Tracking a Financial Benchmark Using a Few Assets *

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Abstract

We study the problem of tracking a financial benchmark — a continuously compounded growth rate or a stock market index — by dynamically managing a portfolio consisting of a small number of traded stocks in the market. In either case, we formulate the tracking problem as an instance of the stochastic linear quadratic control (SLQ), involving indefinite cost matrices. As the SLQ formulation involves a discounted objective over an infinite horizon, we first address the issue of stabilizability. We then use semidefinite programming (SDP) as a computational tool to generate the optimal feedback control. We present numerical examples involving stocks traded at Hong Kong and New York Stock Exchanges, to illustrate the various features of the model and its performance.

Keywords: steady growth-rate tracking, stock index tracking, stochastic linear quadratic control, semidefinite programming, stabilizability.

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1 Introduction

Investment problems can be generally described as to identify and manage a portfolio of assets in order to satisfy certain criteria. In this paper we consider two specific criteria: (i) to track a continuously compounded growth rate, and (ii) to track a market index. For both criteria, we want to be able to track the target by means of using a small, *given* portfolio, in terms of the number of stocks involved. Furthermore, we would like to have an approach that is robust in the sense that the tracking performance will be insensitive to the stocks that constitute the portfolio. (For the purpose of this paper, we do not address the issue of portfolio *selection*, i.e., how to pick the stocks to form the portfolio.)

This tracking problem is clearly of interest to money managers, whose funds, while being actively managed following certain strategies, need not be well-diversified portfolios; whereas their performances will be measured against certain financial benchmarks. Recent research in behavioral finance shows that most investors tend not to mind losses as long as their funds beat or match market indices, but they tend to have very low tolerance towards losses that are worse than market benchmarks, [21]. While index-related funds have arisen dramatically in the past decades, [10], for small-size portfolios or funds that concentrate on a relatively small number of stocks, it is impractical to track a large market index (e.g., S&P 500 or Russell 2000) *literally*, i.e., by holding all constituents stocks in proportion, and continuously adjust their weights.

In this paper we propose a new approach to tracking either a given fixed growth rate or a stochastic market index. Our approach is based on stochastic linear quadratic control (SLQ) involving *indefinite* cost matrices. SLQ, as a natural extension of Kalman's celebrated work in deterministic linear quadratic control theory [13], has a long history pioneered by the work of Wonham [27]. In recent years, SLQ problems with indefinite costs, have attracted extensive interests, [1, 7, 28], as they arise naturally in many financial applications, [16, 30]. (Also see the problem formulation below.) In [28], we have developed a general approach to such SLQ problems using semidefinite programming (SDP), an important tool in optimization (see [2, 19]). SDP has a rich duality/complementarity structure, which, as revealed in [28], connects closely to the stability and optimality of SLQ control. SDP also provides an efficient computational means to SLQ problems when the classical Riccati approach fails to handle the singularity caused by indefinite cost matrices.

To model our tracking problems as SLQ control problems solvable by SDP, we need to overcome several technical difficulties. First, in order to have the optimal feedback control characterized by the solution to an algebraic Ricatti equation, which is then solved by SDP, we need to adopt an infinite time horizon. (In contrast, a finite horizon will result in the optimal control specified by a differential equation, and hence a completely different problem class; refer to [16, 30].) This infinite horizon must be reconciled with the reality that the tracking problem is typically concerned with a relatively short time horizon, as fund performance is usually measured on a quarterly, semi-annual or annual basis. To this end, we introduce a discount factor in the tracking objective. A sufficiently large discount factor effectively forces the control to focus on the near term. The discount also plays the role of a stabilizing factor in the control problem, and we provide theoretical guidance, in terms of sufficient conditions, on the choice of the discount factor so that the control problem is well-defined in the sense that it is stabilizable (Theorems 1 and 2).

Another issue is the modeling of a market index. It would be tempting to model it as a geometric Brownian motion (GBM), i.e., as an aggregated single asset. But this not only is technically objectionable (as a market index is typically a weighted sum of its constituent stocks, it does not follow a GBM even if each of the stocks does), but also results in rather poor tracking performance as we learned from our numerical studies. In this paper, we model the market index as a weighted sum of the constituents, each of which is modeled by a GBM. This, however, leads to an equation (of the index dynamics) that is non-linear in the state variable, and hence outside of the SLQ framework. To overcome this difficulty, we find a way to augment the state space so as to bring the model back into the SLQ realm. (Refer to §3.)

Our model shares certain characteristics of the celebrated Markowitz mean-variance theory [17]. In particular, like the Markowitz model, our tracking objective also penalizes both the over-performance and under-performance of the portfolio with respect to the benchmark. Ever since the inception of the Markowitz theory, various alternative risk measures have been proposed, notably the so-called downside risk where only the return below its mean or a target level is penalized [9, 18, 23]. For a recent survey on the Markowitz model and models with other risk measures, refer to [24]; and see [12] for a recent work on continuous-time portfolio selection with general risk measures, including the semivariance. In this paper, we limit ourselves to what is essentially the classical two-sided objective, so as to stay within the well-studied SLQ framework. Furthermore, our motivation is to develop a reference tool for fund managers to *compare* their performance against benchmarks, rather than an execution tool to *beat* the benchmark.

Other related literature includes Browne [6], which is also concerned with financial tracking but focuses on different objectives: e.g., maximizing the probability of beating a benchmark by a given percentage without going below it by another percentage, and minimizing the expected time until beating the benchmark. In addition, since there is only a few given assets in our portfolio, and we use these assets only, rather than all the available ones in the market, to track the performance of the benchmarks, our model is inherently one in an incomplete market. As such, there is a large related literature on mean-variance portfolio selection and hedging in continuous-time — for both complete and incomplete markets, and in a finite time horizon; see [5, 8, 15, 16, 22, 30].

The remainder of the paper is organized as follows. In the next two sections, Section 2 and Section 3, we introduce the growth rate tracking problem and the market index tracking problem, respectively. Both will be formulated as SLQ problems. In Section 4 we present the SDP technique to solve the control problems. Numerical examples are reported in Section 5 to illustrate the tracking performance and various features of the model. Brief conclusions are summarized in Section 6.

2 Tracking a Given Growth Rate

Consider m listed stocks that are constituents of a market index (e.g., S&P500 or the Hang Seng Index). Assume that the price of each stock $S_i(t)$, $i = 1, ..., m$, follows the multi-dimensional GBM:

$$dS_i(t) = b_i S_i(t)dt + \sum_{j=1}^{m} \sigma_{ij} S_i(t) dW_j(t), \quad S_i(0) = S_{i0}, \tag{1}$$

where $W(t) = (W_1(t), \cdots, W_m(t))^T$ is an m-dimensional standard Brownian motion (with $t \in [0, +\infty)$ and $W(0) = 0$), defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$.

Further assume that there is a riskless asset (e.g. a government bond), the price of which is $S_0(t)$:

$$dS_0(t) = rS_0(t)dt, \quad S_0(0) = S_{00}. \tag{2}$$

Given a portfolio of n ($n \le m$) stocks out of the m constituent stocks, our objective is to control the investment of a given wealth initially valued at x_0, among the n stocks and the bond, via dynamic asset allocation, in such a way that the performance of the investment follows as closely as possible a pre-specified, deterministic, continuously compounded growth trajectory, $x_0 e^{\mu t}$ (where $\mu > 0$ is a given parameter representing the growth factor) over a long time horizon. Here, the number of stocks in the portfolio, n, is typically much smaller than m, the number of stocks in the market index. Thus, we are essentially dealing with a portfolio selection problem in an *incomplete* market. Assume, without loss of generality (up to a re-ordering if necessary) that the first n of the m stocks have been selected for the portfolio.

Let $\pi_i(t)$, $i = 1, \cdots, n$, denote the wealth invested in stock i at time t. That is, $\pi(\cdot) := (\pi_1(\cdot), \cdots, \pi_n(\cdot))^T$ is the composition of the stock portfolio at time t; and it is called a (continuous-time) *portfolio*. In control parlance, $\pi(\cdot)$ is the control. We say the portfolio or control is *admis-*

sible if $\pi(\cdot)$ belongs to $L2_{\mathcal{F}}(\Re^n)$, the space of all $\Re^n$-valued, $\mathcal{F}_t$-adapted measurable processes satisfying $\mathsf{E}\int_0^{+\infty}\|\pi(t)\|^2 dt < +\infty$.

It is well known (e.g., [14]) that, in a self-financed manner, the *wealth process*, $x(\cdot)$, under an admissible control $\pi(\cdot)$, satisfies:

$$dx(t) = \Big\{rx(t) + \sum_{i=1}^{n}[b_i - r]\pi_i(t)\Big\}dt + \sum_{j=1}^{m}\sum_{i=1}^{n}\sigma_{ij}\pi_i(t)dW_j(t), \quad x(0) = x_0. \tag{3}$$

In the control terminology $x(\cdot)$ is the state process under the control $\pi(\cdot)$. Note that $\pi_0(t) := x(t) - \sum_{i=1}^{n}\pi_i(t)$ is the amount invested in the bond, which is uniquely determined by $\pi(\cdot)$ via the above equation. Write

$$b := (b_1 - r, \cdots, b_n - r)^T, \qquad \sigma := (\sigma_{ij})_{m\times m}, \qquad \Gamma := \sigma\sigma^T. \tag{4}$$

Moreover, let σ_n denote the $n \times m$ matrix which is identical to the matrix consisting of the first n rows of σ, and let $\Gamma_n := \sigma_n\sigma_n^T$.

The dynamics in (3) can be rewritten as follows:

$$dx(t) = [rx(t) + b^T\pi(t)]dt + \pi^T\sigma_n dW(t), \qquad x(0) = x_0. \tag{5}$$

Our objective is:

$$\min \mathsf{E}\int_0^{\infty} e^{-2\rho t}[x(t) - x_0e^{\mu t}]^2 dt, \tag{6}$$

where $2\rho > 0$ is a discount factor. The choice of the parameter ρ will be discussed later. At this point we simply remark that ρ is introduced to guarantee the *stabilizability* of the control system; its actual value will have minimal impact on the result. (More on this in the numerical section below.)

Applying a transformation of variables:

$$y(t) := e^{-\rho t}[x(t) - x_0e^{\mu t}], \qquad \bar{\pi}(t) := e^{-\rho t}\pi(t).$$

to turn the above control problem into the following equivalent from:

$$\begin{aligned} \min \quad & \mathsf{E}\textstyle\int_0^{\infty}|y(t)|^2 dt \\ \text{s.t.} \quad & dy(t) = \Big\{(r-\rho)y(t) + b^T\bar{\pi}(t) + (r-\mu)x_0e^{(\mu-\rho)t}\Big\}dt + \bar{\pi}(t)^T\sigma_n dW(t) \\ & y(0) = 0. \end{aligned} \tag{7}$$

The above is a control problem to minimize a quadratic cost functional, with the system dynamics being affine (i.e., linear with a nonhomogeneous term) with respect to the state and control variables. Moreover, the system dynamics are stochastic. Hence, this is a *stochastic linear quadratic control* (SLQ) problem.

A canonical formulation of the SLQ problem is as follows:

$$
\begin{aligned}
\text{(SLQ)} \quad \min \quad & \mathsf{E}\int_0^\infty [y(t)^T Q y(t) + u(t)^T R u(t)]dt \\
\text{s.t.} \quad & dy(t) = [Ay(t) + Bu(t) + f(t)]dt + \textstyle\sum_{j=1}^k [C_j y(t) + D_j u(t) + g_j(t)]dW_j(t), \qquad (8) \\
& y(0) = y_0.
\end{aligned}
$$

In this expression, Q, R, A, B, C_j and D_j, $j = 1, ..., k$, are constant matrices with appropriate dimensions; f and g_j are $\mathcal{F}_t$-adapted processes; $y(\cdot)$ denotes the state, and $u(\cdot)$ the control. Again, the model is defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, involving a k-dimensional standard Brownian motion $W(t)$. Some basic notions and concepts are listed below (also refer to [28]).

- The control $u(\cdot)$ is called *admissible* if $u \in L2_{\mathcal{F}}(\Re^n)$.
- The control $u(\cdot)$ is called (mean-square) *stabilizing* at x_0, if the corresponding state process $x(\cdot)$, following the dynamics specified above with the initial state x_0, satisfies $\lim_{t\to+\infty} \mathsf{E}[x(t)^T x(t)] = 0$.
- A feedback control, $u(t) = Kx(t)$, with K being a constant matrix, is called *stabilizing*, if for every initial state x_0, the corresponding state process $x(\cdot)$ under the control u satisfies $\lim_{t\to+\infty} \mathsf{E}[x(t)^T x(t)] = 0$. The SLQ problem is called *stabilizable* if there exists a stabilizing feedback control as specified above.
- The problem (SLQ) is called *attainable* at x_0, if with the initial state x_0 there exists an optimal admissible control that achieves a finite infimum of the cost functional.

For an extensive coverage on SLQ, we refer to Yong and Zhou [29, Chapter 6].

We first address the issue of stabilizability, which is essential for the SLQ to to be a meaningful problem.

Below we shall use the notion of a pseudo-inverse (refer to [20]). Denote by M^+ pseudo-inverse of a matrix M. It is known that M^+ satisfies the following properties:

$$M^+M = MM^+, \qquad MM^+M = M, \qquad M^+MM^+ = M^+;$$

and also the following when M is a positive semidefinite matrix:

$$M^+ \succeq 0, \qquad (M^+)^T = M^+.$$

Theorem 1 If

$$\rho > \max\{\mu, r - \frac{1}{2} b^T \Gamma_n^+ b\}, \tag{9}$$

then the SLQ problem in (7) is stabilizable.

Proof. First we consider the homogeneous version of the problem:

$$\begin{aligned} \min \quad & \mathsf{E} \int_0^\infty |z(t)|^2 dt \\ \text{s.t.} \quad & dz(t) = \Big\{ (r-\rho) z(t) + b^T \bar{\pi}(t) \Big\} dt + \bar{\pi}(t)^T \sigma_n dW(t) \\ & z(0) = 0. \end{aligned} \tag{10}$$

We then apply the equivalent condition for mean-square stabilizability in [1, Theorem 1], to show that (10) is stabilizable if and only if there exist a vector $v = (v_1, \cdots, v_n)^T$ and a scalar $x > 0$, so that

$$2(r-\rho) + 2b^T v x + v^T \sigma_n \sigma_n^T v x 2 < 0,$$

or, equivalently

$$\begin{aligned} 0 > \quad & v^T \Gamma_n v + 2b^T v + 2(r-\rho) \\ = \quad & (v + \Gamma_n^+ b)^T \Gamma_n (v + \Gamma_n^+ b) + 2b^T (I - \Gamma_n^+ \Gamma_n) v + 2(r-\rho) - b^T \Gamma_n^+ b. \end{aligned} \tag{11}$$

Taking $v = -\Gamma_n^+ b$, we see that the above inequality is satisfied if (9) holds. Moreover, in this case the following control,

$$\bar{\pi}(t) = -\Gamma_n^+ b z(t), \tag{12}$$

is a stabilizing feedback control for the problem in (10). Now, taking the same feedback control matrix $-\Gamma_n^+ b$ and applying $\bar{\pi}(t) = -\Gamma_n^+ b y(t)$ to the non-homogeneous problem in (7), we get

$$\begin{aligned} & dy(t) = \Big\{ (r - \rho - b^T \Gamma_n^+ b) y(t) + (r-\mu) x_0 e^{(\mu-\rho)t} \Big\} dt - b^T \Gamma_n^+ \sigma_n y(t) dW(t) \\ & y(0) = 0. \end{aligned} \tag{13}$$

Applying Ito's formula to $[y(t)]^2$, we obtain

$$d[y(t)]^2 = \Big\{ [2(r-\rho) - b^T \Gamma_n^+ b][y(t)]^2 + 2(r-\mu) x_0 e^{(\mu-\rho)t} y(t) \Big\} dt + \{\cdots\} dW(t). \tag{14}$$

Set $\alpha := 2(r-\rho) - b^T \Gamma_n^+ b < 0$, and fix $\varepsilon > 0$, so that $\varepsilon < -\alpha$ and $\alpha + \varepsilon \neq 2(\mu - \rho)$. Using the general inequality $2c_1 c_2 \leq \frac{1}{\varepsilon} c_1 2 + \varepsilon c_2 2$ for any real numbers c_1 and c_2, we get from (14) that

$$d[y(t)]^2 \leq [(\alpha + \varepsilon)[y(t)]^2 + \frac{1}{\varepsilon} (r-\mu)^2 x_0^2 e^{2(\mu-\rho)t}] dt + \{\cdots\} dW(t). \tag{15}$$

Taking expectations on both sides, we have

$$d\mathsf{E}\{[y(t)]^2\} \leq [(\alpha+\varepsilon)\mathsf{E}\{[y(t)]^2\} + \frac{1}{\varepsilon}(r-\mu)^2 x_0^2 e^{2(\mu-\rho)t}]dt.$$

Multiplying both sides by $e^{-(\alpha+\varepsilon)t}$, we obtain

$$\frac{d}{dt}\left\{e^{-(\alpha+\varepsilon)t}\mathsf{E}[y(t)]^2\right\} \leq \frac{1}{\varepsilon}(r-\mu)^2 x_0^2 e^{2(\mu-\rho)t-(\alpha+\varepsilon)t}.$$

Integrating from 0 to t and going through some algebra lead to

$$\mathsf{E}[y(t)]^2 \leq \frac{(r-\mu)^2 x_0 2}{\varepsilon[2(\mu-\rho)-(\alpha+\varepsilon)]}[e^{2(\mu-\rho)t} - e^{(\alpha+\varepsilon)t}].$$

Taking into account $\alpha+\varepsilon<0$ and $\mu-\rho<0$, we conclude $\mathsf{E}\{[y(t)]^2\} \to 0$ as $t\to\infty$. $\square$

Note that in practice, it only makes sense that the target growth rate exceeds the riskless rate, namely $\mu > r$. Hence, the condition in Theorem 1 is essentially $\rho > \mu$.

Having addressed the stabilizability issue, we now turn to solving the optimization problem in (7). As we noted above, (7) involves a nonhomogeneous term in the drift part. So we need to first reformulate the problem into a homogeneous one in order to apply the semidefinite programming approach later. To do so, let

$$y_0(t) := x_0 e^{(\mu-\rho)t}.$$

Then, the problem in (7) becomes

$$\begin{array}{ll} \min & \mathsf{E}\int_0^\infty |y(t)|^2 dt \\ \text{s.t.} & dy_0(t) = (\mu-\rho)y_0(t)dt \\ & dy(t) = \Big\{(r-\mu)y_0(t) + (r-\rho)y(t) + b^T\bar{\pi}(t)\Big\}dt + \bar{\pi}(t)^T\sigma_n dW(t) \\ & \begin{bmatrix} y_0(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}. \end{array} \tag{16}$$

To relate the above to the general SLQ model in (8), let

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad R = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{n\times n}; \tag{17}$$

$$A = \begin{bmatrix} \mu-\rho & 0 \\ r-\mu & r-\rho \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ b_1-r & b_2-r & \cdots & b_n-r \end{bmatrix}_{2\times n}; \tag{18}$$

and

$$C_j = 0, \qquad D_j = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \sigma_{1j} & \sigma_{2j} & \cdots & \sigma_{nj} \end{bmatrix}_{2\times n}, \qquad f(t) \equiv 0, \;\; g_j(t) \equiv 0; \tag{19}$$

for $j = 1, ..., n$.

In most of the SLQ literature, it is standard to require that the matrix Q be positive semidefinite and R positive definite [3, 4, 27]. To solve the SLQ problem, the centerpiece is to solve a matrix equation known as the *Riccati equation*:

$$\begin{aligned} & Q + A^T P + PA + \sum_{j=1}^{k} C_j^T P C_j \\ & -(B^T P + \sum_{j=1}^{k} D_j^T P C_j)^T (R + \sum_{j=1}^{k} D_j^T P D_j)^{-1} (B^T P + \sum_{j=1}^{k} D_j^T P C_j) = 0, \end{aligned} \tag{20}$$

where the unknown is the matrix P, which must satisfy

$$R + \sum_{j=1}^{k} D_j^T P D_j \succ 0. \tag{21}$$

If the Riccati equation in (20) admits a solution P^* that satisfies the inequality in (21), then the solution to the original SLQ problem is a feedback control:

$$u^*(t) = -(R + \sum_{j=1}^{k} D_j^T P^* D_j)^{-1} (B^T P^* + \sum_{j=1}^{k} D_j^T P^* C_j) y^*(t)$$

provided that the above control is stabilizing; see [1].

In our case, however, Q and R are both singular. Therefore, we are in the realm of the *indefinite* SLQ. In Section 4, we shall resort to a different approach, which uses the semidefinite programming as a tool to solve the SLQ problem.

3 Tracking a Market Index

Consider the same stock market as described in §2. The market index can be represented as follows:

$$I(t) = \sum_{j=1}^{m} \alpha_j S_j(t), \qquad I(0) = I_0 \tag{22}$$

where α_j corresponds to the weight shares of stock j in the index. Our objective here is to track this index using the same portfolio of n given stocks and the bond as in §2. Specifically, the problem is:

$$\min \mathsf{E} \int_0^\infty e^{-2\rho t} [x(t) - I(t)]^2 dt,$$

subject to the wealth equation in (5). Note that by scaling we may assume without loss of generality that $x(0) = I(0)$.

As in the previous section, set

$$y(t) := x(t) - I(t), \quad \bar{\pi}(t) := e^{-\rho t}\pi(t).$$

Also, denote

$$L(t) := [\alpha_1 S_1(t), \cdots, \alpha_m S_m(t)]^T.$$

Then, the above can be transformed into the following problem:

$$\begin{array}{ll} \min & \mathsf{E}\int_0^\infty |y(t)|^2 dt \\ \text{s.t.} & dy(t) = \Big\{(r-\rho)y(t) + b^T\bar{\pi}(t) + e^{-\rho t}[rI(t) - \sum_{i=1}^m \alpha_i b_i S_i(t)]\Big\}dt \\ & \qquad\quad +[\bar{\pi}(t)^T\sigma_n - L(t)^T\sigma]dW(t) \\ & y(0) = 0. \end{array} \tag{23}$$

As in the last section, we first address the issue of stabilizability.

Theorem 2 If

$$\rho > \max\Big\{r - \frac{1}{2}b^T\Gamma_n^+ b; \quad b_i + \frac{1}{2}\sum_{j=1}^m \sigma_{ij}^2,\ i = 1, 2, \cdots, m\Big\}, \tag{24}$$

then the problem in (23) is stabilizable.

Proof. Notice that the homogeneous version of the index-tracking problem in (23) is exactly the same as that of the growth-tracking problem in (7). Following the proof of Theorem 1, we may apply the feedback control there, $\bar{\pi}(t) = -\Gamma_n^{-1}by(t)$, to the problem in (23). Under this control, the dynamics in (23) become

$$\begin{array}{rcl} dy(t) & = & \Big\{(r-\rho-b^T\Gamma_n^+ b)y(t) + e^{-\rho t}[rI(t) - \sum_{i=1}^m \alpha_i b_i S_i(t)]\Big\}dt \\ & & -[b^T\Gamma_n^+\sigma_n y(t) + L(t)^T\sigma]dW(t), \\ y(0) & = & 0. \end{array} \tag{25}$$

Applying Ito's formula to $[y(t)]^2$, we get

$$\begin{array}{rl} d[y(t)]^2 = & \Big\{[2(r-\rho) - b^T\Gamma_n^+ b][y(t)]^2 + 2e^{-\rho t}[rI(t) - \sum_{i=1}^m \alpha_i b_i S_i(t) \\ & +b^T\Gamma_n^+\sigma_n\sigma^T L(t)]y(t) + e^{-2\rho t}L(t)^T\sigma\sigma^T L(t)\Big\}dt + \{\cdots\}dW(t). \end{array} \tag{26}$$

It follows from (1) that

$$\mathsf{E}|S_i(t)| \le S_{i0}e^{(b_i + \frac{1}{2}\sum_{j=1}^m \sigma_{ij}^2)t}, \qquad t \ge 0, \quad i = 1, \cdots, m.$$

As a result,

$$\mathsf{E}|I(t)| \leq \sum_{i=1}^{m} |\alpha_i| S_{i0} e^{(b_i+\frac{1}{2}\sum_{j=1}^{m}\sigma_{ij}^2)t}, \qquad t \geq 0.$$

In view of the preceding two estimations, the rest of the proof is exactly the same as in the proof of Theorem (7), under the given condition in (24). □

To solve the problem in (23), however, we need to reformulate the model as a homogeneous one as before. To this end, we augment the state definition by making the index price $I(t)$ and all individual stock prices in the index as state variables. This way, we have the following SLQ problem:

$$\begin{aligned}
\min \quad & \mathsf{E}\int_0^\infty e^{-2\rho t}[x(t)-I(t)]^2 dt \\
\text{s.t.} \quad & dx(t) = \Big\{ rx(t) + \textstyle\sum_{i=1}^{n}[b_i - r]\pi_i(t)\Big\}dt + \sum_{j=1}^{m}\sum_{i=1}^{n}\sigma_{ij}\pi_i(t)dW_j(t) \\
& dI(t) = \textstyle\sum_{i=1}^{m}\alpha_i b_i S_i(t)dt + \sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i\sigma_{ij}S_i(t)dW_j(t) \\
& dS_i(t) = b_i S_i(t)dt + \textstyle\sum_{j=1}^{m}\sigma_{ij}S_i(t)dW_j(t),\ i=1,\dots,m, \\
& (x(0), I(0), S_i(0)) = (x_0, x_0, S_{i0}),
\end{aligned}$$

where $x(t)$, $I(t)$, and $S_i(t)$, $i=1,...,m$, are the state variables, and $\pi_i(t)$, $i=1,...,n$ are the control variables. We remark here that $I(t)$ and $S_i(t)$'s are independent of the composition of the portfolio under control; as such, they are *uncontrollable* state variables.

Next, apply the following transformation of variables:

$$\begin{aligned}
\bar{x}(t) &:= e^{-\rho t}x(t) \\
\bar{I}(t) &:= e^{-\rho t}I(t) \\
\bar{S}_i(t) &:= e^{-\rho t}S_i(t) \\
\bar{\pi}_i(t) &:= e^{-\rho t}\pi_i(t).
\end{aligned}$$

The optimal control problem then becomes:

$$\begin{aligned}
\min \quad & \mathsf{E}\int_0^\infty [\bar{x}(t)-\bar{I}(t)]^2 dt \\
\text{s.t.} \quad & d\bar{x}(t) = \Big\{ (r-\rho)\bar{x}(t) + \textstyle\sum_{i=1}^{n}[b_i - r]\bar{\pi}_i(t)\Big\}dt + \sum_{j=1}^{m}\sum_{i=1}^{n}\sigma_{ij}\bar{\pi}_i(t)dW_j(t) \\
& d\bar{I}(t) = -\rho\bar{I}(t)dt + \textstyle\sum_{i=1}^{m}\alpha_i b_i \bar{S}_i(t)dt + \sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i\sigma_{ij}\bar{S}_i(t)dW_j(t) \\
& d\bar{S}_i(t) = (b_i-\rho)\bar{S}_i(t)dt + \textstyle\sum_{j=1}^{m}\sigma_{ij}\bar{S}_i(t)dW_j(t),\ i=1,\dots,m, \\
& (\bar{x}(0), \bar{I}(0), \bar{S}_i(0)) = (x_0, x_0, S_{i0}).
\end{aligned}$$

To relate to the canonical (SLQ) in (8), here the state (vector) is

$$y(t) := [\bar{x}(t), \bar{I}(t), \bar{S}_1(t), \cdots, \bar{S}_m(t)]^T$$

and the control (vector) is

$$u(t) := [\bar{\pi}_1(t), \cdots, \bar{\pi}_n(t)]^T.$$

The coefficient matrices take the following values:

$$Q := \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(m+2)\times(m+2)}$$

$$R := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{n\times n}$$

$$A := \begin{bmatrix} r-\rho & 0 & 0 & \cdots & 0 \\ 0 & -\rho & \alpha_1 b_1 & \cdots & \alpha_m b_m \\ 0 & 0 & b_1-\rho & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_m-\rho \end{bmatrix}_{(m+2)\times(m+2)}$$

$$B := \begin{bmatrix} b_1-r & b_2-r & \cdots & b_n-r \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{(m+2)\times n}$$

$$C_j := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_1\sigma_{1j} & \cdots & \alpha_m\sigma_{mj} \\ 0 & 0 & \sigma_{1j} & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_{mj} \end{bmatrix}_{(m+2)\times(m+2)}$$

$$D_j := \begin{bmatrix} \sigma_{1j} & \sigma_{2j} & \cdots & \sigma_{nj} \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{(m+2)\times n}$$

$f(t) \equiv 0; \quad g_j(t) \equiv 0$

where $j = 1, ..., n$. Once again, in this case Q and R are both singular.

4 The Semidefinite Programming Resolution

We have seen in the previous two sections that the two tracking models both belong to the homogeneous version of the canonical form in (8):

$$\begin{array}{lll} \text{(SLQ)} & \min & \mathsf{E}\int_0^\infty [y(t)^T Q y(t) + u(t)^T R u(t)] dt \\ & \text{s.t.} & dy(t) = [Ay(t) + Bu(t)]dt + \sum_{j=1}^k [C_j y(t) + D_j u(t)] dW_j(t), \\ & & y(0) = y_0. \end{array} \tag{27}$$

As mentioned earlier, the conventional approach to the SLQ via solving the Riccati equation does not apply here due to the indefiniteness of Q and R. In this section we adopt the approach developed by Yao, Zhang and Zhou in [28], based on semidefinite programming (SDP). Since in [28] only the case of one-dimensional Brownian motion is treated, we spell out below the necessary details in terms of the multi-dimensional model formulated here.

SDP can be viewed as an extension of linear programming (LP), with the standard nonnegativity constraint in LP replaced by the matrix positive semidefiniteness constraint; refer to a recent handbook on SDP, [26], and the references therein.

In our context, the associated SDP problem takes the following form:

$$\begin{array}{lll} (\mathrm{P}_{\mathrm{SLQ}}) & \max & \langle I, P \rangle \\ & \text{s.t.} & \begin{bmatrix} R + \sum_{j=1}^k D_j^T P D_j, & B^T P + \sum_{j=1}^k D_j^T P C_j \\ PB + \sum_{j=1}^k C_j^T P D_j, & Q + PA + A^T P + \sum_{j=1}^k C_j^T P C_j \end{bmatrix} \succeq 0, \\ & & P \text{ is a symmetric matrix.} \end{array}$$

Here the decision variable is a symmetric matrix P, and the matrix inner product (of two matrices X and Y) is defined as

$$\langle X, Y \rangle := \sum_{i,j} X_{ij} Y_{ij}.$$

The dual SDP, with the matrix Z as the decision variable, is:

$$\begin{array}{lll} (\mathrm{D}_{\mathrm{SLQ}}) & \min & \langle R, Z_{11} \rangle + \langle Q, Z_{22} \rangle \\ & \text{s.t.} & Z_{22} A^T + A Z_{22} + B Z_{12} + Z_{12}^T B^T \\ & & + \sum_{j=1}^k \left[D_j Z_{11} D_j^T + D_j Z_{12} C_j^T + C_j Z_{12}^T D_j^T + C_j Z_{22} C_j^T \right] + I = 0, \\ & & \begin{bmatrix} Z_{11}, & Z_{12} \\ Z_{12}^T, & Z_{22} \end{bmatrix} \succeq 0. \end{array}$$

Let P^* and $Z^* \equiv \begin{bmatrix} Z_{11}^*, & Z_{12}^* \\ Z_{12}^{*T}, & Z_{22}^* \end{bmatrix}$ denote, respectively, the solutions to the primal and the dual SDP's. Then we can construct two feedback controls based on P^* and Z^*, respectively:

$$u_P^*(t) = -(R + \sum_{j=1}^{k} D_j^T P^* D_j)^+ (B^T P^* + \sum_{j=1}^{k} D_j^T P^* C_j) y^*(t) \tag{28}$$

and

$$u_D^*(t) = Z_{12}^* (Z_{22}^*)^{-1} y^*(t). \tag{29}$$

Note that the two controls u_P^* and u_D^* are in general different. An interesting and practically useful fact is that, in most cases, solving ($\mathrm{D_{SLQ}}$) suffices for the purpose of solving (SLQ) completely. This is because, analytically, u_D^* is *automatically* stabilizing as shown in [28, Theorem 3.4] and, computationally, the dual problem is usually the standard input form for most SDP solvers. Hence, in our numerical implementation, we solve ($\mathrm{D_{SLQ}}$) and apply the optimal control u_D^*.

Specifically, we use a package called SeDuMi, developed by Jos Sturm and widely recognized as a very stable and fast general-purpose SDP solver. (Refer to [25] for an introduction to SeDuMi.) To get a proper input format we use Kronecker products. Below we summarize a few basic facts of Kronecker products that are used in our implementation; more details can be found in Chapter 4 of Horn and Johnson [11].

Let $A \in \Re^{m\times n}$ and $B \in \Re^{p\times q}$ be two matrices. The standard "vec" operator on A is:

$$\mathrm{vec}\,(A) = (a_{11}, \cdots, a_{m1}, a_{12}, \cdots, a_{m2}, \cdots, a_{1n}, \cdots, a_{mn})^T,$$

and the Kronecker product between A and B is:

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

The following relations hold (see Chapter 4 of [11]):

- $(A \otimes B)(C \otimes D) = AC \otimes BD$
- $\mathrm{vec}\,(AB) = (I \otimes A)\mathrm{vec}\,B = (B^T \otimes I)\mathrm{vec}\,A$
- $\mathrm{vec}\,(AXB) = (B^T \otimes A)\mathrm{vec}\,X$.

Based on the above, ($\mathrm{D_{SLQ}}$) can be put in the following standard form for SeDuMi application:

$$
(\mathrm{SDP}_{\mathrm{SLQ}}) \quad \min \quad \operatorname{vec}\left(\begin{bmatrix} R & 0 \\ 0 & Q \end{bmatrix}\right)^T \operatorname{vec}(Z)
$$

$$
\begin{aligned}
\text{s.t.} \quad & \Big([B, A] \otimes [0, I] + [0, I] \otimes [B, A] + \textstyle\sum_{j=1}^{k} [D_j, C_j] \otimes [D_j, C_j]\Big) \operatorname{vec}(Z) \\
& = -\operatorname{vec}(I) \\
& Z \succeq 0.
\end{aligned}
$$

(In our implementation, we need to use the symmetric Kronecker products. But this only requires some minor modifications.)

5 Numerical Examples

In this section, we present several numerical examples demonstrating the application of our approach to portfolios of stocks traded at the Hong Kong and New York Stock Exchanges. The examples below are meant to illustrate what the tracking models are capable of doing and some of the insights obtained. While these examples by no means constitute a thorough numerical study, they are indeed representative of a much larger set of examples that we have run.

Before we present the examples, there are two points we want to bring out up front. First, to estimate the required model parameters — specifically, the drift vector and the covariance matrix — we use the relevant stock prices in the 60 days prior to the period in question, i.e., either the entire tracking period when we initialize the algorithm, or the remaining tracking period at an interim updating of the SDP solution and the feedback control. In other words, we use a simple 60-day moving average. We have experimented with both longer and shorter time windows (than 60 days), as well as more sophisticated estimation schemes, but found their impact on the tracking performance quite minimal. (More on this in §5.5.)

Second, we have experimented thoroughly with the choice of the discount factor ρ, and found that it has virtually no impact on the tracking performance, for both growth rate tracking and stock index tracking, as long as it is above the threshold required for stabilizability as stipulated in Theorem 1 (for growth rate tracking) and Theorem 2 (for stock index tracking). In the examples below, we simply choose a ρ value that is slightly above the lower bounds specified in the two theorems.

The risk-free rate is set as $r = 4\%$ per annum. In all examples, we allow short selling, and ignore transaction costs. The SDP algorithm is run at the beginning of the tracking period to generate the optimal feedback control. The updating of the portfolio based on the optimal control is executed once every trading day or once every week (5 trading days) or even longer as

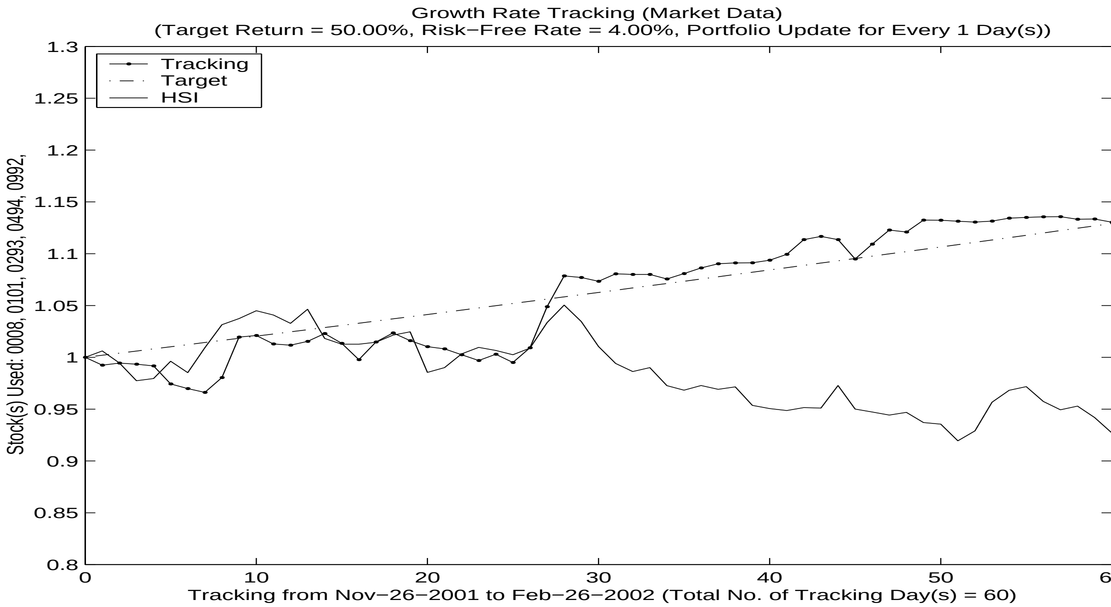

Figure 1: Volatile stocks; trade every day.

specified in the examples. In some cases, we also re-run the SDP algorithm during the interim of the tracking periods, in which case the parameter estimation is also updated (using, again, 60-day moving averages).

5.1 Growth Rate Tracking Using Hang Seng Index Stocks

We focus on the fourth quarter of 2001. During the period, the Hang Seng Index (HSI) dropped about 7%. On the other hand, we want our portfolio to return a positive, annualized 50%, or 12.5% over the period. The portfolio, given below, consists of 5 stocks representing industries such as property, airline, media, international trading, and personal computers . The performance of these stocks was quite volatile during the tracking period.

- Volatile stocks: Amoy Properties (0101), Cathy Pacific (0293), Television Broadcasts (0511), Li and Fung Limited (0494), and Legend Holdings (0992).

The results are summarized in three figures, Figure 1 through Figure 3, for trading (i.e., adjusting the portfolio) every day, every 5 days and every 20 days, respectively. Note that trading every 20 days, i.e., a total of three times only over the tracking period, only slightly degrades the tracking performance.

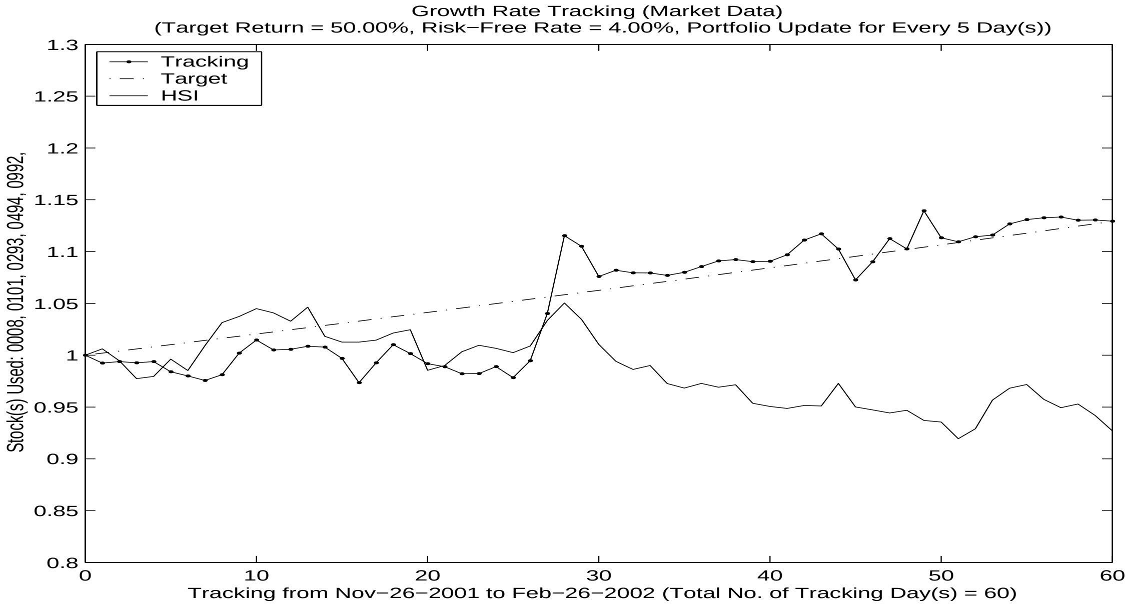

Figure 2: Volatile stocks; trade every 5 days.

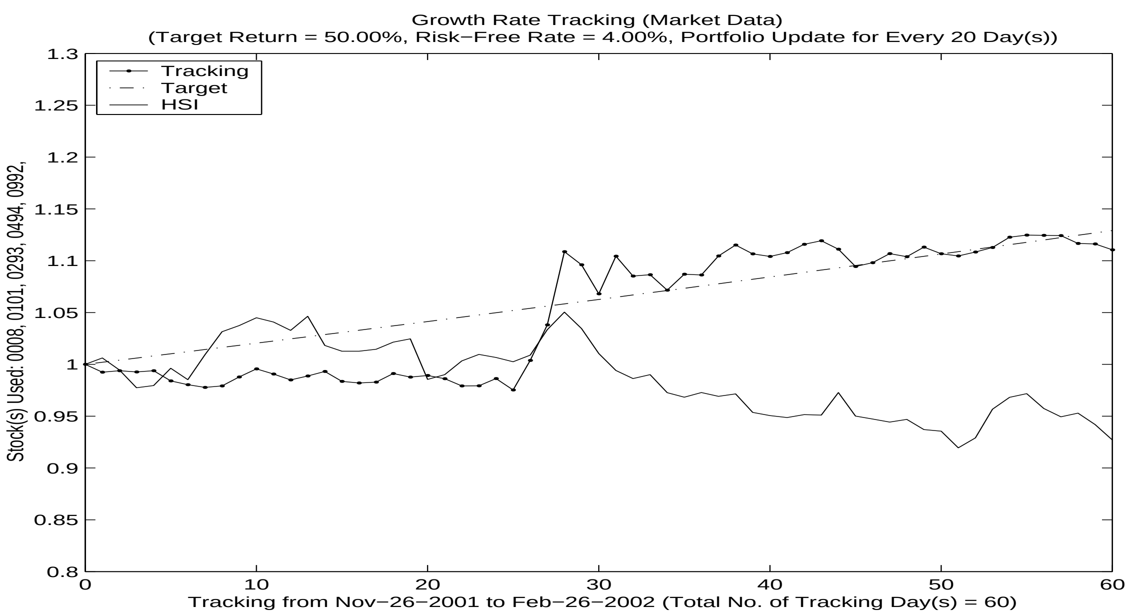

Figure 3: Volatile stocks; trade every 20 days.

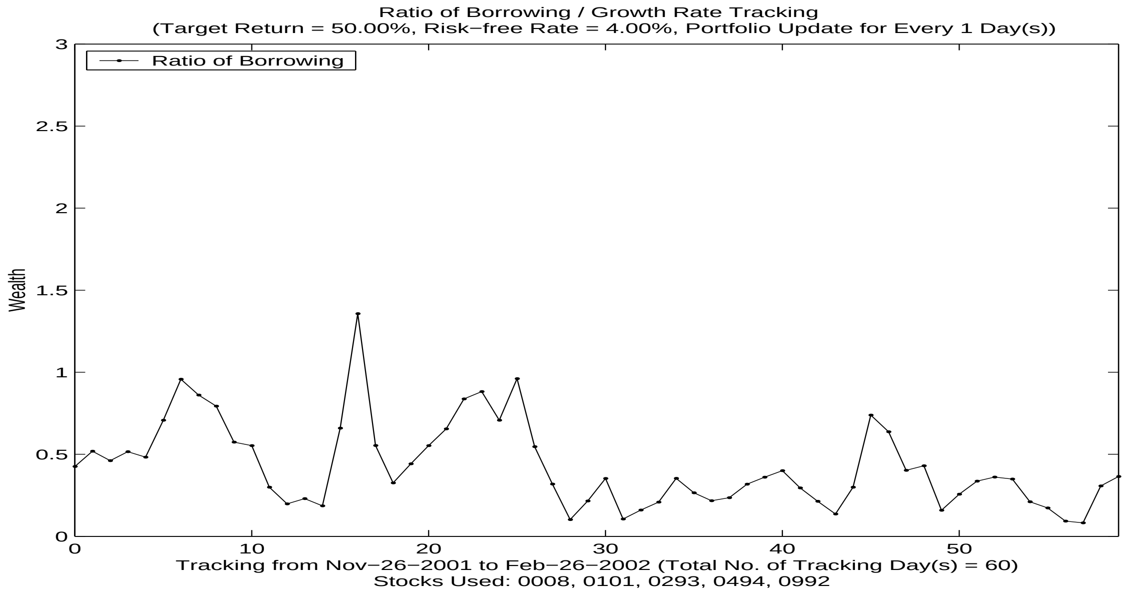

Figure 4: Amount of borrowing: volatile stocks; trade every day.

As the HSI dropped 7% while the portfolio returns 12.5% over the period, one might expect a substantial leverage, i.e., heavy borrowing (shorting). It turns out this is not the case; the level of shorting is quite moderate. Figure 4 through Figure 6, report the ratio of total borrowed amount to the total net wealth. Denote the total short position by S, and the total long position by L. Then, this ratio is $S/(L-S)$. For instance, a value of 1.5 (on the y-axis) in Figure 4 means $S/(L-S) = 1.5$, or $S/L = 0.6$. Note that, interestingly, trading every 20 days not only smoothed but also significantly reduced the amount of borrowing. Indeed, in Figure 6, we have $S/(L-S)$ consistently below 0.5, i.e., S/L consistently below 1/3, and indeed around 1/5 during most of the tracking period.

5.2 Growth Rate Tracking Using Simulated Data

In Figure 7, we present the same growth rate tracking model as above, but use a group of four (relatively) small stocks as follows:

- Small-cap stocks: Hung Lung (0010), Sino Land (0083), Henderson Investment (0097), and China Resources (0291).

Figure 7 shows that using this group, during this period, cannot quite achieve the require growth rate of 12.5% (50% annualized). Towards the end of the tracking period, its performance falls

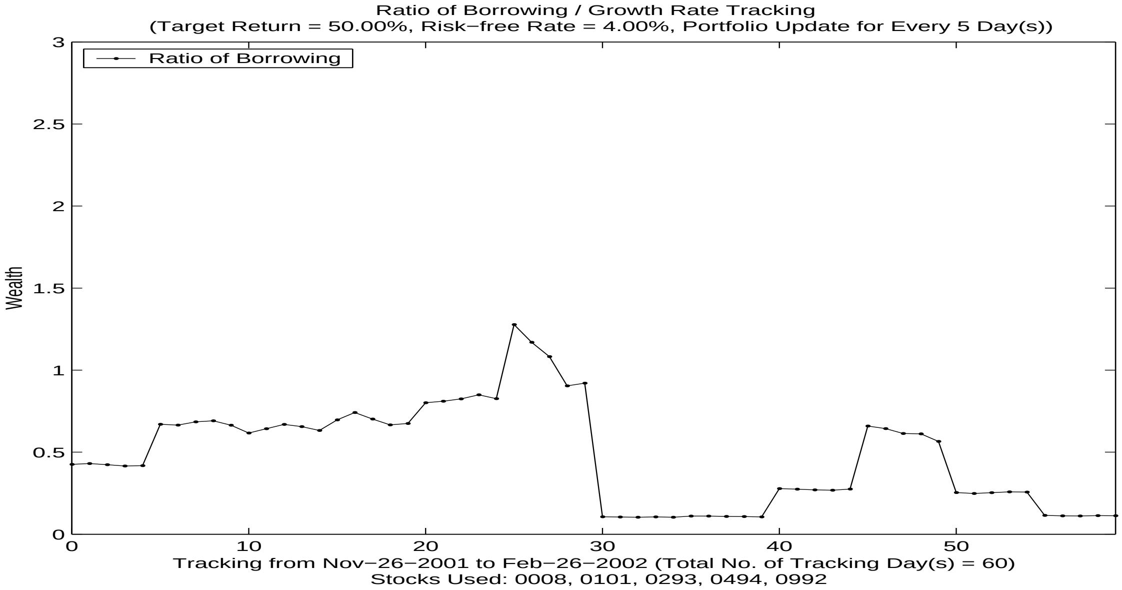

Figure 5: Amount of borrowing: volatile stocks; trade every 5 days.

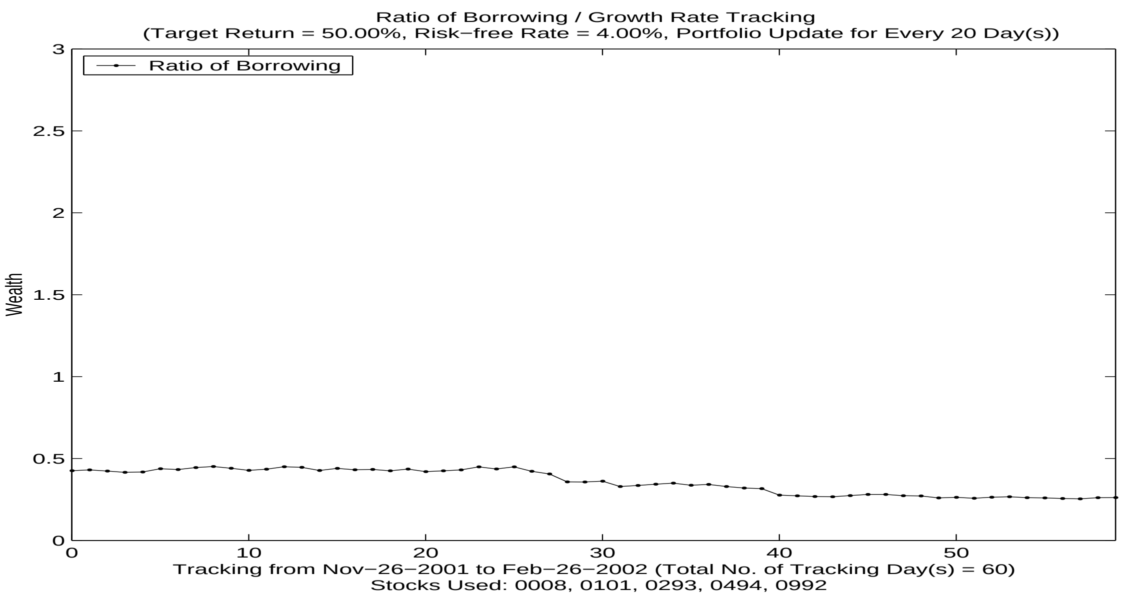

Figure 6: Amount of borrowing: volatile stocks; trade every 20 days.

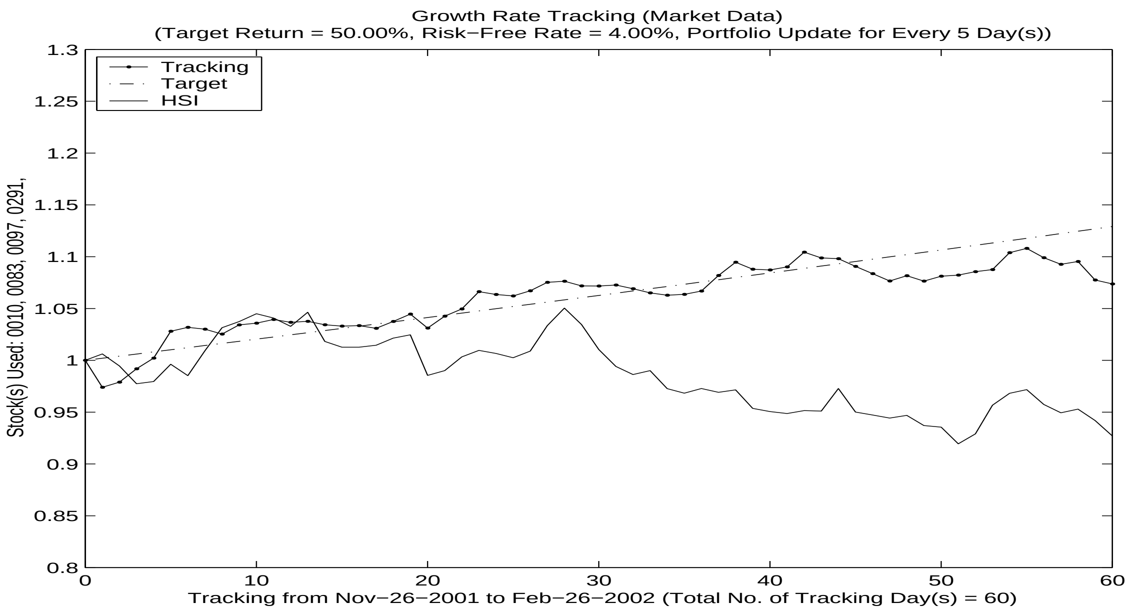

Figure 7: Small-cap stocks; historical data.

a bit short.

Note, however, that this is only a single sample path, represented by the historical market data; whereas the SLQ objective, after all, is in expectation. Hence, we generated 50 additional sample paths using simulation. Averaged over these sample paths, the tracking trajectory is able to follow very closely the required growth rate, as evident from Figure 8.

5.3 Tracking the Hang Seng Index

Figure 9 and 10 show the tracking performance using, respectively, the small-cap stocks and the volatile stocks listed above to track the HSI, during the first quarter of 2002.

Furthermore, from Oct 2, 2002 through Nov 29, 2002, we carried out a real-time tracking on the HSI using a group of four large-cap stocks specified below.

- Large-cap stocks: HSBC Holdings (0005), Hutchison Whampoa (0013), Sun Hung Kai (0016), and China Telecom (0941).

As in the other experiments, we used data from the 60 days prior to the tracking period to estimate the mean vector and the covariance matrix, and run the SDP once at the very beginning of the experiment. Once the experiment started, we adjust the portfolio once every week (5

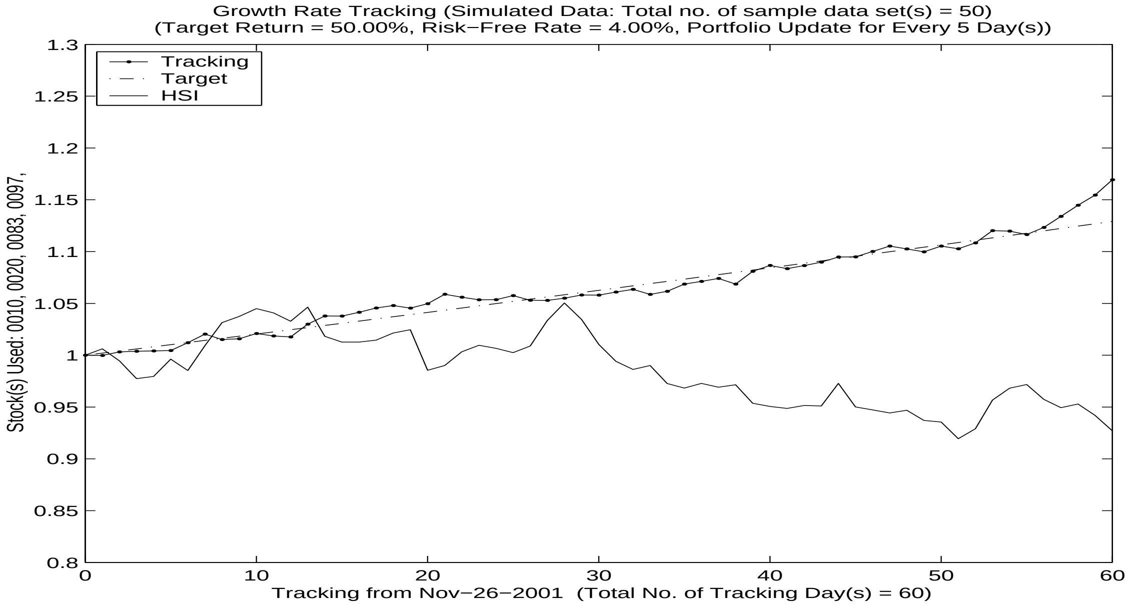

Figure 8: Small-cap stocks; simulation.

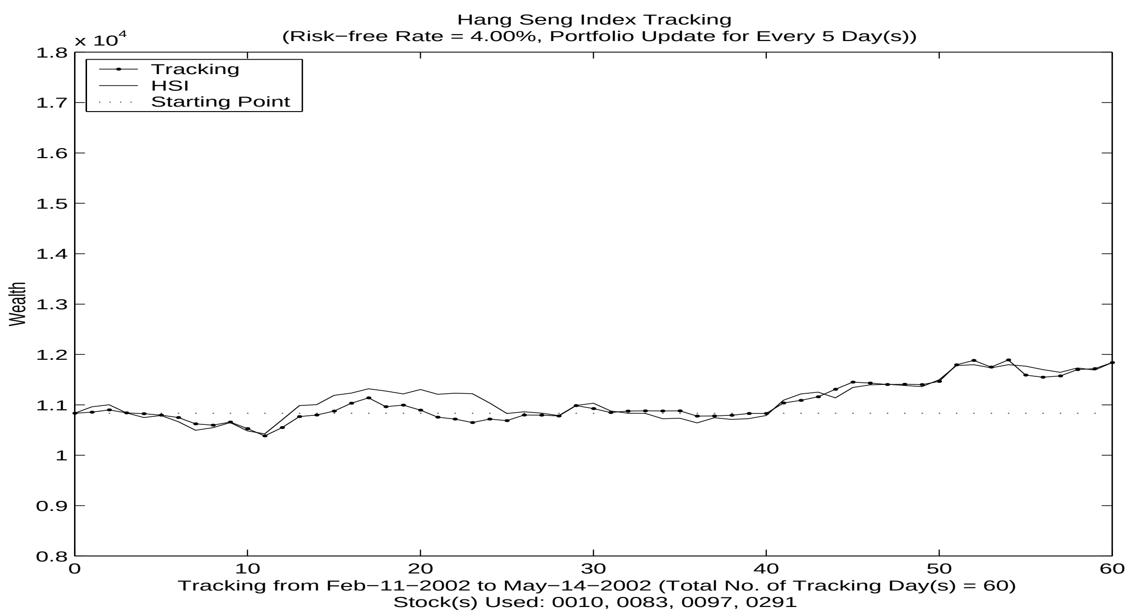

Figure 9: Small-cap stocks; trade every 5 days.

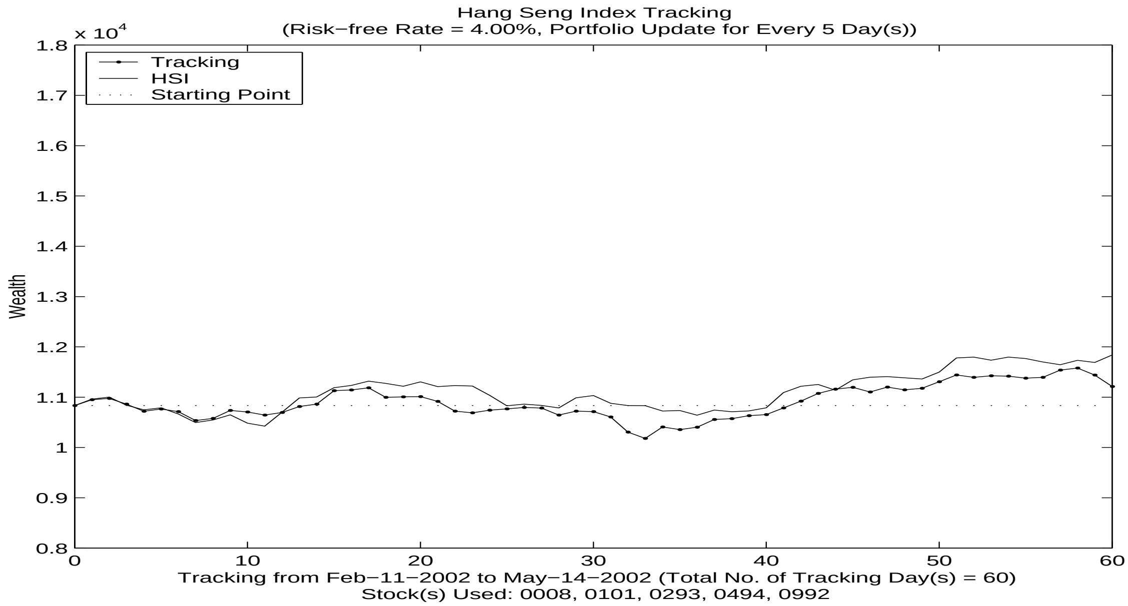

Figure 10: Period B; volatile stocks; trade every 5 days.

trading days) according to the feedback control law specified by the model. Figure 11 reports the outcome.

5.4 Growth-Rate Tracking Using S&P 500 Stocks

Here we consider a group of 5 stocks randomly generated from the constituents of the S&P 500 index:

- S&P500 stocks: J.P. Morgan Chase & Co. (JPM), Eastman Kodak (EK), King Pharmaceuticals (KG), Starwood Hotels & Resorts (HOT) and Jabil Circuit (JBL).

The tracking period starts from October 1, 2002 and lasts for two months.

As before we set the tracking target to be an annualized 50% growth rate. The tracking performance is illustrated in Figure 12, where the portfolio is updated every 5 days, and the SDP is recomputed each time using updated parameters. This is repeated in Figure 13, but with the portfolio updated every day and the SDP recomputed every day as well. Comparing the two figures, we note that the second one only exhibits a slight improvement.

However, suppose we compute the SDP once only, at the beginning of the period, while still adjust the portfolio every day (according to the feedback matrix). The result is shown in Figure 14. In this particular case the tracking performance becomes significantly worse.

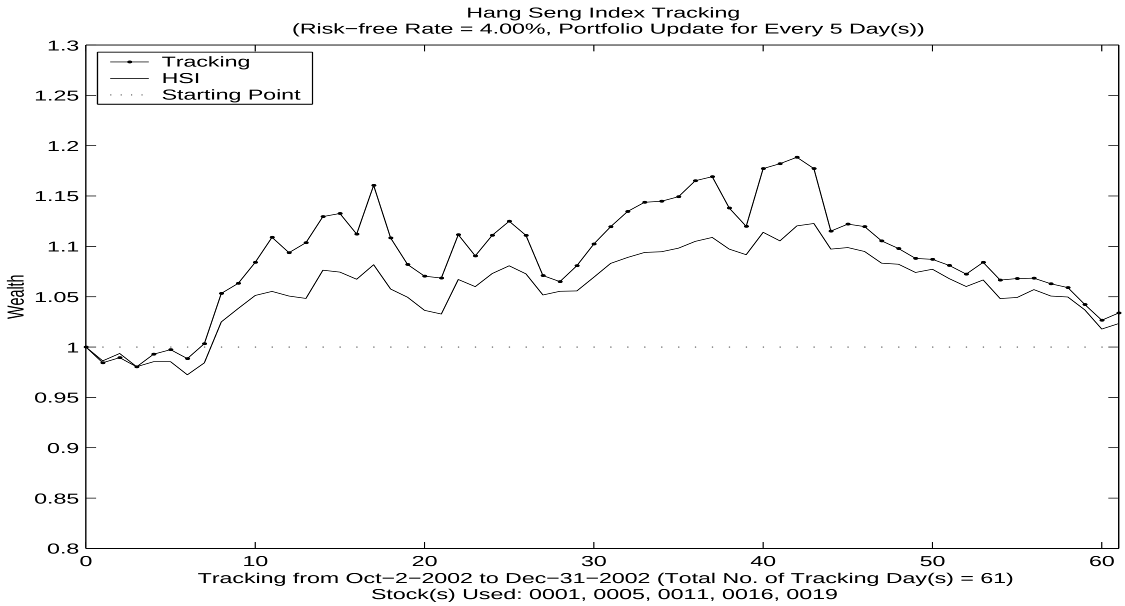

Figure 11: Real-time tracking; large stocks; trade every 5 days.

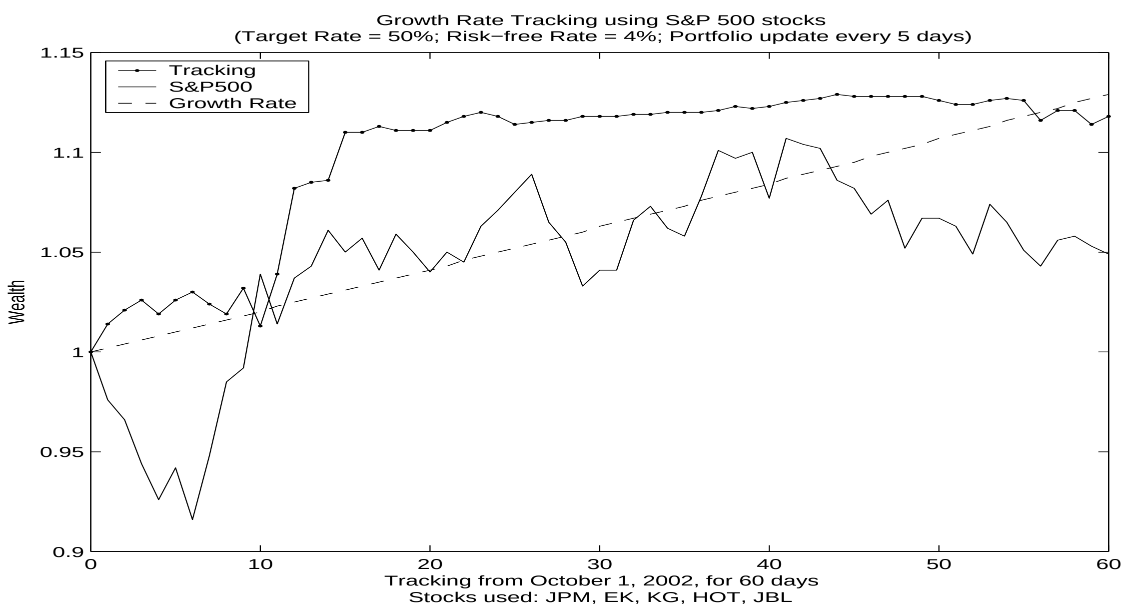

Figure 12: Randomly selected S&P 500 stocks; Period 2; trade every 5 days.

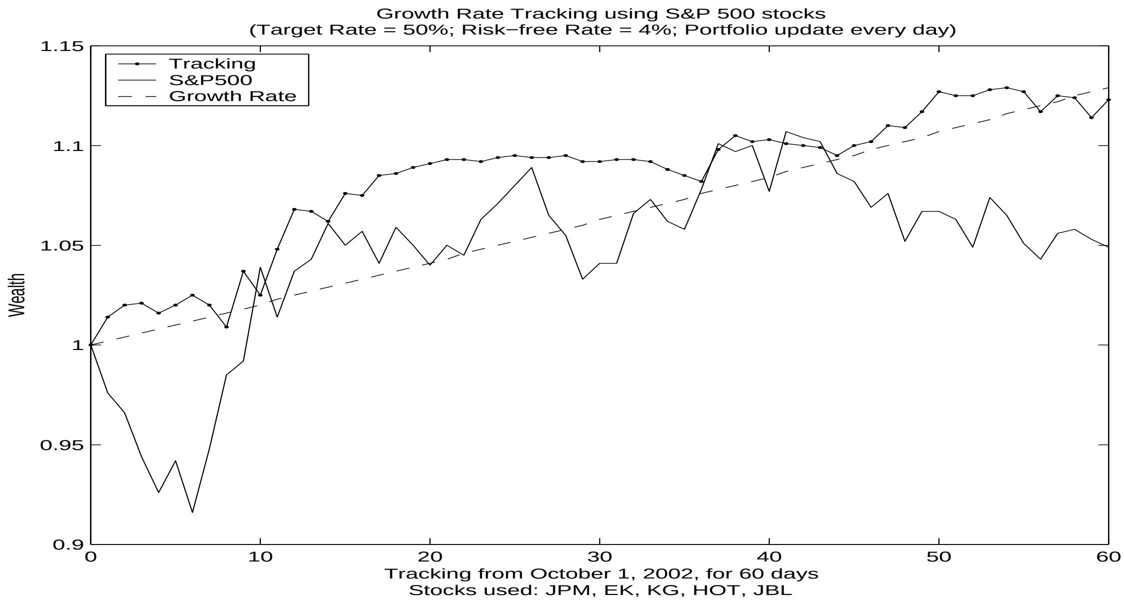

Figure 13: Randomly selected S&P 500 stocks; Period 2; trade every day.

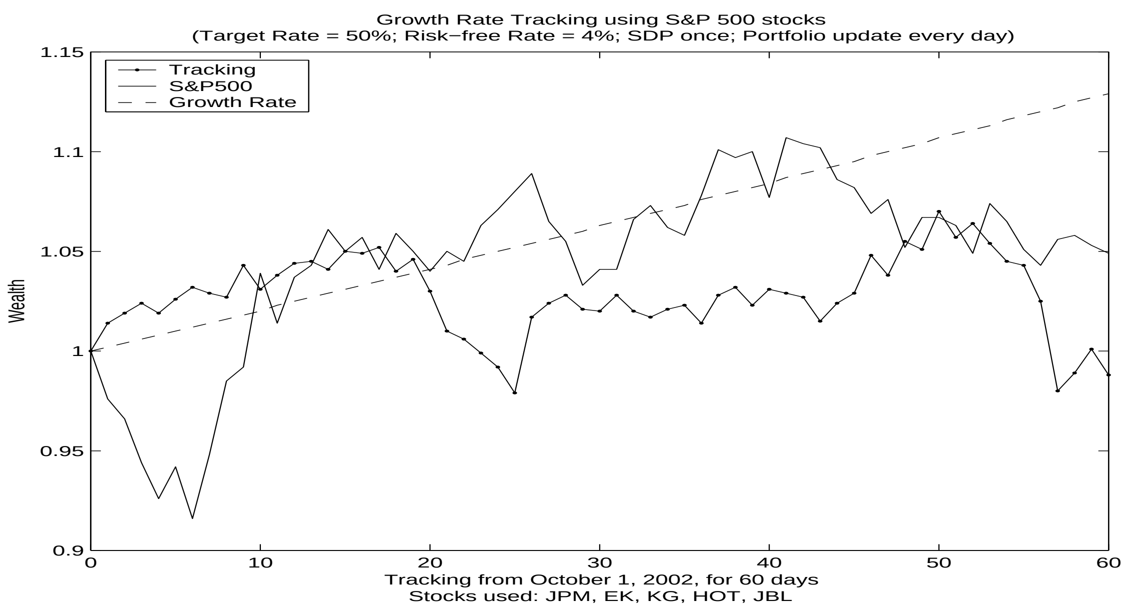

Figure 14: The same parameters as Figure 13, except that the feedback matrix computed only at the beginning, but trade every day.

5.5 Discussions

Several insights can be garnered from the examples illustrated above. First, the optimal feedback control obtained from the SDP solution exhibits a good level of "intelligence" in discerning what stocks to long and what to short, by how much and when. Second, these decisions appear to be rather insensitive to the estimated parameters (the mean and covariance matrices) that are fed into the model. This insensitivity is intuitively appealing: on the one hand, it is widely accepted wisdom that the past history of any stock is not indicative of its future performance; on the other hand, the feedback control law, being a linear function of the state, assures that any dependence on parametric changes is smooth and gradual. This insensitivity is further enhanced by the inclusion into the objective function the discount factor, which dampens reliance on the parameter estimation (from the past) and sharpens the focus on the immediate future.

The last example in §5.4 sheds more light into this issue. It brings out the importance of updating the SDP more often during the tracking period. A closer examination of the five stocks in the portfolio shows that three of them, JPM, EK and JBL, all have a sharp downward trend before the tracking period, while moving substantially up during the tracking period. Updating the SDP from time to time during the tracking period allows timely detection of the new trend and triggers consequent adjustments in the feedback control. In contrast, a single run of the SDP yields a control law that is based on the estimated parameters over a period in which the three stocks behave very differently from the tracking period, and thus results in a sub-par tracking performance.

Third, although there is no constraint on the short position in our model, the amount of shorting needed appears to be quite modest. Indeed, in all of the examples we have examined, including portfolios of randomly selected stocks and required growth rates as high as 50% (per annum), there is not a single case in which the short/long ratio gets even close to 100%; in most cases, this ratio is well below 50%. Of course, one can always construct extreme cases so that the short/long ratio becomes excessive. (For instance, by picking stocks that perform uniformly poorly over the tracking period and requiring unrealistically high growth rate.) This, however, should not be a concern when the model is used properly as a reference or study tool (as opposed to a trading tool). For instance, a fund manager can run the model on his/her portfolio with a growth rate hypothetically set, and then determine whether the resulting short/long ratio is suitable or not.

6 Conclusions

We have presented in this paper a new approach to track either a market index or a constant growth rate using a small number of stocks. The approach is to formulate a stochastic linear quadratic control problem, and to generate the optimal feedback control by means of semidefinite programming.

Numerical examples based on both market data and simulation have shown that our SLQ-via-SDP approach is a theoretically sound, computationally efficient and easy-to-use method. The examples have also demonstrated that the tracking performance appears to be independent of whether the market is up or down, and independent of which stocks are used to track the benchmark; and the required leverage, in terms of the short/long ratio, is quite modest. As the SDP can be efficiently re-solved and the optimal control updated frequently over the tracking period, the model does not require a heavy reliance on parametric estimation based on past data; instead, it focuses on trying to capture the dynamical changes of the asset prices over the tracking period and react accordingly.

On the other hand, any implementation of the model can only update the feedback control at discretized time intervals (as opposed to continuously), and this inevitably incurs sub-optimality. The tradeoff between updating frequency and optimality is an issue that warrants further study, both numerically and analytically. Finally, adding constraints to the short position or the overall wealth will lead to a new class of control problems outside of the realm of the classical SLQ theory. This also calls for new approaches, in both theory and computation.

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