A Stochastic Linear Goal Programming Approach to Multistage Portfolio Management Based on Scenario Generation via Linear Programming [1]

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Abstract

In this paper, a stochastic linear goal programming model for multistage portfolio management is proposed. The model takes into account both the investment goal and risk control at each stage. A scenario generation method, as the basis of the portfolio management model, is also proposed. In particular, by matching the moments and fitting the descriptive features of the asset returns, a linear programming model is used to generate the single stage scenarios. Scenarios for multistage portfolio management are generated by incorporating this single stage method with the time series model for the asset returns. Meanwhile, no arbitrage opportunity exists in the proposed method. A real case is solved via the goal programming model and the scenario generation approach, through which the effectiveness of the model is shown. We also comment on some practical issues of the approach.

Keywords: Stochastic programming; Linear goal programming; Multistage portfolio management; No Arbitrage; Scenario generation

1 Introduction

For a long-term investment, an investor usually adjusts his/her portfolio timely with the varying environment according to his/her risk preference. This is referred to as the dynamic portfolio selection policy. Many authors have considered this problem since 1960's, *e.g.*, Dantzig and Infanger [5], Dumas and Luciano [8], Elton and Gruber [9], Fama [10], Grauer and Hakansson [12], Hakansson [13], Li and Ng [21], Merton [24, 25], Mossin [26], Östermark [30], Samuelson [31], Zhou and Li [41], etc. The dynamic portfolio selection in discrete-time, namely the multistage portfolio selection, allows an investor to adjust dynamically his/her portfolio positions at successive stages. With the development of computing techniques in recent years, the stochastic

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programming approach has claimed some success to solve the multistage portfolio management problems, *e.g.*, the asset/liability management for insurance companies (Cariño et al. [3, 4]) and pension plans (Dert [7], Mulvey [29]), portfolio management for the fixed-income bonds (Zenios et al. [39]), etc. Ziemba and Mulvey [38] surveyed the state-of-the-art of the theory and applications in this field. All these practical applications can not be solved two decades ago due to the constraints of hardware and software ([11]). We refer to [1] for a general introduction to stochastic programming.

In most cases, in dynamic portfolio problems investors are assumed to be concerned with the terminal wealth. The objectives of most existing dynamic portfolio models in the literature try to maximize the expected utility of terminal wealth. Li and Ng [21], and Zhou and Li [41] generalized the static mean-variance model of Markowitz [22, 23] to dynamic ones, and in their models the objective function is only related to the mean and variance of the terminal wealth. Using an endogenously determined worst case risk measure, Zhao and Ziemba [40] presented a stochastic linear programming model for multistage asset allocation, where the objective function is the weighted sum of a deterministic terminal target and the expected surplus over it. In [6], a stochastic linear quadratic control method (SLQ) is employed via semidefinite programming (SDP) to track a financial benchmark, a continuously compounded growth rate or a stock market index. Only by dynamically operating a portfolio of a few traded stocks, an exciting result is derived that the tracking performance in most cases is excellent even with rather infrequent data and is rather insensitive to the market volatility and stock selections.

For a long-term investment, a rational investor usually considers the long-term, medium-term and short-term goals simultaneously. Therefore the model, of which the objective function is only related to the terminal wealth, cannot completely characterize the behavior of the long-term portfolio management. Moreover, Zhu et al. [42] found a phenomenon for multistage portfolio: for both utility function model and mean-variance model, even though the criteria directly related to the terminal wealth behave well (*e.g.*, the expected value is large and the variance is small for the terminal wealth), it is still possible for the wealth at intermediate stages to fluctuate drastically, which indicates that the investor may suffer bankruptcy before the end of planning horizon with a high possibility. So the portfolio selection model, which takes a measure only related to the terminal wealth as the sole goal, conceals great risk. Thus the risk control at each stage is indispensable for the long-term portfolio management. A generalized mean-variance control model was given in [42], which considers the risk control of bankruptcy at intermediate stages. However, the control model is difficult to solve when complex constraints are involved, *e.g.*, non-negative constraints for control variables.

This paper is organized as follows. A stochastic linear goal programming model for multi-stage portfolio management and its deterministic equivalence based on scenarios are proposed in Section 2. A linear programming method is introduced in Section 3 to generate the single stage scenarios based on moments matching and descriptive features fitting. In this section, an approach to detect the arbitrage opportunity is also discussed. In Section 4, a practical case is illustrated, where a multistage scenario generation method is presented by incorporating the Vector Auto-Regression (VAR) model with the single stage scenario generation method discussed in Section 3. We conclude the paper in Section 5 with some remarks.

2 Notations and modelling

Assume that there are n risky assets and a risk-free asset to be invested in a financial market. An investor plans for a T-stage investment, who constructs his/her portfolio at time 0 and adjusts the positions at successive $T-1$ time points. The investment policy is assumed to be self-financing. In the sequel, the stage t indicates the period between time t and $t+1$. To formulate the model, some notations are given as follows:

w_0: the investor's initial wealth at time 0;

r_{ft}: the return of the risk-free asset at stage t;

r_{it}: the return of the i^{th} risky asset at stage t;

x_{ft}: the amount invested in the risk-free asset at time t;

x_{it}: the amount invested in the i^{th} risky asset at time t;

P_{it}: the amount purchased of the i^{th} risky asset at time t;

Q_{it}: the amount sold of the i^{th} risky asset at time t;

α_{it}: the unit transaction cost to purchase the i^{th} risky asset at time t;

β_{it}: the unit transaction cost to sell the i^{th} risky asset at time t;

$$i = 1, 2, \ldots, n,\ t = 0, 1, \ldots, T-1;$$

w_t: the investor's wealth at time t, $t = 1, 2, \ldots, T$.

The budget constraint at time 0 is

$$x_{f0} + \sum_{i=1}^{n} (1 + \alpha_{i0})\, x_{i0} = w_0. \tag{1}$$

The balance of the i^{th} risky asset at time t is

$$x_{i,t-1} r_{i,t-1} + P_{it} - Q_{it} = x_{it} \qquad i = 1, \cdots, n, \quad t = 1, 2, \cdots, T-1. \tag{2}$$

The dynamic variation of risk-free asset at time t is

$$x_{f,t-1} r_{f,t-1} + \sum_{i=1}^{n} (1 - \beta_{it})\, Q_{it} - \sum_{i=1}^{n} (1 + \alpha_{it})\, P_{it} = x_{ft} \qquad t = 1, 2, \cdots, T-1. \tag{3}$$

The investor's wealth at time t is

$$w_t = x_{f,t-1} r_{f,t-1} + \sum_{i=1}^{n} x_{i,t-1} r_{i,t-1} \qquad t = 1, 2, \cdots, T. \tag{4}$$

Denote G_t as the goal of the investor's wealth at time t, d_t^+, d_t^- as the positive and negative deviations between the realized wealth and the target wealth G_t respectively. The equation holds as follows:

$$w_t + d_t^- - d_t^+ = G_t \qquad t = 1, 2, \cdots, T. \tag{5}$$

Furthermore, we denote

$$\begin{aligned}
x_f &= \{x_{ft},\ t = 0, 1, \cdots, T-1\} \\
x &= \{x_{it},\ i = 1, 2, \cdots, n;\ t = 0, 1, \cdots, T-1\} \\
P &= \{P_{it},\ i = 1, 2, \cdots, n;\ t = 1, 2, \cdots, T-1\} \\
Q &= \{Q_{it},\ i = 1, 2, \cdots, n;\ t = 1, 2, \cdots, T-1\} \\
w &= \{w_t,\ t = 1, 2, \cdots, T\} \\
d^+ &= \left\{d_t^+,\ t = 1, 2, \cdots, T\right\} \\
d^- &= \left\{d_t^-,\ t = 1, 2, \cdots, T\right\}.
\end{aligned}$$

For the multistage investment problem, a rational investor tries to achieve the investment goals as close as possible. More specifically, the investor tries to achieve the target wealth G_t ($t = 1, \cdots, T$), *i.e.*, to minimize the expected downside deviation from the target wealth, we propose the following portfolio management model:

$$\begin{aligned}
(GP1) \qquad \min \quad & \sum_{t=1}^{T} \lambda_t \frac{E\left(d_t^-\right)}{\prod_{i=0}^{t-1} r_{fi}} \\
s.t. \quad & \text{Eqs. } (1), (2), (3), (4) \text{ and } (5) \\
& x, P, Q, d^+, d^- \geq 0 \\
& x_f, w \text{ free},
\end{aligned}$$

where $x \geq 0$ indicates no short-sale is allowed for the risky assets, $\lambda_t \in [0, +\infty)$ is the weight measuring the risk aversion, a larger value of this parameter implies a higher risk aversion at period $t-1$. The expected value of negative deviations at each stage are discounted by risk-free rate. Clearly, the model considers not only the investment goals but also the risk control at each stage, which is necessary for the long-term investment.

The description of the future is the key in modelling and decision making under uncertainty. Scenario analysis is an effective tool to model the dynamics of the uncertainty. Denote $\{r_t, t = 0, 1, \cdots, T-1\}$ as the discrete stochastic process of the uncertain returns of n risky assets at T stages, where r_t is an n-dimension random vector. Assume that the state space of stochastic process is discrete and finite, any realization $\left\{r_0^s, r_1^s, \cdots, r_{T-1}^s\right\}$ is called a "scenario", which is only one of the possible state of the world in future.

Figure 1 is a two-stage scenario tree. A scenario tree consists of nodes and branches. A_0, the initial node, has three possible states at time 1, *i.e.*, B_1, B_2, B_3 with probability p^1, p^2, p^3 ($p^1 + p^2 + p^3 = 1$) respectively. B_1 has two successors at time 2, *i.e.*, C_{11}, C_{12} with probability

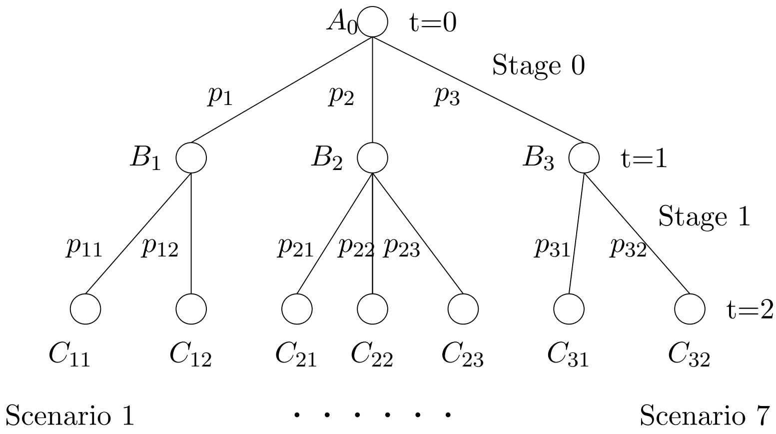

Figure 1: Two-stage scenario tree

p^{11}, p^{12} ($p^{11} + p^{12} = 1$) respectively. In a scenario tree, any intermediate node has multiple successors but a unique predecessor (no predecessor for the root node and no successors for the leaf nodes). Any path in the scenario tree shows a scenario, *e.g.*, $A_0 - B_1 - C_{11}$ as scenario 1 with probability $p^1 p^{11}$. A detailed discussion on scenario generation is given in the next section.

For simplicity, the symmetrical scenario tree is usually used in practice. If there are s_t possible states for each node at stage t with probability constraint $\sum_{s=1}^{s_t} p^s = 1 (p^s \geq 0)$, there will be $S_t = \prod_{i=0}^{t} s_i$ scenarios at stage t. Now the deterministic equivalence of $(GP1)$ can be formulated as follows:

$$
\begin{aligned}
(GP2) \qquad & \min \quad \sum_{t=1}^{T} \lambda_t \frac{\sum\limits_{s=1}^{S_t} p_t^s d_t^{-s}}{\prod_{i=0}^{t-1} r_{fi}} \\
& s.t. \quad x_{f0} + \sum_{i=1}^{n} (1+\alpha_{i0})\, x_{i0} = w_0 \\
& \qquad x_{i,t-1}^{\hat{s}} r_{i,t-1}^{\hat{s}} + P_{it}^s - Q_{it}^s = x_{it}^s \\
& \qquad\qquad i = 1, 2, \cdots, n, \quad t = 1, 2, \cdots, T-1, \quad s = 1, 2, \cdots, S_t \\
& \qquad x_{f,t-1}^{\hat{s}} r_{f,t-1} + \sum_{i=1}^{n} (1-\beta_{it})\, Q_{it}^s - \sum_{i=1}^{n} (1+\alpha_{it})\, P_{it}^s = x_{ft}^s \\
& \qquad\qquad t = 1, 2, \cdots, T-1, \quad s = 1, 2, \cdots, S_t \\
& \qquad x_{f,t-1}^{\hat{s}} r_{f,t-1} + \sum_{i=1}^{n} x_{i,t-1}^{\hat{s}} r_{i,t-1}^{\hat{s}} + d_t^{-s} - d_t^{+s} = G_t \\
& \qquad\qquad t = 1, 2, \cdots, T, \quad s = 1, 2, \cdots, S_t \\
& \qquad x_{it}^s, P_{it}^s, Q_{it}^s, d_t^{-s}, d_t^{+s} \geq 0 \\
& \qquad x_{ft}^s \text{ free}
\end{aligned}
$$

Remarks:

(a) The size of the model expands exponentially with the numbers of stages and scenarios. Generally, it is a large-scale sparse linear programming problem. The more the stages and scenarios are, the sparser the structure of the model is.

(b) Denote P_{it}^{s*}, Q_{it}^{s*} as the optimal transaction amounts for asset i at time t under state s, then $P_{it}^{s*} Q_{it}^{s*} = 0$ holds due to the existence of transaction cost, which means that purchasing and selling will not happen simultaneously for the same asset. Otherwise, the additional transaction cost will improve the objective value if it is invested in the risk-free asset.

(c) If the initial wealth w_0 and the goal G_t $(t = 1, 2, \cdots, T)$ vary in the same proportion, the optimal solution will varies in the same proportion. Therefore, the model can be usually treated with $w_0 = 1$.

(d) The risk preference is embodied not only by λ_t but also by G_t. Therefore, the risk control can be implemented by setting proper values for these two types of parameters.

(e) This model can be extended, *e.g.*, appending the upper bound constraints to the holding volume, etc.

3 Scenario generation

Scenario generation is a key step in financial modelling by stochastic programming. The number and structure of scenarios are directly related to the complexity and reliability of the model. The dynamics of asset returns are usually depicted by Vector Auto-Regression Model (VAR) [2, 7, 19, 40], Binomial Tree Model [39] and the hybrid factor model [4]. Generating scenarios with these models by Monte Carlo simulation is widely adopted in practice. Cariño et al. [4] presented a simple method to generate scenarios referred to as "variance-adjustment method", and they also pointed out that the subjective judgement of the decision maker is of much use in scenario generation. Mulvey [29] designed an integrated system named CAP:Link to generate scenarios for asset/liability management for pension funds.

In a different way, Kouwenberg and Vorst [18] employed two systems of equations (linear and non-linear) to describe the statistical features (mean and covariance) of asset returns and the no-arbitrage conditions, and then determined the scenarios by recursively solving these two systems. More generally, Høyland and Wallace [16] proposed an optimization method to generate scenarios that satisfy some specified statistical features. In Høyland and Wallace's model, both the outcomes and the associated probabilities are treated as decision variables, which may allow of more freedom to choose a suitable scenario tree. However, it results in a nonconvex and nonlinear programming problem. So it is difficult to derive "accurate" solutions from the model.

From our experience to determine the scenarios by fitting the specified statistical features, we find that it is not enough to consider only the quantitative statistical features. For example, the "normal-like" unimodal distribution of the return of an asset is often discretized to a multimodal distribution if only the quantitative statistical features are considered. The underlying reason is that a multimodal distribution may have the same quantitative statistical features as a "normal-like" unimodal distribution. This will be illustrated in the sequel. Therefore, to generate reasonable scenarios by incorporating the descriptive features, such as "normal-like" unimodal, is an important task. Furthermore, the methodology of discretizing continuous distribution by fitting statistical features has been discussed in the literature ([27, 35], etc). See [16] for references.

In this section, we propose a linear programming approach to generate scenarios by predetermining the outcomes of returns of assets, where the quantitative features as well as descriptive features are considered. For simplicity, a single stage scenario generation method is discussed in details. If the returns of assets between different stages are uncorrelated, the multistage scenarios can be easily generated step by step with the single stage method. Otherwise, the approach given below should be adjusted, which will be illustrated by means of a practical case in Section 4. The approach to generate the single stage scenarios is as follows:

Algorithm 1

1) *Determine the distribution space of returns of assets according to historical data, and then partition it into some sub-spaces. Pick out a point in each sub-space as a possible outcome of returns of assets;*

2) *Detect no-arbitrage between outcomes. Go back 1) for new partition if arbitrage opportunity exists;*

3) *Select statistical features as fitting objectives, an optimization model is solved to determine the associated probability of each outcome.*

The following notations are used to transfer the algorithm of single stage scenario generation into an optimization model.

S: the number of total scenarios;

$\bar{r}$: the expected return vector of risky assets;

Σ: the covariance matrix of risky assets;

M_3: the third-order center moment vector of risky assets;

M_4: the fourth-order center moment vector of risky assets.

R^s: the s^{th} outcome vector of risky assets, $s = 1, \cdots, S$. Denote $R = \left(R^1, R^2, \cdots, R^S\right)$;

p^s: the probability of the s^{th} outcome vector of risky assets, $s = 1, \cdots, S$. Denote $p = \left(p^1, p^2, \cdots, p^S\right)'$.

Generally, it is enough to use the first four moments as the statistical features to be matched in scenario generation. The absolute deviations are employed to measure the approximation degree of the matching. The optimization model of the moment matching is as follows:

$$
\begin{aligned}
(MM) \qquad \min \qquad & \sum_{i=1}^{n} \mu_i^0 \left(\bar{r}_i^- + \bar{r}_i^+\right) + \sum_{i,j=1}^{n} \mu_{ij}^1 \left(\Sigma_{ij}^- + \Sigma_{ij}^+\right) \\
& + \sum_{i=1}^{n} \mu_i^2 \left(M_{3i}^- + M_{3i}^+\right) + \sum_{i=1}^{n} \mu_i^3 \left(M_{4i}^- + M_{4i}^+\right) \\
s.t. \qquad & Rp + \bar{r}^- - \bar{r}^+ = \bar{r} \\
& \sum_{s=1}^{S} \left(R^s - Rp\right)\left(R^s - Rp\right)' p^s + \Sigma^- - \Sigma^+ = \Sigma \\
& \sum_{s=1}^{S} \left(R^s - Rp\right)^3 p^s + M_3^- - M_3^+ = M_3 \\
& \sum_{s=1}^{S} \left(R^s - Rp\right)^4 p^s + M_4^- - M_4^+ = M_4 \\
& \sum_{s=1}^{S} p^s = 1 \\
& \bar{r}_i^+,\ \bar{r}_i^-,\ \Sigma_{ij}^+,\ \Sigma_{ij}^-,\ M_{3i}^+,\ M_{3i}^-,\ M_{4i}^+,\ M_{4i}^- \geq 0, \qquad i,j = 1,\cdots,n \\
& p^s \geq 0, \qquad s = 1,\cdots,S
\end{aligned}
$$

where $\mu_i^0, \mu_{ij}^1, \mu_i^2, \mu_i^3$ $(i,j = 1,\cdots,n)$ are given weights, $\bar{r}_i^+$ and $\bar{r}_i^-$ are respectively positive and negative deviations from the expected return vector $\bar{r}$ (analogous for other deviation variables), the 3 and 4-power operations are defined on the elements of the vector.

(MM) is essentially a linear programming model, as we shall see later.This allows for more efficient computation if proper restrictions are appended to the outcomes.

3.1 Partition of the distribution space

It can be seen that the first step in Algorithm 1 plays an important role in the realization of scenario generation. The quality of the predetermined outcomes directly influences the description of the future uncertainty. So it is an important issue of how to properly confine the return distribution space and partition it into some sub-spaces. The approach adopted in this paper will be introduced in this section.

According to historical data, each risky asset has an approximate return interval. These n return intervals are incorporated into an n-dimension space to confine the outcomes of scenarios. For the $i^{th}(i = 1,2,\cdots,n)$ risky asset, partition its return interval into m_i sub-intervals, pick out one point $x^{ij}(j = 1,2,\cdots,m_i)$ in each sub-interval as a possible outcome of the i^{th} risky asset. Then select one point from each asset's outcomes and incorporate them into an n-dimension return vector $\left(x^{1j}, x^{2k}, \cdots, x^{nl}\right)'$, which is used as a possible outcome of n risky assets. Thus there will be $m_1 m_2 \cdots m_n$ possible outcomes to describe the future possibilities. The return interval of risky asset i can be reasonably defined as $[\bar{r}_i - 3\sqrt{\Sigma_{ii}},\ \bar{r}_i + 3\sqrt{\Sigma_{ii}}]$, since the return distribution of risky asset is always approximately normal.

As can be seen from (MM), if only the first equation of the constraints holds with zero deviations, *i.e.*, $Rp = \bar{r}$, then (MM) can be simplified into the following linear programming model:

$$(LP1) \qquad \min \quad \sum_{i,j=1}^{n} \mu_{ij}^1 \left(\Sigma_{ij}^- + \Sigma_{ij}^+\right) + \sum_{i=1}^{n} \mu_i^2 \left(M_{3i}^- + M_{3i}^+\right) + \sum_{i=1}^{n} \mu_i^3 \left(M_{4i}^- + M_{4i}^+\right)$$

$$s.t. \quad Rp = \bar{r} \tag{6}$$

$$\sum_{s=1}^{S} (R^s - \bar{r})(R^s - \bar{r})' p^s + \Sigma^- - \Sigma^+ = \Sigma \tag{7}$$

$$\sum_{s=1}^{S} (R^s - \bar{r})^3 p^s + M_3^- - M_3^+ = M_3 \tag{8}$$

$$\sum_{s=1}^{S} (R^s - \bar{r})^4 p^s + M_4^- - M_4^+ = M_4 \tag{9}$$

$$\sum_{s=1}^{S} p^s = 1 \tag{10}$$

$$\Sigma_{ij}^+,\ \Sigma_{ij}^-,\ M_{3i}^+,\ M_{3i}^-,\ M_{4i}^+,\ M_{4i}^- \geq 0 \qquad i,j = 1,\cdots,n \tag{11}$$

$$p^s \geq 0, \qquad s = 1,\cdots,S \tag{12}$$

Proposition 1: *Provided that a couple of vectors of outcomes are symmetrical to the expected return vector, then there exists a solution to* $(LP1)$.

Proof: $(LP1)$ has a solution iff the following linear equations have a non-negative solution.

$$\begin{cases} Rp = \bar{r} \\ \sum_{s=1}^{S} p^s = 1 \end{cases}$$

Without loss of generality, suppose R^1 and R^2 are two symmetrical outcomes with respect to the expected return vector $\bar{r}$. Clearly, $p' = (0.5, 0.5, 0, \cdots, 0)$ is a non-negative solution for the above linear equations ■

For simplicity, equidistant partition is used and the midpoint of each sub-interval is selected as a possible outcome of an asset in this paper. Obviously, this partition method guarantees the correct application of model $(LP1)$.

The scale of the portfolio management model $(GP2)$ is determined by the number of scenarios. Thus a dense partition will result in a large scale problem. The denser the partition, the larger the size of the model. Fortunately, our experience shows that the moments can be well matched with a sparse partition. Moreover, an important feature of this approach is that the extreme cases, *i.e.*, the best and the worst outcomes, are always included in the scenarios, which is important for the portfolio analysis.

Table 1: Comparisons between statistical features of scenarios generated by $(LP1)$ and $(LP2)$ under (8,8,8)-partition (each '8' denotes a number of partitioned sub-intervals of an asset return interval)

features	historical data	scenario by $(LP1)$	scenario by $(LP2)$
mean	(1.0022 1.0083 1.0103)	(1.0022 1.0083 1.0103)	(1.0022 1.0083 1.0103)
covariance	$\begin{pmatrix} 0.0074 & 0.0051 & 0.0052 \\ 0.0051 & 0.0053 & 0.0055 \\ 0.0052 & 0.0055 & 0.0065 \end{pmatrix}$	$\begin{pmatrix} 0.0074 & 0.0051 & 0.0052 \\ 0.0051 & 0.0053 & 0.0055 \\ 0.0052 & 0.0055 & 0.0065 \end{pmatrix}$	$\begin{pmatrix} 0.0074 & 0.0051 & 0.0052 \\ 0.0051 & 0.0059 & 0.0048 \\ 0.0052 & 0.0048 & 0.0065 \end{pmatrix}$
skewness	(0.4706 0.6461 0.5538)	(0.4706 0.6461 0.5538)	(0.4708 0.5570 0.5539)
kurtosis	(2.8926 3.4140 3.1929)	(2.8926 3.4140 3.1929)	(2.8927 3.0429 3.0377)

It should be mentioned that the partition method given above is unsuitable for generating too few scenarios. In multistage portfolio management, however, a few number of scenarios are possible for some stages due to the "curse of dimension". It is hard to reflect the variation and correlation of the uncertain returns by the above partition method if only a few scenarios are to be generated. Instead, it is better to select points in the diagonals of the multi-dimension space confined by return intervals as possible outcomes. This method is adopted in this paper.

3.2 The influence of descriptive features

A practical example is introduced in this section. We choose three risky assets from Shanghai Stock Exchange (SSE): real estate index (ReE_INDEX), industry index (IND_INDEX) and business index (BUS_INDEX). The historical data from Feb. 1990 till Dec. 2002 can be found from http://www.stockstar.com. Some statistical features from the historical data are given in Table 1.

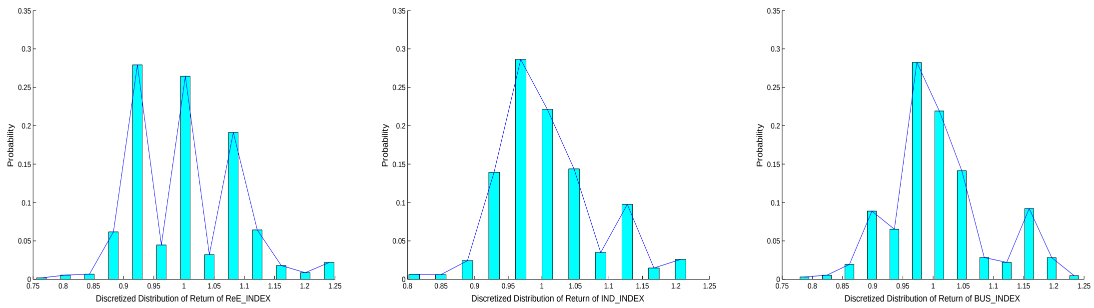

Figure 2: Marginal distributions of scenarios by $(LP1)$ under (13,11,13)-partition

It can be seen from Table 1 that the quantitative statistical features (estimated by the monthly historical data) are perfectly matched by model $(LP1)$. However, the associated scenarios always distribute multimodally (see Figure 2). This contradicts the fact that the distribution of the return of risk asset is approximately normal, and may result in unreasonable decisions, especially to those investors who use probability criteria. Thus the descriptive feature of "normal-like" unimodal should be considered in $(LP1)$. This can be done by assigning lager probabilities for the outcomes near the expected return, whereas the smaller probabilities for those far from the expected return. For example, fix the outcomes of the second till the n^{th} risky assets, incorporate them with m_1[5] possible outcomes of the first asset to comprise m_1 outcomes. Denote $p^1, p^2, \cdots, p^{m_1}$ as the associated probabilities of those m_1 scenarios respectively, then the descriptive feature of "normal-like" unimodal can be formulated as follows:

$$\begin{cases} p^1 \le p^2 \le \cdots \le p^{\frac{m_1+1}{2}-1} \le p^{\frac{m_1+1}{2}} \ge p^{\frac{m_1+1}{2}+1} \ge \cdots \ge p^{m_1-1} \ge p^{m_1}, & \text{if } m_1 \text{ is odd} \\ p^1 \le p^2 \le \cdots \le p^{\frac{m_1}{2}-1} \le p^{\frac{m_1}{2}};\ p^{\frac{m_1}{2}+1} \ge p^{\frac{m_1}{2}+2} \ge \cdots \ge p^{m_1-1} \ge p^{m_1}, & \text{else} \end{cases}$$

These constraints are named as the "*Descriptive Constraints*" in the paper. Thus $(LP1)$ can be revised as follows:

$$\begin{array}{lll} (LP2) & \min & \sum_{i,j=1}^{n} \mu_{ij}^1 \left(\Sigma_{ij}^- + \Sigma_{ij}^+ \right) + \sum_{i=1}^{n} \mu_i^2 \left(M_{3i}^- + M_{3i}^+ \right) + \sum_{i=1}^{n} \mu_i^3 \left(M_{4i}^- + M_{4i}^+ \right) \\ & \text{s.t.} & (6) - (12) \ \ \textit{and Discriptive Constraints.} \end{array}$$

It is easy to verify (as Proposition 1) that there exists a solution to $(LP2)$. Scenarios generated by $(LP2)$ well characterize the "normal-like" unimodal feature (see Figure 3). Of course, the matching accuracy of the higher moments is a little influenced (see Table 1). Experiments show us an interesting phenomenon that $(LP1)$ allocates non-zero probabilities for almost all the outcomes predetermined by the partition, but much less for $(LP2)$ (about one-fourth of the former). Thus, $(LP2)$ efficiently reduces the size of the problem.

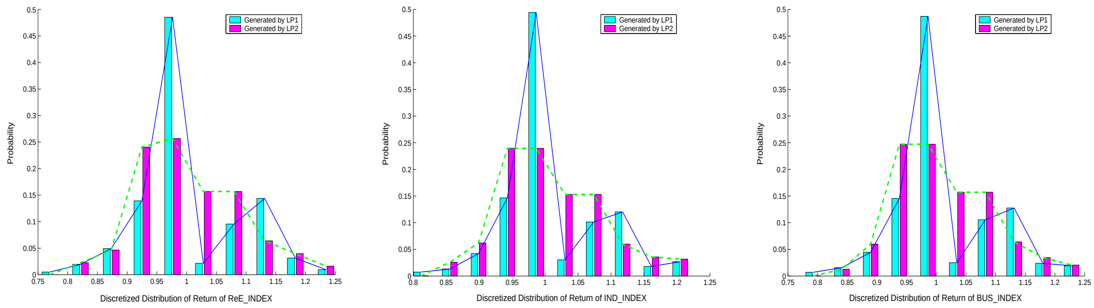

Figure 3: Marginal distributions of scenarios under (10,10,10)-partition

Another phenomenon is observed that the matching deviations may concentrate on part of the specified moments. A feasible method to overcome this drawback is to set different weights to avoid asymmetrical deviations from the moments. Searching algorithm, such as the simulated annealing, is employed to adaptively adjust the weights, which proves to be of some improvement, but at the cost of heavy operations. However, sometimes it deserves.

[5]Without loss of generality, m_1 possible outcomes are assumed to be sorted from the smallest to the biggest.

3.3 Detecting the arbitrage opportunity

Arbitrage means that, for any of the future states, the current non-positive investment yields non-negative payoff and a strictly positive payoff in at least one state. It is well known that arbitrage opportunities do not exist in any efficient and equilibrium market, which is consistent with the asset pricing theory. So the scenarios generated by the model are required to be arbitrage-free. In this paper, the arbitrage opportunity is detected by solving the linear programming problem below:

$$
\begin{aligned}
(LP\_AT) \qquad \max \quad & \sum_{s=1}^{S}\left(\sum_{i=1}^{n} r_i^s x_i + r_f x_f\right) \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_i + x_f \le 0 \\
& \sum_{i=1}^{n} r_i^s x_i + r_f x_f \ge 0, \qquad\qquad s = 1, \cdots, S \\
& \sum_{s=1}^{S}\left(\sum_{i=1}^{n} r_i^s x_i + r_f x_f\right) \le 1.
\end{aligned}
$$

According to the definition of arbitrage, the optimal value of (LP_AT) is 0 for no-arbitrage and 1 for arbitrage. New partition will be performed if arbitrage opportunity occurs.

Klassen [17] gave an example to construct an arbitrage opportunity in scenarios generated by Høyland and Wallace's [16] method. By our approach, experiences show that arbitrage opportunity is always precluded when the partitioned return sub-interval of each asset is no less than 2. Harrion and Krep [15] addressed the necessary and sufficient condition for the absence of arbitrage opportunities: if there exists a risk-neutral probability measure such that the expected return is identical for all assets, and identical to the risk-free return r_f if there exists one. In other words, if r_f lies in the convex hull of the outcomes determined by the scenario generation method, the arbitrage opportunity will be precluded. Because r_f is an interior point of interval $[\bar{r}_i - 3\sqrt{\Sigma_{ii}},\ \bar{r}_i + 3\sqrt{\Sigma_{ii}}](i = 1, 2, \cdots, n)$ (otherwise, the interval should be enlarged), the no-arbitrage condition is guaranteed by the symmetrically predetermined possible outcomes.

4 A practical case

In this section, a practical case is presented to illustrate the portfolio management model. The above single stage scenario generation method is incorporated with the VAR model to generate multistage scenarios. Three indexes, Hang Seng commercial and industrial index (IN_INDEX), utilities index (UT_INDEX), and finance index (FI_INDEX), are selected to construct the index portfolio. Figure 4 exhibits the monthly returns from Jan.1990 till Jul.2003. For simplicity, the risk-free return is assumed to be a constant.

In Section 3, we proposed a linear programming approach to generate a single stage scenarios. For a multistage investment problem, however, it is not reasonable to assume that the returns

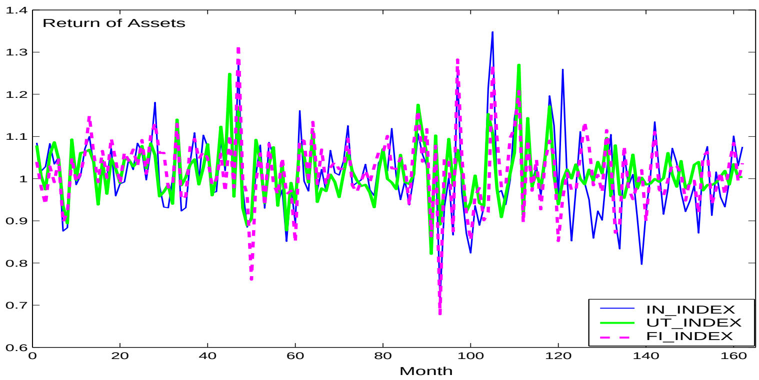

Figure 4: The monthly returns of three risky assets (1990.1-2003.7)

are independently distributed between different stages for all the problems. It is necessary to consider the correlations of returns between stages. The VAR model is just for this end to describe the linear relationships between the values at a certain time point and its lagged time points. Any stationary time series process can be well modelled by a VAR process. Sims [34] argued that the times series by means of VAR model is more suitable than econometric model in economic data modelling. So the VAR model and the above single stage scenario generation method are incorporated to generate multistage scenarios in this section.

Generally, a l order VAR model for time series $\{R_t\}$ of asset returns are given as

$$R_t = \sum_{i=1}^{l} \Theta_{t-i} R_{t-i} + \epsilon_t, \qquad \epsilon_t \quad \text{i.i.d.}$$

The conditional distribution of R_t can be easily specified by the above VAR model provided that $R_{t-i}(i = 1, \cdots, l)$ are fixed. By using the previous single stage scenario generation method to discretize the conditional distribution of R_t, we can generate the multistage scenarios of the asset returns that follow the above VAR model stage by stage. The reader can refer to [14] for details on VAR.

By statistical test, the following second order VAR model is employed to describe the dynamics of return vector $R_t = (R_{t1}, R_{t2}, R_{t3})'$ of those three risky assets[6]:

$$R_t = \begin{pmatrix} 0.8636 \\ 1.0561 \\ 0.6798 \end{pmatrix} + \begin{pmatrix} 0.4155 & 0 & -0.4396 \\ 0.1660 & 0 & -0.2326 \\ 0.2132 & 0 & -0.1725 \end{pmatrix} R_{t-1} + \begin{pmatrix} -0.2545 & 0.4231 & 0 \\ -0.1645 & 0.1900 & 0 \\ -0.2346 & 0.5278 & 0 \end{pmatrix} R_{t-2} + \epsilon_t$$

$$\epsilon_t \sim N(\mathbf{0}, \Sigma) \quad \text{i.i.d}, \qquad \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 0.0078 & 0.0035 & 0.0049 \\ 0.0035 & 0.0040 & 0.0030 \\ 0.0049 & 0.0030 & 0.0061 \end{pmatrix}$$

where the regression coefficients are estimated by the historical data, which are statistically significant at level 5% except two at level 10%.

[6] R_{t1}, R_{t2}, and R_{t3} denote the returns of the three risky assets: IN_INDEX, UT_INDEX, and FI_INDEX.

Table 2: Comparisons between criteria under different λ's with scenario tree (95,7,3) and $G = (1.05, 1.1, 1.2)$

(A)	λ^1= (1,1,1)			λ^2= (1,5,10)			λ^3= (1,5,15)		
	$t=1$	$t=2$	$t=3$	$t=1$	$t=2$	$t=3$	$t=1$	$t=2$	$t=3$
ESG	0.0434	0.0819	0.1209	0.0460	0.0843	0.1167	0.0460	0.0865	0.1158
PSG	0.8456	0.5141	0.6733	0.8456	0.4974	0.6199	0.8456	0.5068	0.6079
EW	1.0069	1.0090	1.0911	1.0099	1.0125	1.1059	1.0099	1.0120	1.1210
ESEW	0.0120	0.0428	0.0636	0.0172	0.0482	0.0716	0.0172	0.0493	0.0806
PSEW	0.8456	0.4800	0.5110	0.8042	0.4703	0.6030	0.8042	0.4816	0.5917
(B)	λ^1= (1,1,1)			λ^2= (10,5,1)			λ^3= (15,5,1)		
	$t=1$	$t=2$	$t=3$	$t=1$	$t=2$	$t=3$	$t=1$	$t=2$	$t=3$
ESG	0.0434	0.0819	0.1209	0.0415	0.0798	0.1285	0.0412	0.0806	0.1289
PSG	0.8456	0.5141	0.6733	0.8828	0.8750	0.6791	0.9019	0.8899	0.6794
EW	1.0069	1.0090	1.0911	1.0046	1.0089	1.0671	1.0041	1.0080	1.0663
ESEW	0.0120	0.0428	0.0636	0.0084	0.0152	0.0537	0.0077	0.0141	0.0535
PSEW	0.8456	0.4800	0.5110	0.8085	0.7682	0.6077	0.7946	0.7689	0.6129

In our experiments, we take one month for each stage, 1.0057 for risk-free return, 0.001 for unit transaction cost and 1 for initial wealth. The first two starting states of returns are $(1.1208, 1.0753, 1.0366)'$ and $(0.9999, 1.0304, 0.9945)'$. Much valuable information can be feeded back to the investors by solving model $(GP2)$, such as the expected shortfall with respect to (w.r.t) goal (ESG), the associated probability of shortfall w.r.t goal (PSG), the expected wealth (EW), the expected shortfall w.r.t expected wealth (ESEW) and the associated probability of shortfall w.r.t expected wealth (PSEW), etc. Table 2[7] exhibits the partial results of a 3-stage portfolio problem, where the values at each stage are discounted by the risk-free return[8].

The problem is solved by linear programming solvers LINDO 6.1 (Schrage [32]) and COPL 1.0 (Ye [37]) on a PC (CPU Pentium 1.1G Hz, RAM 256M). The model $(GP2)$ is usually a large-scale sparse linear programming problem. Figure 5 illustrates the structure of the non-zero elements of a 3-stage model of 3 risky assets with scenario tree $(12, 7, 3)$. The size of the constraint coefficient matrix is 733×1660, and the non-zero elements approximate to 0.39%. For a 3-stage model of 3 risky assets with scenario tree (s_1, s_2, s_3), there will be $1+5s_1+5s_1s_2+s_1s_2s_3$ constraints and $4 + 12s_1 + 12s_1s_2 + 2s_1s_2s_3$ variables. The larger the model size is, the sparser the structure will be.

It shows in Table 2 that there are 15 evaluation criteria (5 for each stage) for a 3-stage

[7]The notation "scenario tree (s_1, s_2, s_3)" denotes that there are s_t $(t = 1, 2, 3)$ branches (or successors) at each node of stage $t - 1$.

[8]In Table 2 - 6, for the convenience of comparison, all the values except the probabilities at each stage are discounted by the risk-free return with respect to time 0.

Figure 5: The non-zero structure of 3-stage model ($GP2$) of 3 risky assets with scenario tree (12,7,3)

portfolio problem, of course some of them may be omitted and other valuable criteria can be considered, *e.g.*, the expected surplus w.r.t goal and the associated probability, etc. Any group of the criteria in Table 2 can not dominate others in the sense of pareto optimality. The investor can select the "optimal" portfolio policy according to his/her preference via multi-criteria decision method. An important information from Table 2 is that the weight λ has a great impact on the results. Different investors emphasize different stages, and thus induce different results. For example, provided that the investor considers much of the terminal stage (Table 2 (A)), the expected shortfall w.r.t. the goal (ESG) decreases with the increasing weight at the terminal stage, *i.e.*, $1 < 10 < 15$ induces $0.1209 > 0.1167 > 0.1158$. Analogously for the first stage (Table 2 (B)), *i.e.*, $1 < 10 < 15$ induces $0.434 > 0.415 > 0.412$.

The investment goals also have much influence on portfolio decision. See Table 3 for the comparisons between different criteria under different target wealth. One of the important aspects is that the higher the investor's goal, the bigger the expected wealth, and of course the lager the expected shortfall w.r.t. the goal. It is clear that the investor's risk preference is embodied not only by the weight λ but also by the goal G.

Table 4 is a special case, where the investor only emphasizes the terminal wealth. From the investment policy derived from solving model ($GP2$), the investor tries to invest his/her initial wealth in the asset with the highest expected return to pursue higher expected terminal wealth, which distinguishes the wealth-decentralized investment policy. An important information is exhibited in Table 4 that the expected wealth at the intermediate stage is less than others, which implies a big expected loss of wealth at the intermediate stage. If the risks at intermediate stages are not controlled for a multistage portfolio problem, the investor will be confronted with the danger of bankruptcy before the end of planning horizon. Zhu et al. [42] once pointed out this phenomenon when the objective is a utility function or a mean-variance quadratic function. Therefore the portfolio model, which takes the measure of terminal wealth as the sole goal, conceals great risk. The risk control at each stage is indispensable for a long-term portfolio management problem.

The asset return distribution is approximately continuous. The more scenarios, the more

Table 3: Comparisons between criteria under different target wealths with scenario tree (95,7,3) and λ=(15,5,1)

(A)	G= (1.03, 1.1, 1.2)			G=(1.04,1.1,1.2)			G=(1.05,1.1,1.2)		
	$t=1$	$t=2$	$t=3$	$t=1$	$t=2$	$t=3$	$t=1$	$t=2$	$t=3$
ESG	0.0226	0.0824	0.1298	0.0319	0.0815	0.1291	0.0412	0.0806	0.1289
PSG	0.9019	0.8878	0.6802	0.9019	0.8879	0.6779	0.9019	0.8899	0.6794
EW	1.0023	1.0062	1.0811	1.0032	1.0072	1.0680	1.0041	1.0080	1.0663
ESEW	0.0042	0.0122	0.0635	0.0060	0.0130	0.0548	0.0077	0.0141	0.0535
PSEW	0.7946	0.7897	0.6215	0.7946	0.7821	0.6164	0.7946	0.7689	0.6129
(B)	G= (1.03, 1.2, 1.5)			G= (1.04,1.2,1.5)			G=(1.05,1.2,1.5)		
	$t=1$	$t=2$	$t=3$	$t=1$	$t=2$	$t=3$	$t=1$	$t=2$	$t=3$
ESG	0.0227	0.1776	0.3448	0.0321	0.1766	0.3436	0.0412	0.1762	0.3438
PSG	0.8828	0.8894	0.6805	0.9019	0.8836	0.6780	0.9019	0.8904	0.6787
EW	1.0024	1.0113	1.1747	1.0034	1.0126	1.2147	1.0041	1.0130	1.1873
ESEW	0.0044	0.0238	0.1431	0.0061	0.0259	0.1689	0.0077	0.0255	0.1506
PSEW	0.8085	0.7967	0.6173	0.7946	0.7971	0.6655	0.7946	0.7895	0.6600

Table 4: Comparisons between criteria under different target wealths with scenario tree (95,7,3) and λ=(0,0,1)

(A)	G= (1.03, 1.1, 1.2)			G= (1.04,1.1,1.2)			G=(1.05,1.1,1.2)		
	$t=1$	$t=2$	$t=3$	$t=1$	$t=2$	$t=3$	$t=1$	$t=2$	$t=3$
ESG	0.0305	0.1262	0.1109	0.0389	0.1266	0.1109	0.0474	0.1261	0.1110
PSG	0.8456	0.4998	0.3110	0.8456	0.4955	0.3092	0.8456	0.4914	0.3102
EW	1.0114	1.0066	1.1076	1.0114	1.0062	1.0936	1.0114	1.0068	1.0956
ESEW	0.0197	0.0870	0.0889	0.0197	0.0872	0.0849	0.0197	0.0871	0.0854
PSEW	0.8456	0.4201	0.2919	0.8456	0.4232	0.2890	0.8456	0.4336	0.2939
(B)	G= (1.03, 1.2, 1.5)			G= (1.04,1.2,1.5)			G=(1.05,1.2,1.5)		
	$t=1$	$t=2$	$t=3$	$t=1$	$t=2$	$t=3$	$t=1$	$t=2$	$t=3$
ESG	0.0305	0.3143	0.3049	0.0389	0.3109	0.3049	0.0474	0.3121	0.3052
PSG	0.8456	0.5436	0.3298	0.8456	0.5465	0.3326	0.8456	0.5392	0.3349
EW	1.0114	0.9974	1.2357	1.0114	0.9994	1.2930	1.0114	0.9981	1.2536
ESEW	0.0197	0.2166	0.2267	0.0197	0.2134	0.2448	0.0197	0.2155	0.2319
PSEW	0.8456	0.4558	0.3214	0.8456	0.4554	0.3274	0.8456	0.4631	0.3240

Table 5: Stability of the criteria of the first stage with G=(1.05,1.1,1.2) and λ=(15,5,1)

scenario tree	ESG			PSG			EW		
(s_1, s_2, s_3)	t=1	t=2	t=3	t=1	t=2	t=3	t=1	t=2	t=3
(8,7,3)	0.0401	0.0741	0.1062	0.5000	0.7497	0.5092	1.0039	1.0188	1.5979
(12,7,3)	0.0408	0.0797	0.1136	0.8346	0.8546	0.5346	1.0046	1.0096	1.1851
(15,7,3)	0.0412	0.0784	0.1262	0.8137	0.8550	0.6529	1.0053	1.0111	1.1081
(23,7,3)	0.0413	0.0784	0.1253	0.8011	0.8788	0.6418	1.0053	1.0106	1.1060
(53,7,3)	0.0413	0.0803	0.1287	0.8759	0.9009	0.6779	1.0044	1.0079	1.0986
(71,7,3)	0.0414	0.0801	0.1291	0.8603	0.8877	0.6693	1.0045	1.0085	1.1539
(89,7,3)	0.0414	0.0799	0.1289	0.8603	0.8833	0.6603	1.0045	1.0087	1.0857
(95,7,3)	0.0412	0.0806	0.1289	0.9019	0.8899	0.6794	1.0041	1.0080	1.0663
(119,7,3)	0.0413	0.0806	0.1290	0.8997	0.8901	0.6658	1.0041	1.0079	1.1016
(128,7,3)	0.0414	0.0807	0.1295	0.8996	0.8865	0.6831	1.0040	1.0080	1.0649
(143,7,3)	0.0413	0.0806	0.1293	0.9009	0.8813	0.6597	1.0041	1.0078	1.0849
(160,7,3)	0.0414	0.0803	0.1290	0.8971	0.8919	0.6630	1.0042	1.0083	1.1009
(179,7,3)	0.0414	0.0807	0.1304	0.8774	0.8852	0.6720	1.0043	1.0083	1.0826
(200,7,3)	0.0413	0.0802	0.1299	0.8997	0.8853	0.6610	1.0043	1.0084	1.1788

accurate of the description of the uncertainty. However, the size of the problem expands drastically with the increasing scenarios. A very important issue in successful portfolio management is how to select the suitable number and structure for the scenarios. A simple criterion for this purpose is the stability of the objective function value. The number of scenarios is viewed to be suitable if the objective function value changes little with the extra scenarios increased. Another criterion is the stability of the solutions, but it is unsuitable when there are multiple optimal solutions. Even if there is a unique solution, it is not easy to compare the investment policy under different scenarios. A compromise method is to observe the stabilities of several evaluation criteria, such as ESG, PSG, EW, ESEW and PSEW, instead of the objective function value only. It shows in Table 5 and Figure 6 that the criteria of the first stage incline to be stable with the increasing scenarios. We validate Dert's [7] experiment conclusion that 100 successors for each node suffice for satisfactory simulation results.

A successful current stage investment is the basis for performing the future investment, and the description of uncertainty of the current stage is the key to this issue. Due to the limit of the model size, a scenario tree with decreasing successors is usually employed, *e.g.*, for a 3-stage model with scenario tree (s_1, s_2, s_3), we generally require $s_1 \gg s_2 > s_3$. This strategy guarantees the adequate description of the uncertainty of the first stage. In practice, the investor should adjust his/her investment policy by rerunning the optimization model according to the realization at the beginning of each stage, which is referred to as the rolling horizon approach. An exciting result from Table 6 is that the number of successors of the scenario tree at latter stages has little impact on the evaluation criteria. Thus the model size can be efficiently controlled by

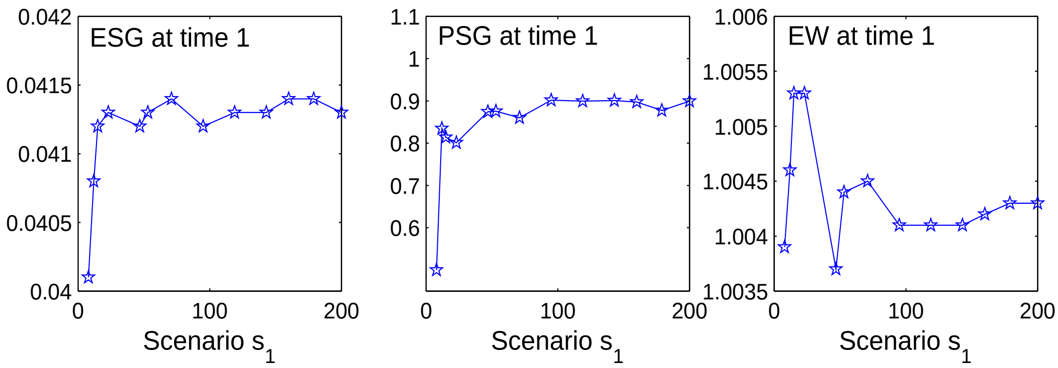

Figure 6: Part of the criteria at the end of the first stage with scenario tree (s_1,7,3)

taking less successors at latter stages.

Figure 7[9] exhibits a part of the scenario tree (12,7,3). Figure 8 exhibits the wealth allocation along two different paths according to Figure 7. The two groups of investment policy are illustrated in Table 7. We can see that the risk-free asset at the second stage of first group is negative (-0.3464), which means that the investor, if this state is realized, will borrow the risk-free asset and invest it in the risky assets for higher profit.

5 Concluding remarks

In this paper, a stochastic linear goal programming model for multistage portfolio is presented, which emphasizes the investor's goal and risk preference at each stage. To avoid the nonlinearity and non-convexity in scenario generation by moment matching, a linear programming approach is proposed based on properly predetermining the possible outcomes of scenarios. This method can accurately match the moments. However, it is evidenced in the paper that considering moment matching only can not well characterize the descriptive features, such as the shape of "normal-like" unimodal distribution. Thus the descriptive features must be considered in scenario generation, since they may result in incorrect scenario aggregation. "Descriptive constraints" are added for this purpose. It is also illustrated how to generate a scenario tree correlated between stages by incorporating the linear programming approach and the VAR model. Experiments shows that arbitrage opportunities almost never occur by our scenario generation method. It is not difficult to understand that this is guaranteed by reasonably selecting the return intervals. Therefore, arbitrage elimination is not incorporated in the scenario generation model, but as a supplement step of the scenario generation method.

A practical case is illustrated as an application of the model and scenario generation approach. By a number of experiments, we find that the evaluation criteria converge as the number of scenarios increases. One of the important findings is that the number of successors

[9] In this figure, the sign "prob" denotes the occurring probability of this branch, and the other three denote the return outcomes of the three risky assets.

Table 6: Impact of the successor number of latter stages of scenario tree with G=(1.05,1.1,1.2) and λ=(15,5,1)

scenario tree	ESG			PSG			EW		
(s_1, s_2, s_3)	t=1	t=2	t=3	t=1	t=2	t=3	t=1	t=2	t=3
(95,5,2)	0.0415	0.0788	0.1495	0.8828	0.9108	0.5626	1.0044	1.0131	1.0348
(95,5,3)	0.0415	0.0788	0.1407	0.8828	0.9112	0.7125	1.0044	1.0131	1.0574
(95,7,3)	0.0412	0.0806	0.1289	0.9019	0.8899	0.6794	1.0041	1.0080	1.0663
(95,7,5)	0.0412	0.0806	0.1334	0.9019	0.8917	0.6734	1.0041	1.0081	1.1144
(95,8,3)	0.0415	0.0780	0.1223	0.8828	0.8716	0.5501	1.0044	1.0129	1.0989
(95,8,4)	0.0415	0.0781	0.1398	0.8828	0.8708	0.5890	1.0044	1.0132	1.0757
(95,8,5)	0.0415	0.0780	0.1304	0.8828	0.8699	0.6229	1.0044	1.0131	1.0834
(95,12,3)	0.0415	0.0789	0.1382	0.8828	0.8897	0.7029	1.0044	1.0139	1.0591
(95,12,5)	0.0415	0.0781	0.1409	0.8828	0.8929	0.6908	1.0044	1.0130	1.0604
(95,12,7)	0.0415	0.0794	0.1345	0.8828	0.9030	0.6464	1.0044	1.0152	1.0591
(95,15,2)	0.0415	0.0783	0.1495	0.8818	0.9197	0.6062	1.0044	1.0132	1.0332
(95,15,3)	0.0415	0.0783	0.1392	0.8828	0.9128	0.6870	1.0044	1.0139	1.0740
(95,15,4)	0.0415	0.0790	0.1481	0.8818	0.9131	0.5655	1.0044	1.0153	1.0682

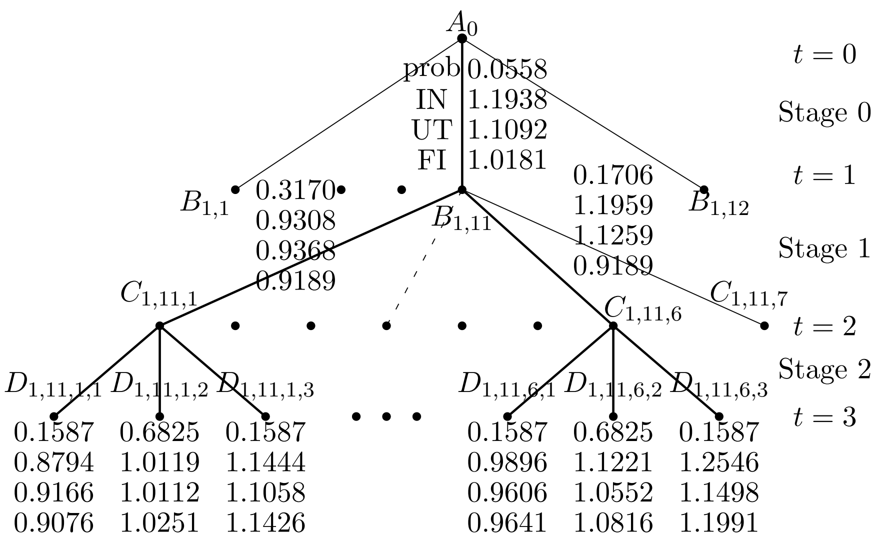

Figure 7: Scenario tree (12,7,3)

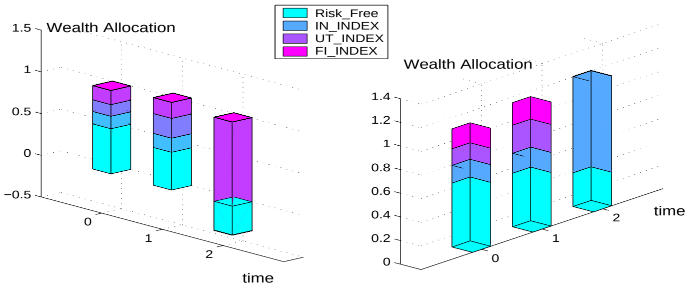

Figure 8: Wealth allocation along different paths of scenario tree (12,7,3)

Table 7: Part of the investment policies according to Figure 7

time	0	1	2	2
node	A_0	$B_{1,11}$	$C_{1,11,1}$	$C_{1,11,6}$
Risk_Free	0.5409	0.4525	-0.3464	0.2845
IN_INDEX	0.1497	0.1692	0	0.8145
UT_INDEX	0.1399	0.2372	0	0
FI_INDEX	0.1690	0.1910	1.3551	0

at the latter stages has little impact on the evaluation criteria. Therefore, by selecting less successors for the latter stages, it is possible to control the model size as well as guarantee a reliable investment analysis for the current stage. Throughout the paper, the model and approach are performed by linear programming, which overcome the difficulty in solving the large-size nonlinear and non-convex problems.

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