On a profit maximizing location model

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In this paper we discuss a locational model with a profit-maximizing objective. The model can be illustrated by the following situation. There is a set of potential customers in a given region. A firm enters the market and wants to sell a certain product to this set of customers. The location and demand of each potential customer are assumed to be known. In order to maximize its total profit, the firm has to decide: 1) where to locate its distribution warehouse to serve the customers; 2) the price for its product. Due to existence of competition, each customer holds a reservation price for the product. This reservation price is a decreasing function in the distance to the warehouse. If the actual price is higher than the reservation price, then the customer will turn to some other supplier and hence is lost from the firm's market. The problem of the firm is to find the best location for its warehouse and the best price for its product at the same time in order to maximize the total profit. We show that under certain assumptions on the complexity counts, a special case of this problem can be solved in polynomial time.

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1. Introduction

Traditionally the literature on locational models under the microeconomic marketing environment is quite limited. See Chapter 15 of [2] and [3] for a survey. One reason for this could be that most locational models have an accent of minimizing costs of some type. One of the most celebrated models in location science is certainly the Weber model introduced by A. Weber in 1909 [7]. This model can be briefly described as follows. There is a set of customers to be served. Each customer is known by its location and the quantity of the demand.

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The problem is to find a location for the distribution warehouse to serve these customers, where the objective is to minimize the total transportation costs. This model has been extensively studied in the literature, because it captures an essence common in many locational problems. At the same time, it receives criticisms as well. One of the criticisms was raised by Lösch in [5]: "Weber's solution for the problem of location proves to be incorrect as soon as not only cost but also sales possibilities are considered. His fundamental error consists in seeking the place of lowest cost. This is as absurd as to consider the point of largest sales as the proper location. Every such one-sided orientation is wrong. Only search for the place of greatest profit is right".

In Chapter 15 of [2] Peeters and Thisse discussed several operational models which combine location problems with profit-maximization objectives. Moreover, it is assumed that there is no competition in the market and that the demand is a decreasing affine linear function in the price, as in many other microeconomic pricing models. We refer the reader to Chapter 4 of [4] for a survey of various pricing models.

The emphasis of the model to be discussed in this paper is different. We shall take competition into account. Consider the following situation. There is a firm which produces a certain product. This product will be brought to a warehouse for distribution. There are n potential customers who are interested in this product. The locations of these n customers are known to be a_i with $i = 1, \cdots, n$. Moreover, it is known that the demand quantity of customer i is Q_i for $i = 1, \cdots, n$. For each customer, it is attractive if the warehouse is located in his/her close neighborhood since the customer bears the transportation costs. Due to the existence of competition in the market, the reservation price (the maximum price up to which a customer is willing to pay) of each customer is a strictly decreasing function in the distance to the warehouse. As soon as the actual price goes beyond a customer's reservation price, he/she will turn to some other supplier, and thus is lost completely from the market of the firm. The problem of the firm is to choose the location for its warehouse and the price for the product in order to maximize the total profit.

The goal of this paper is to show that if finding roots of a polynomial with a fixed degree is counted as one operation, then there is a polynomial time solution method for solving a special case of this problem. Certainly, the model is not supposed to be immediately operational due to its simplicity in nature. However, many extensions of the model are possible.

2. The model

Mathematically, the problem discussed at the beginning of the previous section can be formulated as

$$\begin{array}{l} \text{maximize } p\sum_{i=1}^n Q_i \chi_{\{p \le r_i(\|x-a_i\|)\}} \\ \text{subject to } p \in \Re^1_+ \text{ and } x \in \Re^2 \end{array}$$

where $r_i(\|x-a_i\|)$ is the reservation price of the customer i given the location of the warehouse at x. In this expression χ_s stands for the characteristic function of a statement s:

$$\chi_s = \begin{cases} 1, \text{if } s \text{ is true} \\ 0, \text{otherwise.} \end{cases}$$

According to our model description, the function r_i is a strictly decreasing function.

This model looks quite messy in the sense that it is neither convex nor concave in its decision variables p and x. Even for fixed p, the objective is in general not continuous in x. It is interesting to see, however, that the problem can be solved when p is fixed as a parameter. To illustrate this, let us introduce

$$F_p(x) = p\sum_{i=1}^n Q_i \chi_{\{p \le r_i(\|x-a_i\|)\}}.$$

The original model can be rewritten as

$$\max_{p\in\Re^1_+} \max_{x\in\Re^2} F_p(x).$$

For the moment it is not even clear whether this problem has an attainable optimal solution, since the objective is discontinuous in its decision variables. However, we will show later that an optimal solution indeed exists.

As a first step we now pose the following question: *Can we efficiently evaluate* $\max_{x\in\Re^2} F_p(x)$ *when* p *is fixed as a parameter?*

To keep analysis simple, we consider the case where the norm is Euclidean. In that case, the above problem is equivalent to the following combinatorial disk covering problem. Before going into details, first we give a few words on the notion of *disk*: a disk is assumed to be closed, including its boundary which is called a circle. In some places of the paper these two concepts, viz. disks and circles, are used interchangeably for convenience. Whenever applicable, a disk is denoted by D and a circle is denoted by C respectively.

(Disk Covering) We are given n disks $\{D_i \mid 1 \le i \le n\}$ in the plane. Disk D_i is centered at a_i with radius R_i, $i = 1, \cdots, n$. It is assumed that if a point is in D_i then this point receives a weight w_i with $w_i \ge 0$. The total weight of a point x is denoted by $w(x)$ and is simply the summation of all weights from the disks that the point is contained in, i.e.

$$w(x) = \sum_{\{i|x \in D_i\}} w_i.$$

The problem is to find a point in the plane with the maximum total weight. Namely, we wish to find $x \in \Re^2$ that maximizes $w(x)$.

In the picture below we consider an example of the disk covering problem.

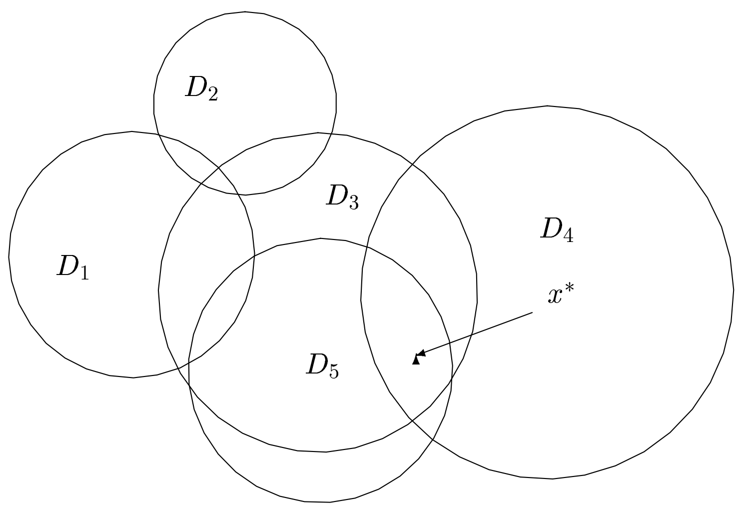

Figure 1. Disk covering.

In this instance, we let $w_1 = 7$, $w_2 = 6$, $w_3 = 10$, $w_4 = 8$ and $w_5 = 9$. Therefore, an optimal solution x^* lies in the intersection of D_3, D_4 and D_5.

This problem was discussed by Drezner in [1]. In fact the method proposed by Drezner [1] is very similar to the one that we are going to discuss now. In [6] Mehrez and Stulman also discussed a special case of this problem, in which all the weights are equal. The motivation for our discussion on this particular problem is that we will use this as a basis for solving the pricing/location problem.

Observe that for each pair of circles C_i and C_j, boundaries of D_i and D_j, there can be maximally two intersection points. Let them form a set $I_{\{i,j\}}$. To be more precise, we define

$$I_{\{i,j\}} = \begin{cases} \text{The two intersection points,} & \text{if } C_i \text{ and } C_j \text{ intersect,} \\ \text{The touching point,} & \text{if } C_i \text{ and } C_j \text{ tangentially touch at a point,} \\ \emptyset, & \text{if } C_i \text{ and } C_j \text{ do not intersect.} \end{cases}$$

Let the set of all intersection points be

$$I = \cup_{1 \le i < j \le n} I_{\{i,j\}}.$$

We define J to be the union of I and the centers of all circles, i.e.

$$J = I \cup \{a_i \mid i = 1, ..., n\}.$$

The next lemma shows that J must contain an optimal solution.

Lemma 1. For the circle covering problem, it holds that

$$\max_{x \in \Re^2} w(x) = \max_{x \in J} w(x).$$

Proof. Let x^* be an optimal solution. We may assume $x^* \in D_i$ for at least one i. If x^* is contained in more than one disk, then one can find a disk D_j, $j \neq i$, such that $x^* \in D_j$ and $\tilde{x} \in I_{\{i,j\}} \subseteq I \subseteq J$ has the property that $w(\tilde{x}) = w(x^*)$, and hence the lemma follows. Now assume that $x^* \notin D_j$ for all $j \in \{1, 2, ..., n\} \setminus i$. In that case, if $D_i \cap D_j = \emptyset$ for all $j \in \{1, 2, ..., n\} \setminus i$, then $w(x^*) = w(a_i)$ and so $a_i \in J$ is optimal. If there is $j \in \{1, 2, ..., n\} \setminus i$ with $D_i \cap D_j \neq \emptyset$, then we let $y \in I_{\{i,j\}}$ and using $w_j \ge 0$ it follows that $w(y) \ge w(x^*)$ and so $y \in I \subseteq J$ is optimal.

□

We remark that in the disk covering problem we assumed $w_i \ge 0$ for all i. If this condition is dropped, i.e. w_i's may take negative values, then the above result no longer holds. However, an extension is possible by properly enlarging the size of the candidate set J.

Below we shall describe a procedure for solving the problem. Note that in our case $|J| \le n^2$ and that checking the weight of a given point amounts to $O(n)$ operations. Hence, checking the objective value for all the points in J yields an optimal solution in $O(n^3)$ operations.

This computational complexity can be further improved by a more careful sorting.

First check the objective value of all the points in $\{a_i \mid i = 1, 2, ..., n\}$. What remains is to select the best point in I.

Let the circles be ordered by their natural indices:

$$C_1, C_2, \cdots, C_n.$$

First, take the circle C_1. Sort all of its intersection points with other circles, i.e. sort the points in the set I_1 with

$$I_1 := \cup_{j=2,...,n} I_{\{1,j\}}$$

in the clockwise direction with respect to C_1. For simplicity, one may start sorting with an intersection point between C_1 and C_2.

Take one intersection point in I_1 in that order.

Let d be an n-dimensional vector defined as

$$d_i = \begin{cases} 1, \text{if the current point is contained in } C_i, \\ 0, \text{otherwise.} \end{cases}$$

Adding w_i's with $d_i = 1$ gives the total weight for the point under consideration.

Now, take the next point in I_1 according to the ordering described previously. Note that the d vector of the next point differs only in one position from the previous one. Hence, it takes only constant time to update the total weight. We remark that it involves a sorting for the points with respect to their positions in C_1, and so this gives an computational complexity of order $O(n \log n)$.

After we are done with the points in I_1, skip I_1 from I and continue the procedure with C_2. This will be repeated for all circles. In total the procedure will require $O(n^2 \log n)$ basic operations to find a point with maximum disk covering weight.

A similar algorithm was described in [1] which leads to the same computational complexity, namely $O(n^2 \log n)$. However, the description of the method above is better suited for solving our pricing/location model.

3. A procedure for solving the pricing/location model

In this section we discuss the original problem in which price p plays the role as a part of the decision variable. Moreover, we assume that the reservation price of a customer is affine, i.e.

$$r_i(\|x - a_i\|) = \max\{u_i - v_i \|x - a_i\|, 0\}$$

where $u_i > 0$ and $v_i > 0$. This formula can be explained as follows. Suppose that apart from the firm we are considering, the best price customer i gets elsewhere is u_i (including the transportation cost). Moreover, if customer i patronizes the current firm, then the transportation cost is assumed to be $v_i \|x - a_i\|$. Therefore, customer i will stay with the firm as long as $p \leq r_i(\|x - a_i\|)$, i.e.

$$p + v_i \|x - a_i\| \leq u_i.$$

As a matter of notation we denote $(x)_+ := \max\{x, 0\}$, and denote $D(a; R)$ (resp. $C(a; R)$) to be a disk (resp. circle) centered at a with radius R.

The pricing/location problem can be reformulated as follows:

$$\max_{p \in \Re^1_+} \max_{x \in \Re^2} F_p(x)$$

with

$$F_p(x) = p \sum_{i=1}^{n} Q_i \chi_{\{\|x - a_i\| \leq (u_i - p)_+ / v_i\}}.$$

Let

$$\rho_i(p) = (u_i - p)_+ / v_i$$

for $i = 1, 2, ..., n$. Define

$$D_i(p) = \begin{cases} D(a_i; \rho_i(p)), & \text{if } p \leq u_i, \\ \emptyset, & \text{if } p > u_i \end{cases}$$

for $i = 1, 2, ..., n$.

Clearly, for each given value of p, solving $\max_{x \in \Re^2} F_p(x)$ is equivalent to solving the disk covering problem discussed in the previous section with disks $D_i(p)$ and weights pQ_i, $i = 1, 2, ..., n$. Let the corresponding optimal value be

$$f(p) := \max_{x \in \Re^2} F_p(x). \tag{1}$$

Consider a pair of disks $D_i(p)$ and $D_j(p)$. These two disks will come into existence when $p \leq \min\{u_i, u_j\}$. For $p \leq \min\{u_i, u_j\}$, their relative position can be determined explicitly as follows:

- if $\rho_i(p) + \rho_j(p) = \|a_i - a_j\|$, then $D_i(p)$ and $D_j(p)$ tangentially touch from outside at one point;

- if $\rho_i(p) - \rho_j(p) = \|a_i - a_j\|$, then $D_i(p)$ contains $D_j(p)$ and they tangentially touch from inside at one point;
- if $\rho_i(p) - \rho_j(p) > \|a_i - a_j\|$, then $D_i(p)$ properly contains $D_j(p)$;
- if $\rho_j(p) - \rho_i(p) = \|a_i - a_j\|$, then $D_j(p)$ contains $D_i(p)$ and they tangentially touch from inside at one point;
- if $\rho_j(p) - \rho_i(p) > \|a_i - a_j\|$, then $D_j(p)$ properly contains $D_i(p)$;
- if $\rho_i(p) + \rho_j(p) < \|a_i - a_j\|$, then $D_i(p)$ and $D_j(p)$ intersect at two points.

Therefore, there can be nine different states regarding the positions of $D_i(p)$ and $D_j(p)$, namely: 1) none of them exists; 2) only $D_i(p)$ exists; 3) only $D_j(p)$ exists; 4)–9) they both exist, in which case each of the above six situations may occur, so it adds up to nine different states in total. Because both $\rho_i(p)$ and $\rho_j(p)$ are linear functions for $p \le \min\{u_i, u_j\}$, we conclude that there are maximally five values of p, two of them being u_i and u_j, such that the open intervals formed by these values and the values themselves represent different states of the circles. Moreover, these p values can be computed explicitly.

Next we consider the values of p for which the boundaries of at least three of the disks $D_1(p)$, $D_2(p)$, ..., $D_n(p)$ will intersect. It is elementary that for a triangle with edge lengths x, y and z, the area of the triangle, denoted by $S(x, y, z)$, is given by the following formula:

$$S(x, y, z) = \frac{1}{4}\sqrt{2x^2y^2 + 2x^2z^2 + 2y^2z^2 - x^4 - y^4 - z^4}.$$

Consider a triple of disks $D_i(p)$, $D_j(p)$ and $D_k(p)$. Their boundaries will intersect at a single point if and only if one of the following equations holds:

$$\begin{aligned} S(\|a_i - a_j\|, \|a_i - a_k\|, \|a_j - a_k\|) &= S(\|a_i - a_j\|, \rho_i(p), \rho_j(p)) \\ &+ S(\|a_i - a_k\|, \rho_i(p), \rho_k(p)) + S(\|a_j - a_k\|, \rho_j(p), \rho_k(p)); \end{aligned}$$

or

$$\begin{aligned} S(\|a_i - a_j\|, \|a_i - a_k\|, \|a_j - a_k\|) &+ S(\|a_i - a_j\|, \rho_i(p), \rho_j(p)) \\ &= S(\|a_i - a_k\|, \rho_i(p), \rho_k(p)) + S(\|a_j - a_k\|, \rho_j(p), \rho_k(p)); \end{aligned}$$

or

$$\begin{aligned} S(\|a_i - a_j\|, \|a_i - a_k\|, \|a_j - a_k\|) &+ S(\|a_i - a_k\|, \rho_i(p), \rho_k(p)) \\ &= S(\|a_i - a_j\|, \rho_i(p), \rho_j(p)) + S(\|a_i - a_j\|, \rho_j(p), \rho_k(p)); \end{aligned}$$

or

$$S(\|a_i - a_j\|, \|a_i - a_k\|, \|a_j - a_k\|) + S(\|a_j - a_k\|, \rho_j(p), \rho_k(p))$$
$$= S(\|a_i - a_j\|, \rho_i(p), \rho_j(p)) + S(\|a_i - a_k\|, \rho_i(p), \rho_k(p)).$$

Since $\rho_i(p)$, $\rho_j(p)$ and $\rho_k(p)$ are decreasing linear functions in their respective domains, the solution to each of the above equations can be found by searching the roots of some polynomials obtained by, e.g., squaring the equations above, and then re-arranging the terms and squaring again, and then one more re-arrange the terms and squaring. The polynomials thus obtained will be of degree 16. From the geometric meaning of these equations it is evident that for each triple of disks there can be at most four valid roots. In our computational complexity model, we count finding the roots of such a polynomial as one *basic operation*. We remark that this assumption is stronger than the usual definition of a basic operation such as computing the square-root of a number. In practice, Newton's method is expected to work very fast for the root-finding purpose.

Now, for each triple of the disks we compute all the values of p satisfying one of the above equations. Together with the value of p regarding positions of each pair of the circles, which we computed before, we call them the *critical p values*. In total there can be maximally $O(n^3)$ different critical p values. Finding all of them can be done in $O(n^3)$ time according to our computational model.

Let all the critical p values be

$$0 = p_0 < p_1 < p_2 < \cdots < p_m = \max\{u_1, ..., u_n\}$$

with $m = O(n^3)$.

Lemma 2. The value function $f(p)$ defined in (1) is increasing and linear in each interval $(p_{j-1}, p_j]$ where $1 \le j \le m$.

Proof.

By the definition of the critical p values, for all $p \in (p_{j-1}, p_j)$ the relative positions of the disks $D_1(p)$, $D_2(p)$, ..., $D_n(p)$ remain unchanged, where $j = 1, 2, ..., m$. Observe that

$$f(p) = p \max_{x \in \Re^2} \sum_{i=1}^{n} Q_i \chi_{\{x \in D(a_i; \rho_i(p))\}}.$$

For an arbitrarily fixed $\bar{p} \in (p_{j-1}, p_j)$ we apply Lemma 1 and obtain that the value $f(\bar{p})$ is achieved by an intersection point or a center of the circles $C(a_i; \rho_i(\bar{p}))$, $i =$

$1, 2, ..., n$. If this point, denoted by $x(\bar{p})$, is an intersection between $C(a_k; \rho_k(\bar{p}))$ and $C(a_l; \rho_l(\bar{p}))$ (similar arguments can be applied if it is a center of a circle), then, because I is invariant for all $p \in (p_{j-1}, p_j)$ the same intersection point of $C(a_k; \rho_k(p))$ and $C(a_l; \rho_l(p))$ will attain the maximum value for all $p \in (p_{j-1}, p_j)$. This shows that $f(p)$ is an increasing linear function in (p_{j-1}, p_j). Now observe that the radius $\rho_i(p)$ is continuous and non-increasing in p for all i. Therefore, $x(p)$ converges as $p \to p_j$. Let the limit be $x(p_j)$. For $p_{j-1} < p < p_j$, $x(p) \in C(a_i; \rho_i(p))$ if and only if $x(p_j) \in C(a_i; \rho_i(p_j))$. This shows that

$$\lim_{p \uparrow p_j} f(p) = f(p_j),$$

i.e. f is right-continuous. The lemma is thus proven.

□

Using the above lemma it is clear that the pricing/location problem has an optimal solution, and this solution can be obtained by evaluating $f(p)$ at $p = p_1, p_2, ..., p_m$ and then selecting the one with maximum value. Since finding a solution for each $f(p_j)$ is a circle covering problem with nonnegative weights, we may apply the result of the previous section. It will require $O(n^5 \log n)$ basic operations in total. This complexity can be reduced quite significantly by observing that for $p_{j-1} < p < p_j$ and $p = p_j$ the relative positions of $D(a_i; \rho_i(p))$, $i = 1, 2, ..., n$, differ only slightly.

In total there can be at most $n(n+1)/2$ intersection points and centers for $C(a_i; \rho_i(p))$, $i = 1, 2, ..., n$, for all p values. Let $K = n(n+1)/2$. Construct now an $n \times K$ matrix $W = (w^1, w^2, ..., w^K)$. Initially, let $j = 0$ and let

$$w_i^k := \begin{cases} 1, & \text{if this point is contained in } D(a_i; \rho_i(p_j)), \\ 0, & \text{if this point is not contained in } D(a_i; \rho_i(p_j)), \\ \#, & \text{if this point does not exist.} \end{cases}$$

The construction of this initial matrix costs $O(n^3)$ operations.

At the same time we also keep record of the value (scaled by p) by a K-dimensional vector v with

$$v_k = \sum_{\{i | w_i^k = 1\}} Q_i, \text{ for } k = 1, 2, ..., K.$$

Next, we compute all the critical p values and sort these values in increasing order. This process takes $O(n^3 \log n)$ basic operations according to our model.

In the process of calculating each p_j, $j = 0, 1, 2, ..., m$, we keep the information about which intersection point is appearing, or is disappearing, or is touching another circle, at this value of p. This means that from p_{j-1} to p_j it takes only constant time to update W, and consequently it takes constant time to update v. Now starting from p_0, we move from p_0 to p_1, p_1 to p_2, and so on. Finally, we move from p_{m-1} to p_m. In this process the matrix W and the vector v are updated accordingly. This phase in fact takes only $O(m) = O(n^3)$ time. Let $v^j_{\max}$ be the maximum value in all v_k vectors at the j-th step. By keeping the highest value in the v vector during this process, we have a sequence $\{v^j_{\max} \mid j = 1, 2, ..., m\}$. Simply letting $j^* = \arg\max_{1 \leq j \leq m} p_j v^j_{\max}$ we find a solution for $\max_{p \in \Re_+} f(p)$ due to Lemma 2.

To summarize we have arrived at the following theorem:

Theorem 3. If we count finding roots of a polynomial with a fixed degree as a basic operation, then the pricing/location model can be solved in at most $O(n^3 \log n)$ basic operations.

4. Discussions

The algorithm is based on a real-number computational model. Moreover, finding roots of a polynomial with a fixed degree is counted as one basic operation. One may apply similar arguments if any other explicit form of norms are used, including the L_q-norms or polyhedral gauges. The form of r_i can be generalized as well. It is important, however, that one is able to detect when disks change their relative positions.

In a more general form we may consider a model in which several firms start competing in a given region for the same set of potential customers. It can be interesting to investigate, for instance, whether or not an equilibrium exists.

We remark that the problem we discuss is a planar one. It remains a topic for further research how to solve the problem in polynomial time as the dimension becomes more than two.

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