.

A mild regularity condition, which is assumed throughout the paper, is that the system in (2) is stabilizable as defined at the end of §1. In terms of SDP, this is equivalent to (D) satisfying Slater's condition. Refer to the lemma below.

Lemma 2.2 ([1, Theorem 5.2]) The system in (2) is stabilizable if and only if Problem (D) satisfies Slater's condition.

In the non-singular setting (i.e., when $Q \succ 0, R \succ 0$), the following lemma can be derived from the classical result about Riccati equation; see e.g. [12].

Lemma 2.3 Suppose $Q \succ 0, R \succ 0$. Then, there exists a *maximal* solution $P^* \succ 0$ to the Riccati equation $F(P^*) = 0$ (i.e., $P^* - P \succeq 0$ for any symmetric P that satisfies $F(P) \succeq 0$). And, P^* is also the unique optimal solution to the SDP problem (P). In this case, (LQ) has an optimal feedback control $u(t) = -R^{-1}B^T P^* x(t)$, with an optimal value $x_0^T P^* x_0$ for any initial state x_0.

It should be noted that the non-singularity of Q is necessary for the constructed control to be stabilizing. For example, take $n = 1, A = 0, B = 1, Q = 0$ and $R = 1$. In this case the Riccati equation is $P^2 = 0$ having the only solution $P = 0$. The corresponding control suggested by the above lemma is $u^*(t) = 0$, which is not stabilizing if the initial state is non-zero.

Underlying the elegant simplicity of the results summarized in Lemma 2.3 is the fact that in the non-singular setting both primal and dual SDP's satisfy Slater's condition. To see this, note that

$P^0 = 0$ is strictly feasible for the primal problem (as evident from $\mathcal{L}(P) \succ 0$ following (5), taking into account $Q \succ 0$ and $R \succ 0$), while the dual is strictly feasible by virtue of the system in (2) being stabilizable as discussed earlier. Hence complementary duality holds automatically in the non-singular setting. This leads to a constructive way of solving the Riccati equation through solving the SDP, for which efficient interior point codes are available (e.g., [15]). However, with the possible singularity of R, the situation becomes more complicated, as the primal problem may no longer satisfy Slater's condition, and consequently, complementary duality may fail. In contrast to Lemma 2.3, we have the following result, which will also be used later.

Proposition 2.4 Suppose $Q \succeq 0, R \succeq 0$. Then (P) has a unique optimal solution $P^* \succeq 0$.

Proof. Consider the following perturbed problem of (P) along with its dual, where $\epsilon > 0$:

$$\begin{array}{lll} (\mathrm{P}_\epsilon) & \max & \langle I, P\rangle \\ & \text{s.t.} & \left[\begin{array}{cc} R+\epsilon I, & B^T P \\ PB, & Q+\epsilon I + A^T P + PA \end{array}\right] \succeq 0 \end{array}$$

and

$$\begin{array}{lll} (\mathrm{D}_\epsilon) & \min & \langle R+\epsilon I, Z_b\rangle + \langle Q+\epsilon I, Z_n\rangle \\ & \text{s.t.} & I + Z_u^T B^T + BZ_u + Z_n A^T + AZ_n = 0 \\ & & Z := \left[\begin{array}{cc} Z_b, & Z_u \\ Z_u^T, & Z_n \end{array}\right] \succeq 0. \end{array}$$

Both problems satisfy Slater's condition, and therefore complementary optimal solutions exist. Observe that the feasible set of (D_ϵ) is independent of ϵ. Take any dual feasible solution Z^0. By weak duality, we have

$$\langle I, P\rangle \le \langle R+\epsilon I, Z_b^0\rangle + \langle Q+\epsilon I, Z_n^0\rangle. \tag{6}$$

Further, by Lemma 2.3, the unique optimal solution for (P_ϵ), denoted P_ϵ^*, is positive definite: $P_\epsilon^* \succ 0$. This, together with (6), implies in particular that P_ϵ^* is contained in a compact set, where $0 \le \epsilon \le \epsilon_0$, and $\epsilon_0 > 0$ is a pre-determined constant. Now, take a convergent subsequence such that $\lim_{i\to\infty} P_{\epsilon_i}^* = P_0^* \succeq 0$, with $\epsilon_i \to 0$ as $i \to \infty$. Clearly, P_0^* is a feasible solution of (P) since the feasible region of (P_ϵ) monotonically shrinks as $\epsilon \downarrow 0$. Now it suffices to show that $P_0^* = P^*$. Indeed, since P^* is feasible for (P_ϵ), and by Lemma 2.3 P_ϵ^* is the maximal solution to the corresponding Riccati equation, we have $P_\epsilon^* \succeq P^*$, resulting in $P_0^* \succeq P^*$. But P^* is optimal; hence, $\langle I, P^*\rangle \ge \langle I, P_0^*\rangle$. Therefore, we have $P^* = P_0^* \succeq 0$. The uniqueness is evident from the above argument. □

3 A Unified LQ Theory: Equivalent Conditions

In this section we present results where the optimal control of (LQ) is explicitly constructed in terms of the solutions to (P), the primal SDP. For the most part of this section we assume $Q \succ 0$. The following Theorem 3.1 summarizes our main results. Towards the end of the section, in Theorem 3.6, we point out the necessary modifications when allowing $Q \succeq 0$.

Theorem 3.1 Suppose $Q \succ 0$. The following three statements are equivalent:

(A) (P) and (D) have complementary optimal solutions.

(B) (P) has an optimal solution P^* which satisfies the generalized Riccati equation $F(P) = 0$.

(C) (LQ) has an attainable optimal feedback control,

$$u^*(t) = -R^+ B^T P^* x^*(t), \tag{7}$$

where P^* is an optimal solution to (P).

As discussed earlier, in the non-singular setting when $Q \succ 0$ and $R \succ 0$, complementary duality, and hence **(A)** is automatically satisfied. Therefore, Theorem 3.1 reduces to Lemma 2.3.

Among the three equivalent conditions in the above theorem, **(C)** concerns the original (LQ) problem, whereas **(A)** and **(B)** are easy to verify numerically — via SDP. (Notice that **(B)** does not require solving the generalized Riccati equation, which could be a difficult task; rather, it only verifies if an optimal solution to (P) satisfies the Riccati equation.)

From another angle, **(A)** and **(C)** equate verifying complementary duality of the SDP to solving the original (LQ) problem, whereas **(B)** plays the role of an intermediary between the two, with the generalized Riccati equation substituting for the dual SDP. Note that computationally **(B)** is not needed, as most SDP codes are primal-dual based, which directly solves (verifies) **(A)**.

For ease of exposition we break the proof of Theorem 3.1 into four parts, each with a different implication result. These parts are of interest in their own rights, and are listed below as Lemmas 3.2, 3.3, 3.4 and 3.5. First we show **(A)⇒(B)**, which asserts that complementary duality is the key for the solvability of the generalized Riccati equation.

Lemma 3.2 If (P) and (D) have complementary optimal solutions P^* and Z^*, respectively, then P^* must satisfy the generalized Riccati equation: $F(P^*) = 0$.

Proof. By Lemma 2.1, we have $(I - RR^+)B^T P^* = 0$. Thus, the following Schur decomposition holds true:

$$\mathcal{L}(P^*) = \begin{bmatrix} I, & 0 \\ P^*BR^+, & I \end{bmatrix} \begin{bmatrix} R, & 0 \\ 0, & F(P^*) \end{bmatrix} \begin{bmatrix} I, & R^+B^TP^* \\ 0, & I \end{bmatrix}. \tag{8}$$

From the relation $\mathcal{L}(P^*)Z^* = 0$, it follows that

$$\begin{bmatrix} R, & 0 \\ 0, & F(P^*) \end{bmatrix} \begin{bmatrix} I, & R^+B^TP^* \\ 0, & I \end{bmatrix} \begin{bmatrix} Z_b^*, & Z_u^* \\ (Z_u^*)^T, & Z_n^* \end{bmatrix}$$
$$= \begin{bmatrix} R(Z_b^* + R^+B^TP^*(Z_u^*)^T), & R(Z_u^* + R^+B^TP^*Z_n^*) \\ F(P^*)(Z_u^*)^T, & F(P^*)Z_n^* \end{bmatrix} = \begin{bmatrix} 0, & 0 \\ 0, & 0 \end{bmatrix}.$$

Therefore

$$F(P^*)(Z_u^*)^T = 0 \quad \text{and} \quad F(P^*)Z_n^* = 0,$$

and hence,

$$Z_u^* F(P^*) = 0 \quad \text{and} \quad Z_n^* F(P^*) = 0.$$

Since Z^* is dual feasible, we have

$$I + (Z_u^*)^T B^T + BZ_u^* + Z_n^* A^T + AZ_n^* = 0.$$

Multiplying $F(P^*)$ on both sides above yields

$$0 \;=\; F(P^*)(I + (Z_u^*)^T B^T + BZ_u^* + Z_n^* A^T + AZ_n^*)F(P^*) = F(P^*)^2,$$

which implies $F(P^*) = 0$. □

Next, we establish **(B)⇒(C)**, which relates the SDP to the original (LQ) problem.

Lemma 3.3 If (P) has an optimal solution P^* satisfying $F(P^*) = 0$, then (LQ) has an attainable optimal feedback control as determined by (7).

Proof. To start with, consider any primal feasible solution P, and any admissible (therefore stabilizing) control $u(\cdot) \in U^{x_0}$. We have,

$$\begin{aligned} \frac{d}{dt}(x(t)^T Px(t)) &= (Ax(t) + Bu(t))^T Px(t) + x(t)^T P(Ax(t) + Bu(t)) \\ &= x(t)^T (A^T P + PA)x(t) + 2u(t)^T B^T Px(t). \end{aligned} \tag{9}$$

Integrating (9) over $[0, \infty)$ and making use of the fact that $x(t)^T Px(t) \to 0$ as $t \to \infty$, we have

$$0 = x_0^T Px_0 + \int_0^\infty \Big[x(t)^T (A^T P + PA)x(t) + 2u(t)^T B^T Px(t)\Big]dt.$$

Therefore,

$$\begin{aligned} & J(x_0, u(\cdot)) = \int_0^\infty \Big[x(t)^T Qx(t) + u(t)^T Ru(t)\Big]dt \\ = \;& x_0^T Px_0 + \int_0^\infty \Big[x(t)^T (A^T P + PA + Q)x(t) + 2u(t)^T B^T Px(t) + u(t)^T Ru(t)\Big]dt \\ = \;& x_0^T Px_0 + \int_0^\infty \Big[(u(t) + R^+ B^T Px(t))^T R(u(t) + R^+ B^T Px(t)) + x(t)^T F(P)x(t)\Big]dt. \end{aligned} \tag{10}$$

Since P is feasible, we have $F(P) \succeq 0$. This means

$$J(x_0, u(\cdot)) \geq x_0^T Px_0, \tag{11}$$

for any P feasible to (P) and for any admissible control $u(\cdot) \in U^{x_0}$. On the other hand, under the feedback control $u^*(t) = -R^+B^TP^*x(t)$, taking into account $P^* \succeq 0$ (Proposition 2.4), we have

$$\begin{aligned}
0 \;\leq\; & J(x_0, u^*(\cdot)) = \int_0^\infty \left[x(t)^TQx(t) + u^*(t)^TRu^*(t)\right]dt \\
= \;& \lim_{t\to\infty} \int_0^t \left[x(\tau)^TQx(\tau) + u^*(\tau)^TRu^*(\tau)\right]d\tau \\
= \;& \lim_{t\to\infty} \{x_0^TP^*x_0 - x(t)^TP^*x(t) \\
& + \int_0^t \left[x(\tau)^T(A^TP^* + P^*A + Q)x(\tau) + 2u(\tau)^TB^TP^*x(\tau) + u^*(\tau)^TRu^*(\tau)\right]d\tau\} \\
\leq \;& x_0^TP^*x_0 \\
& + \lim_{t\to\infty} \int_0^t \left[(u^*(\tau) + R^+B^TP^*x(\tau))^TR(u^*(\tau) + R^+B^TP^*x(\tau)) + x(\tau)^TF(P^*)x(\tau)\right]d\tau \\
= \;& x_0^TP^*x_0. \qquad (12)
\end{aligned}$$

First of all the above shows that the feedback control $u^*(\cdot)$ incurs a finite cost (w.r.t. any initial state x_0), then it must be stabilizing (and hence admissible). This is because a finite cost in (1) implies $\lim_{t\to+\infty} x^*(t)^TQx^*(t) = 0$, where $x^*(\cdot)$ is the corresponding state trajectory; and since $Q \succ 0$, we must have $\lim_{t\to+\infty} x^*(t) = 0$. On the other hand, (12) yields $J(x_0, u^*(\cdot)) \leq x_0^TP^*x_0$. Thus, in view of (11) we conclude that $u^*(\cdot)$ is an optimal control. □

The last piece in establishing the equivalence relations in Theorem 3.1 is to show **(C)⇒(A)**. To do so, we need to first establish another result, which is useful in its own right. We want to show that **(A)** is, in fact, implied by a weaker version of **(B)**. That is, complementary duality is actually *necessary* for any non-negative and *feasible* (as opposed to optimal) solution of (P) to satisfy the generalized Riccati equation.

Lemma 3.4 If (P) has a feasible solution P^* satisfying $P^* \succeq 0$ and $F(P^*) = 0$, then there exist complementary optimal solutions to (P) and (D); and in particular, P^* is optimal to (P).

Proof. Denote $K := -R^+B^TP^*$. First we show that the feedback control given by $u(t) = Kx(t)$ must be stabilizing. Indeed, going through the same calculation as (12) and noting the assumption that $P^* \succeq 0$, we conclude that $u(\cdot)$ incurs a finite cost with respect to any initial state. Hence it must be stabilizing (as shown in the proof of Lemma 3.3). It follows that the following Lyapunov equation

$$(A + BK)Y + Y(A + BK)^T + I = 0$$

has a positive solution; let it be $Y^* \succ 0$. Let

$$Z_n^* = Y^*, \qquad Z_u^* = KY^*, \qquad Z_b^* = KY^*K^T.$$

By this construction, we can easily verify the following:

$$\begin{bmatrix} Z_b^*, & Z_u^* \\ (Z_u^*)^T, & Z_n^* \end{bmatrix} = \begin{bmatrix} I, & K \\ 0, & I \end{bmatrix} \begin{bmatrix} 0, & 0 \\ 0, & Z_n^* \end{bmatrix} \begin{bmatrix} I, & 0 \\ K^T, & I \end{bmatrix} \succeq 0,$$

and

$$I + (Z_u^*)^T B^T + B Z_u^* + Z_n^* A^T + A Z_n^* = 0.$$

Therefore, Z^* is a feasible solution of (D). Moreover,

$$\begin{aligned}
&\mathcal{L}(P^*)\begin{bmatrix} Z_b^*, & Z_u^* \\ (Z_u^*)^T, & Z_n^* \end{bmatrix} \\
= &\begin{bmatrix} I, & 0 \\ -K^T, & I \end{bmatrix}\begin{bmatrix} R, & 0 \\ 0, & F(P^*) \end{bmatrix}\begin{bmatrix} I, & -K \\ 0, & I \end{bmatrix}\begin{bmatrix} Z_b^*, & Z_u^* \\ (Z_u^*)^T, & Z_n^* \end{bmatrix} \\
= &\begin{bmatrix} I, & 0 \\ -K^T, & I \end{bmatrix}\begin{bmatrix} R(Z_b^* - K(Z_u^*)^T), & R(Z_u^* - K Z_n^*) \\ F(P^*)(Z_u^*)^T, & F(P^*) Z_n^* \end{bmatrix} = \begin{bmatrix} 0, & 0 \\ 0, & 0 \end{bmatrix}.
\end{aligned}$$

This means that P^* and Z^* are complementary solutions. In particular, P^* is optimal to (P).

□

We are now ready to close the loop of equivalence, to show **(C)⇒(A)**, which indicates that complementary duality is not only sufficient but also necessary for solving (LQ).

Lemma 3.5 If (LQ) has an attainable optimal control w.r.t. any initial condition x_0, then (P) and (D) must have complementary optimal solutions.

Proof. Since (LQ) has an attainable optimal control w.r.t. any initial condition x_0, it is known ([3, p.21]) that there exists $M \succeq 0$ such that

$$\inf_{u(\cdot) \in U^{x_0}} J(x_0, u(\cdot)) = x_0^T M x_0.$$

For the time being, suppose the matrix M is a feasible solution to (P). Fix an initial x_0 and let $u^*(\cdot)$ be the optimal control w.r.t. x_0. Since M is feasible to (P), using (10), we obtain the following identity:

$$\begin{aligned}
J(x_0, u^*(\cdot)) &= x_0^T M x_0 \\
&+ \int_0^\infty \Big[(u^*(t) + R^+ B^T M x(t))^T R (u^*(t) + R^+ B^T M x(t)) + x(t)^T F(M) x(t)\Big] dt.
\end{aligned}$$

Since $J(x_0, u^*(\cdot)) = x_0^T M x_0$, we have

$$\int_0^\infty \Big[(u^*(t) + R^+ B^T M x(t))^T R (u^*(t) + R^+ B^T M x(t)) + x(t)^T F(M) x(t)\Big] dt = 0. \qquad (13)$$

Thus, $x(t)^T F(M) x(t) = 0$ for all $t \in [0, \infty)$. Since x_0 can be chosen arbitrarily, we conclude that $F(M) = 0$. The desired result then follows from Theorem 3.4. Furthermore, we know M must be optimal to (P).

What remains is to show the primal feasibility of M. To this end we consider the perturbed problem (P_ϵ) and its dual (D_ϵ) introduced in the proof of Proposition 2.4. For the optimal solution of (P_ϵ), denoted P_ϵ^*, there is a convergent subsequence such that $\lim_{i\to\infty} P_{\epsilon_i}^* = P_0^*$, with $\epsilon_i \to 0$ as $i \to \infty$. Clearly, P_0^* is a feasible solution of (P), since the feasible region of (P_ϵ) shrinks as $\epsilon \to 0$. We now show that $P_0^* = M$.

First, it follows from Lemma 2.3 that $\inf_{u(\cdot)\in U^{x_0}} J_{\epsilon_i}(x_0, u(\cdot)) = x_0^T P_{\epsilon_i}^* x_0$ for all i, where

$$J_\epsilon(x_0, u(\cdot)) = \int_0^\infty [x(t)^T(Q + \epsilon I)x(t) + u(t)^T(R + \epsilon I)u(t)]dt.$$

Let $\bar{u}(\cdot)$ be the optimal control of (LQ) w.r.t. x_0 and $\bar{x}(\cdot)$ be the corresponding state. Then, we have

$$\begin{aligned} 0 &\le \inf_{u(\cdot)\in U^{x_0}} J_\epsilon(x_0, u(\cdot)) - \inf_{u(\cdot)\in U^{x_0}} J(x_0, u(\cdot)) = \inf_{u(\cdot)\in U^{x_0}} J_\epsilon(x_0, u(\cdot)) - J(x_0, \bar{u}(\cdot)) \\ &\le J_\epsilon(x_0, \bar{u}(\cdot)) - J(x_0, \bar{u}(\cdot)) = \epsilon \int_0^\infty (\|\bar{x}(t)\|^2 + \|\bar{u}(t)\|^2)dt. \end{aligned}$$

It then follows that

$$\lim_{i\to\infty} x_0^T P_{\epsilon_i}^* x_0 \equiv \lim_{i\to\infty} \inf_{u(\cdot)\in U^{x_0}} J_{\epsilon_i}(x_0, u(\cdot)) = \inf_{u(\cdot)\in U^{x_0}} J(x_0, u(\cdot)) \equiv x_0^T M x_0.$$

The above yields $x_0^T P_0^* x_0 = x_0^T M x_0$ for all x_0, implying $M = P_0^*$. This shows that M is indeed a primal feasible solution, and consequently (P) and (D) have complementary optimal solutions as we discussed before. □

To conclude this section we discuss the more general case of $Q \succeq 0$. The key advantage with a non-singular Q, as we have observed above, is that for any primal feasible solution P^* satisfying $P^* \succeq 0$ and $F(P^*) = 0$, the control in (7) is automatically stabilizing (see the proof of Lemma 3.3). This stability is no longer guaranteed when Q is possibly singular.

Theorem 3.6 Suppose $Q \succeq 0$.

(i) If (P) and (D) have complementary optimal solutions P^* and Z^*, respectively, then P^* must satisfy the generalized Riccati equation, $F(P) = 0$.

(ii) If (P) has a feasible solution P^* satisfying $P^* \succeq 0$ and $F(P^*) = 0$, and if the control $u(t) = -R^+B^TP^*x(t)$ is stabilizing, then it must be optimal to (LQ).

(iii) If (LQ) has an attainable optimal control, w.r.t. any initial condition x_0, then (P) must have an optimal solution P^* satisfying $F(P^*) = 0$. Moreover, if the feedback control $u(t) = -R^+B^TP^*x(t)$ is stabilizing, then (P) and (D) must have complementary optimal solution-s.

Proof. (i) This follows from Lemma 3.2, as the proof there does not require Q to be non-singular. (ii) In the proof of Lemma 3.3, we established that the control in (7) is stabilizing due to Q being non-singular. Here, the stability of the control is assumed. On the other hand, note that the proof of Lemma 3.3 does not require the optimality of P^*; it only utilizes the fact that $P^* \succeq 0$ (which is implied by the optimality of P^* via Proposition 2.4). Hence the proof of Lemma 3.3 applies to the present case.

(iii) The proof of Lemma 3.5 implies that there is a primal feasible solution M with

$$\inf_{u(\cdot)\in U^{x_0}} J(x_0, u(\cdot)) = x_0^T M x_0, \text{ and } F(M) = 0.$$

On the other hand, in view of (11), we have

$$x_0^T M x_0 \equiv \inf_{u(\cdot)\in U^{x_0}} J(x_0, u(\cdot)) \geq x_0^T P x_0,$$

for any P feasible to (P). Hence M is optimal for (P). Let $P^* = M$. However, *a priori*, we do not know whether the control $u(t) = -R^+B^TP^*x(t)$ is stabilizing because (13) does not necessarily imply that the optimal control stated in (iii) must be $u(t) = -R^+B^TP^*x(t)$ (due to the possible singularity of R); hence, its assumed stabilizing property is needed to guarantee the complementary optimal solutions of (P) and (D) as in the proof of Lemma 3.4. (Note, in Lemma 3.4, this stabilizing property is automatic, since $Q \succ 0$.) □

Now let us illustrate the above results by an example.

Example 3.7 Let

$$A = \begin{bmatrix} 1, & 1 \\ 4, & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1, & 1 \\ 1, & -2 \end{bmatrix}, \qquad Q = \begin{bmatrix} 12, & 6 \\ 6, & 3 \end{bmatrix} \succeq 0, \qquad R = \begin{bmatrix} 1, & 0 \\ 0, & 0 \end{bmatrix} \succeq 0.$$

This system is easily seen to be stabilizable. To identify a non-negative feasible solution P^* to the primal SDP that satisfies the generalized Riccati equation $F(P^*) = 0$, first consider the constraint $(I - RR^+)B^TP^* = 0$ as stipulated by Lemma 2.1. This gives rise to $P^* = \begin{bmatrix} 2p, & p \\ p, & \frac{p}{2} \end{bmatrix}$ for some p. Substituting the above into the generalized Riccati equation yields

$$A^TP^* + P^*A + Q - P^*BR^+B^TP^* \equiv \begin{bmatrix} -9p^2 + 12p + 12, & -\frac{9}{2}p^2 + 6p + 6 \\ -\frac{9}{2}p^2 + 6p + 6, & -\frac{9}{4}p^2 + 3p + 3 \end{bmatrix} = 0.$$

Solving for p leads to $p = 2$. Thus, $P^* = \begin{bmatrix} 4, & 2 \\ 2, & 1 \end{bmatrix} \succeq 0$ is a primal feasible solution that satisfies $F(P^*) = 0$. On the other hand, in this case $A - BR^+B^TP^* = \begin{bmatrix} -5, & -2 \\ -2, & -2 \end{bmatrix}$, which has eigenvalues -1 and -6. Hence the control $u^*(t) = -R^+B^TP^*x^*(t)$ is stabilizing. By Theorem 3.6-(ii), the control must be optimal as well.

4 Concluding Remarks

We have developed a unified theory, based on primal-dual SDP, to LQ control allowing the cost matrices Q and R to be singular. The unified theory presents a complete solution to the problem: it either derives the optimal feedback control or determines that the problem possesses no attainable optimal control.

In a sequel to this paper, we study the SDP approach to the *stochastic* LQ control problem, where the cost matrices may even be *indefinite* (negative definite, in particular). Indeed, the fact that the SDP approach is extendable to the stochastic case further demonstrates its power and versatility. As a matter of fact, it is not clear how other approaches such as distributional control could be adaptable to the stochastic case. For stochastic systems, stability will play a more critical role, as the "dual" of optimality; and while the main results here hold, mutatis mutandis, in the stochastic setting, the analysis and treatment of the stochastic problem will have to be quite different.

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