Complex Quadratic Optimization and Semidefinite Programming

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Abstract

In this paper we study the approximation algorithms for a class of discrete quadratic optimization problems in the Hermitian complex form. A special case of the problem that we study corresponds to the max-3-cut model used in a recent paper of Goemans and Williamson. We first develop a closed-form formula to compute the probability of a complex-valued normally distributed bivariate random vector to be in a given angular region. This formula allows us to compute the expected value of a randomized (with a specific rounding rule) solution based on the optimal solution of the complex SDP relaxation problem. In particular, we present an $[m^2(1-\cos\frac{2\pi}{m})/8\pi]$-apprxomation algorithm, and then study the limit of that model, in which the problem remains NP-hard. We show that if the objective is to maximize a positive semidefinite Hermitian form, then the randomization-rounding procedure guarantees a worst-case performance ratio of $\pi/4 \approx 0.7854$, which is better than the ratio of $2/\pi \approx 0.6366$ for its counter-part in the real case due to Nesterov. Furthermore, if the objective matrix is real-valued positive semidefinite with non-positive off-diagonal elements, then the performance ratio improves to 0.9349.

Keywords: Hermitian quadratic functions, approximation ratio, randomized algorithms, complex SDP relaxation.

MSC subject classification: 90C20, 90C22.

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1 Introduction

The pioneering work of Goemans and Williamson [8] has caused a great deal of excitement in the field of optimization, as it used a new tool (SDP) in continuous optimization, through randomization and probabilistic analysis, to yield an excellent approximation ratio for a classical combinatorial optimization problem, known as the *max-cut* problem. This ground-breaking work has been extended in various ways since its first appearance. Among others, Frieze and Jerrum [6] extended the method to solve the general max-k-cut problem. Bertsimas and Ye [4] introduced another randomization scheme using normal distributions, to achieve the same approximation result as in Goemans and Williamson's original paper [8]. The Bertsimas-Ye analysis makes use of an important result in statistics, which states that the probability of a bivariate (2-dimensional) normally distributed random vector to be in the first orthant can be expressed analytically using elementary functions. This is impossible however, for any dimension higher than three; see [1]. Recently, Goemans and Williamson [9] proposed another novel approach to solve the max-3-cut problem, using the unit circle in the complex plane as a key modelling ingredient. In this paper we show that it is possible to compute the probability of the bivariate *complex-valued* normally distributed random vector to be in a specific angular region in $\mathbf{C}^2$ (see Section 2). We then consider the following quadratic optimization problem in complex variables: maximizing $z^{\mathrm{H}}Qz$, subject to $z_k^m = 1$, where z_k is a complex variable and is the k-th component of the vector z, and $m \geq 2$ is an integer parameter of the model. Thanks to the new probability result developed in Section 2, we are able to compute the expected quality of a particular randomized solution for solving the above quadratic optimization model. The model of Goemans and Williamson for max-3-cut ($m = 3$) turns out to be a special case of this general model. It is interesting to study the limit of this model; that is, the case where $m \to \infty$, and the constraints become $|z_k| = 1$. It turns out that the problem remains NP-hard. However, the corresponding complex SDP relaxation yields an approximation ratio of $\pi/4 \approx 0.7854$, whereas for its counter-part in the real case, the ratio is $2/\pi \approx 0.6366$ as shown by Nesterov [11]. If the off-diagonal elements of the objective matrix are real-valued and non-positive, then the approximation ratio is actually 0.9349.

This paper is organized as follows. In Section 2 we discuss the computation of the probability for the complex-valued normal distributions. In Section 3 we apply the results developed in Section 2 to solve complex-valued quadratic optimization problems. In particular, Subsection 3.1 discusses the Hermitian quadratic function maximization problem, where the complex decision variables take discrete values. Subsection 3.2 presents an approximation algorithm for the problem. Subsection 3.3 considers the continuous version of the problem. Subsection 3.4 considers a special case where a sign restriction on the objective matrix is observed. Finally, we conclude the paper in Section 4.

Notation. Throughout, we denote by $\bar{a}$ the conjugate of a complex number a, by $\mathbf{C}^n$ the space of n-dimensional complex vectors. For a given vector $z \in \mathbf{C}^n$, z^{H} denotes the conjugate transpose of

z. The space of $n \times n$ real symmetric and complex Hermitian matrices are denoted by $\mathcal{S}^n$ and $\mathcal{H}^n$, respectively. For a matrix $Z \in \mathcal{H}^n$, we write Re Z and Im Z for the real and imaginary part of Z, respectively. Matrix Z being Hermitian implies that Re Z is symmetric and Im Z is skew-symmetric. We denote by $\mathcal{S}^n_+$ ($\mathcal{S}^n_{++}$) and $\mathcal{H}^n_+$ ($\mathcal{H}^n_{++}$) the cones of real symmetric positive semidefinite (positive definite) and complex Hermitian positive semidefinite (positive definite) matrices, respectively. The notation $Z \succeq (\succ 0)$ means that Z is positive semidefinite (positive definite). For two complex matrices Y and Z, their inner product $Y \bullet Z = \text{Re }(\text{tr } Y^{\text{H}} Z) = \text{tr } \left[(\text{Re } Y)^{\text{T}}(\text{Re } Z) + (\text{Im } Y)^{\text{T}}(\text{Im } Z)\right]$, where tr denotes the trace of a matrix and $^{\text{T}}$ denotes the transpose of a matrix.

2 Complex Bivariate Normal Distribution

It is well known that the density function of an n-dimensional real-valued multivariate normal distribution is given as follows

$$f(x) = \frac{1}{(2\pi)^{n/2}\sqrt{\det \Omega}} \exp\left(-\frac{1}{2}(x-\mu)^{\text{T}}\Omega^{-1}(x-\mu)\right),$$

where $\mu \in \Re^n$ is the mean and $\Omega \in \mathcal{S}^n_{++}$ is the covariance matrix.

Let us consider a complex-valued normally distributed random variable in $\mathbf{C}$, with the mean value $z_0 \in \mathbf{C}$ and variance $\sigma^2 \in \Re_+$. (For more information on the complex-valued normal distributions, we refer the reader to [2]). Similar as in the real-valued case, its density function can be written as

$$f(z) = \frac{1}{\pi\sigma^2} \exp\left(-|z-z_0|^2/\sigma^2\right), \; z \in \mathbf{C}.$$

We denote the complex-valued normal distribution by $\mathcal{N}_c(z_0, \sigma^2)$ with mean z_0 and variance σ^2.

Using Euler's formula, i.e., letting $z - z_0 = \rho e^{\boldsymbol{i}\theta}$, we have

$$f(\rho, \theta) = \frac{\rho}{\pi\sigma^2} \exp\left(-\frac{\rho^2}{\sigma^2}\right), \text{ with } (\rho,\theta) \in [0,+\infty) \times [0, 2\pi),$$

where the variable transformation is

$$\begin{cases} \text{Re }(z - z_0) &= \rho\cos\theta \\ \text{Im }(z - z_0) &= \rho\sin\theta. \end{cases}$$

As a matter of terminology, ρ is usually called the modulus of $z - z_0$, also denoted as $|z - z_0|$; θ is called the argument of $z - z_0$, denoted as Arg $(z - z_0)$.

The density of the joint (complex-valued) normal distribution $z = (z_1, z_2, ..., z_n)$, with $z_k \in \mathbf{C}$, $k = 1, ..., n$, has the following form

$$f(z) = \frac{1}{(\pi)^n \det \Omega} \exp\left(-(z-\mu)^{\text{H}}\Omega^{-1}(z-\mu)\right),$$

where $z, \mu \in \mathbf{C}^n$, and $\Omega \in \mathcal{H}_{++}^n$; μ is the mean vector, and Ω is the covariance matrix.

Let us denote the above complex-valued normal distribution as $\mathcal{N}_c(\mu, \Omega)$.

The bivariate case is of particular interest to us. Consider a complex-valued, bivariate normal random vector. Suppose that it has zero-mean. Furthermore, suppose that its covariance matrix is

$$\Omega = \begin{bmatrix} 1 & \lambda \\ \bar{\lambda} & 1 \end{bmatrix} \succ 0$$

where $\bar{\lambda} \in \mathbf{C}$ denotes the conjugate of $\lambda \in \mathbf{C}$. In particular, let $\lambda = \gamma e^{\boldsymbol{i}\alpha}$, and so $\bar{\lambda} = \gamma e^{-\boldsymbol{i}\alpha}$. Since $\Omega \succ 0$, it follows that $1 - \gamma^2 > 0$. Moreover,

$$\Omega^{-1} = \frac{1}{1-\gamma^2} \begin{bmatrix} 1 & -\gamma e^{\boldsymbol{i}\alpha} \\ -\gamma e^{-\boldsymbol{i}\alpha} & 1 \end{bmatrix}.$$

Then, by letting $z_1 = \rho_1 e^{\boldsymbol{i}\theta_1}$ and $z_2 = \rho_2 e^{\boldsymbol{i}\theta_2}$, we may rewrite the density function as

$$\begin{aligned} f(\rho_1, \rho_2, \theta_1, \theta_2) &= \frac{\rho_1 \rho_2}{\pi^2(1-\gamma^2)} \exp\left(-\frac{1}{1-\gamma^2} \begin{bmatrix} \rho_1 e^{\boldsymbol{i}\theta_1} \\ \rho_2 e^{\boldsymbol{i}\theta_2} \end{bmatrix}^{\mathrm{H}} \begin{bmatrix} 1 & -\gamma e^{\boldsymbol{i}\alpha} \\ -\gamma e^{-\boldsymbol{i}\alpha} & 1 \end{bmatrix} \begin{bmatrix} \rho_1 e^{\boldsymbol{i}\theta_1} \\ \rho_2 e^{\boldsymbol{i}\theta_2} \end{bmatrix} \right) \\ &= \frac{\rho_1 \rho_2}{\pi^2(1-\gamma^2)} \exp\left(-\frac{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\gamma\cos(\alpha + \theta_2 - \theta_1)}{1-\gamma^2} \right), \end{aligned}$$

where the domain of the variables is given as

$$(\rho_1, \rho_2, \theta_1, \theta_2) \in [0, +\infty)^2 \times [0, 2\pi)^2.$$

Now let us further introduce a variable transformation

$$\begin{cases} \rho_1 &= \rho\cos\xi \\ \rho_2 &= \rho\sin\xi \end{cases}$$

with the domain $(\rho, \xi) \in [0, +\infty) \times [0, \pi/2]$. The density function can be further written as

$$\begin{aligned} f(\rho, \xi, \theta_1, \theta_2) &= \frac{\rho^3 \cos\xi \sin\xi}{\pi^2(1-\gamma^2)} \exp\left(-\frac{\rho^2 - 2\gamma\rho^2\cos\xi\sin\xi\cos(\alpha + \theta_2 - \theta_1)}{1-\gamma^2} \right) \\ &= \frac{\rho^3 \sin 2\xi}{2\pi^2(1-\gamma^2)} \exp\left(-\frac{\rho^2 - \rho^2\gamma\sin 2\xi\cos(\alpha + \theta_2 - \theta_1)}{1-\gamma^2} \right), \end{aligned}$$

and the domain is $(\rho, \xi, \theta_1, \theta_2) \in [0, +\infty) \times [0, \pi/2] \times [0, 2\pi)^2$.

Consider $0 \le \theta_1^b < \theta_1^e \le 2\pi$ and $0 \le \theta_2^b < \theta_2^e \le 2\pi$. Below we shall compute the probability that $(\theta_1, \theta_2) \in [\theta_1^b, \theta_1^e] \times [\theta_2^b, \theta_2^e]$.

Let us denote

$$
\begin{aligned}
P: &= \text{Prob}\,\{\theta_1^b \le \theta_1 \le \theta_1^e;\ \theta_2^b \le \theta_2 \le \theta_2^e\} \\
&= \int_{\theta_1^b}^{\theta_1^e}\int_{\theta_2^b}^{\theta_2^e}\int_0^{\pi/2}\left[\int_0^{\infty}\frac{\rho^3\sin 2\xi}{2\pi^2(1-\gamma^2)}\exp\left(-\frac{\rho^2-\rho^2\gamma\sin 2\xi\cos(\alpha+\theta_2-\theta_1)}{1-\gamma^2}\right)d\rho\right]d\xi d\theta_2 d\theta_1
\end{aligned}
$$

To compute the above integration, we note the following facts:

Lemma 2.1 *(i) For a given $a>0$, it holds that*

$$\int_0^{\infty}\rho^3\exp(-a\rho^2)d\rho = \frac{1}{2a^2}.$$

(ii) Suppose that $-1<b<1$ is a given real number. Then, with respect to the variable θ, it holds that

$$\int\frac{\sin\theta}{(1-b\sin\theta)^2}d\theta = -\frac{\cos\theta}{(1-b^2)(1-b\sin\theta)}+\frac{2b}{(1-b^2)^{3/2}}\arctan\frac{\tan(\theta/2)-b}{\sqrt{1-b^2}}+C.$$

(iii) With respect to the variable θ, it holds that

$$\int\left[\frac{1}{1-\gamma^2\cos^2(\theta)}+\frac{\gamma\cos\theta\arccos(-\gamma\cos\theta)}{(1-\gamma^2\cos^2(\theta))^{3/2}}\right]d\theta = \frac{1}{1-\gamma^2}\left(\theta+\frac{\gamma\sin\theta\arccos(-\gamma\cos\theta)}{\sqrt{1-\gamma^2\cos^2(\theta)}}\right)+C.$$

(iv) With respect to the variable θ, it holds that

$$\int\left[\frac{\gamma\sin(\beta-\theta)\arccos(-\gamma\cos(\theta-\beta))}{\sqrt{1-\gamma^2\cos^2(\theta-\beta)}}\right]d\theta = \frac{1}{2}\left(\arccos(-\gamma\cos(\theta-\beta))\right)^2+C.$$

Part (i) of the lemma is straightforward, and the rest of the lemma can be readily verified by differentiation.

Applying Lemma 2.1 (i) and (ii), we get

$$
\begin{aligned}
P &= \frac{1}{4\pi^2(1-\gamma^2)}\int_{\theta_1^b}^{\theta_1^e}\int_{\theta_2^b}^{\theta_2^e}\left[\int_0^{\pi/2}\sin 2\xi\left(\frac{1-\gamma^2}{1-\gamma\sin 2\xi\cos(\alpha+\theta_2-\theta_1)}\right)^2 d\xi\right]d\theta_2 d\theta_1 \\
&= \frac{1-\gamma^2}{4\pi^2}\int_{\theta_1^b}^{\theta_1^e}\int_{\theta_2^b}^{\theta_2^e}\left[\int_0^{\pi/2}\frac{\sin 2\xi}{(1-\gamma\cos(\alpha+\theta_2-\theta_1)\sin 2\xi)^2}d\xi\right]d\theta_2 d\theta_1 \\
&= \frac{1-\gamma^2}{4\pi^2}\int_{\theta_1^b}^{\theta_1^e}\int_{\theta_2^b}^{\theta_2^e}\left[\frac{1}{1-\gamma^2\cos^2(\alpha+\theta_2-\theta_1)}+\right. \\
&\quad\left.+\frac{\gamma\cos(\alpha+\theta_2-\theta_1)\arccos\left(-\gamma\cos(\alpha+\theta_2-\theta_1)\right)}{(1-\gamma^2\cos^2(\alpha+\theta_2-\theta_1))^{3/2}}\right]d\theta_2 d\theta_1.
\end{aligned}
$$

Using Lemma 2.1 (iii) we obtain

$$P = \frac{1}{4\pi^2}\left[(\theta_1^e-\theta_1^b)(\theta_2^e-\theta_2^b)+\int_{\theta_1^b}^{\theta_1^e}\frac{\gamma\sin(\theta_2^e+\alpha-\theta_1)\arccos(-\gamma\cos(\theta_2^e+\alpha-\theta_1))}{\sqrt{1-\gamma^2\cos^2(\theta_2^e+\alpha-\theta_1)}}d\theta_1 - \int_{\theta_1^b}^{\theta_1^e}\frac{\gamma\sin(\theta_2^b+\alpha-\theta_1)\arccos(-\gamma\cos(\theta_2^b+\alpha-\theta_1))}{\sqrt{1-\gamma^2\cos^2(\theta_2^b+\alpha-\theta_1)}}d\theta_1\right],$$

and further using Lemma 2.1 (iv), we have

$$P = \frac{(\theta_1^e-\theta_1^b)(\theta_2^e-\theta_2^b)}{4\pi^2}+\frac{1}{8\pi^2}\left[(\arccos(-\gamma\cos(\theta_1^e-\theta_2^e-\alpha)))^2-\Big(\arccos(-\gamma\cos(\theta_1^b-\theta_2^e-\alpha))\Big)^2 + \Big(\arccos(-\gamma\cos(\theta_1^b-\theta_2^b-\alpha))\Big)^2-\Big(\arccos(-\gamma\cos(\theta_1^e-\theta_2^b-\alpha))\Big)^2\right].$$

Summarizing, we have proven the following result by a limiting argument.

Theorem 2.2 *For the complex-value bivariate normal random vector* $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathcal{N}_c(\mu,\Omega)$ *with*

$$\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \Omega = \begin{bmatrix} 1 & \gamma e^{\boldsymbol{i}\alpha} \\ \gamma e^{-\boldsymbol{i}\alpha} & 1 \end{bmatrix} \in \mathcal{H}_+^2,$$

it holds that

$$\begin{aligned}&\text{Prob}\,\{\theta_1^b \le \text{Arg } z_1 \le \theta_1^e;\ \theta_2^b \le \text{Arg } z_2 \le \theta_2^e\}\\ =\ & \frac{(\theta_1^e-\theta_1^b)(\theta_2^e-\theta_2^b)}{4\pi^2}+\frac{1}{8\pi^2}\left[(\arccos(-\gamma\cos(\theta_1^e-\theta_2^e-\alpha)))^2-\Big(\arccos(-\gamma\cos(\theta_1^b-\theta_2^e-\alpha))\Big)^2 + \Big(\arccos(-\gamma\cos(\theta_1^b-\theta_2^b-\alpha))\Big)^2-\Big(\arccos(-\gamma\cos(\theta_1^e-\theta_2^b-\alpha))\Big)^2\right].\end{aligned}$$

3 Quadratic Programs and Complex SDP Formulations

3.1 Discrete Complex Quadratic Optimization

Suppose that Q is a Hermitian matrix. Consider the following quadratic programming problem with discrete decision variables,

$$\begin{array}{lll}(\text{P}) & \max & z^{\text{H}}Qz \\ & \text{s.t.} & z_k \in \{1,\omega,\dots,\omega^{m-1}\},\ k=1,\dots,n,\end{array}$$

where $m \ge 2$ and $\omega = e^{\boldsymbol{i}\frac{2\pi}{m}} = \cos\frac{2\pi}{m} + \boldsymbol{i}\sin\frac{2\pi}{m}$. As we shall see later that this model is an extension of the Goemans and Williamson's model for solving the max-3-cut problem; see [9].

Denote the optimal value of (P) to be $v(P)$. Consider the following complex-valued mapping F_m

$$F_m(z) := \frac{m(2-\omega^{-1}-\omega)}{8\pi^2}\sum_{j=0}^{m-1}\omega^j(\arccos(-\text{Re }(\omega^{-j}z)))^2.$$

For a Hermitian matrix Z with $|Z_{kl}| \leq 1$ for all k, l, define the componentwise matrix function

$$F_m(Z) := (F_m(Z_{kl}))_{n\times n} \in \mathcal{H}^n.$$

It is easy to verify that $F_m(\bar{z}) = \overline{F_m(z)}$. Therefore, if Z is Hermitian, then so is $F_m(Z)$.

Lemma 3.1 *We have*

$$1 = \frac{m(2-\omega^{-1}-\omega)}{8\pi^2}\sum_{j=0}^{m-1}\omega^j(\arccos(-\text{Re }(\omega^{-j})))^2.$$

Moreover, $F_m(z) = z$ *for any* $z \in \{1, \omega, \ldots, \omega^{m-1}\}$.

Proof. We observe that

$$\begin{aligned}
& \frac{m(2-\omega^{-1}-\omega)}{8\pi^2}\sum_{j=0}^{m-1}\omega^j(\arccos(-\cos(\frac{j}{m}2\pi)))^2 \\
= & \frac{m(2-\omega^{-1}-\omega)}{8\pi^2}\sum_{j=0}^{m-1}\omega^j\pi^2(1-\frac{2j}{m})^2 \\
= & \frac{2-\omega^{-1}-\omega}{8m}\left(4\sum_{j=0}^{m-1}j^2\omega^j - 4m\sum_{j=0}^{m-1}j\omega^j\right). \qquad (1)
\end{aligned}$$

Moreover, we have

$$\sum_{j=0}^{m-1}j^2\omega^j = \frac{m^2(\omega-1)-2m\omega}{(\omega-1)^2} \text{ and } \sum_{j=0}^{m-1}j\omega^j = \frac{m}{\omega-1}.$$

Substituting the above equations into (1) yields the intended result.

Suppose $z = \omega^{j_0}$ for some $j_0 \in \{1, \dots, n\}$. Then,

$$\begin{aligned}
& \frac{m(2-\omega^{-1}-\omega)}{8\pi^2} \sum_{j=0}^{m-1} \omega^j (\arccos(-\mathrm{Re}\ (\omega^{-j} z)))^2 \\
= \;& \frac{m(2-\omega^{-1}-\omega)}{8\pi^2} \sum_{j=0}^{m-1} \omega^j (\arccos(-\cos(\frac{j_0 - j}{m} 2\pi)))^2 \\
= \;& \frac{m(2-\omega^{-1}-\omega)}{8\pi^2} \sum_{j=0}^{m-1} \omega^j (\arccos(-\cos(\frac{j - j_0}{m} 2\pi)))^2 \\
= \;& \omega^{j_0} \frac{m(2-\omega^{-1}-\omega)}{8\pi^2} \sum_{j=-j_0}^{m-1-j_0} \omega^j (\arccos(-\cos(\frac{j}{m} 2\pi)))^2 \\
= \;& \omega^{j_0} = z.
\end{aligned}$$

This completes the proof for Lemma 3.1. □

Hence we can rewrite (P) as

$$\begin{array}{ll}
\max & Q \bullet F_m(zz^{\mathrm{H}}) \\
\text{s.t.} & z_k \in \{1, \omega, \dots, \omega^{m-1}\},\ k = 1, \dots, n.
\end{array}$$

Consider the following nonlinear complex semidefinite programming problem

$$\begin{array}{lll}
\text{(SP)} & \max & Q \bullet F_m(Z) \\
& \text{s.t.} & Z_{kk} = 1,\ k = 1, \dots, n, \\
& & Z \succeq 0.
\end{array}$$

Let $v(SP)$ denote the optimal value of (SP).

Theorem 3.2 *It holds that $v(P) = v(SP)$.*

Proof. Let $\hat{z}$ is optimal to (P), then, by Lemma 3.1, $\hat{Z} = \hat{z}\hat{z}^{\mathrm{H}}$ is a feasible solution to (SP) and $F_m(\hat{Z}) = \hat{Z}$. Therefore, $v(SP) \geq Q \bullet F_m(\hat{Z}) = Q \bullet \hat{Z} = v(P)$.

On the other hand, for every feasible solution Z of (SP), we randomly generate a complex vector ξ

such that $\xi \in \mathcal{N}_c(0, Z)$, and assign

$$z_k = \sigma(\xi_k) = \begin{cases} 1, & \text{if Arg } \xi_k \in [0, \frac{1}{m}2\pi) \\ \omega, & \text{if Arg } \xi_k \in [\frac{1}{m}2\pi, \frac{2}{m}2\pi) \\ \vdots & \\ \omega^j, & \text{if Arg } \xi_k \in [\frac{j}{m}2\pi, \frac{j+1}{m}2\pi) \\ \vdots & \\ \omega^{m-1}, & \text{if Arg } \xi_k \in [\frac{m-1}{m}2\pi, 2\pi) \end{cases} \tag{2}$$

for $k = 1, \ldots, n$. Suppose that $Z_{kl} = \gamma e^{\boldsymbol{i}\alpha}$. Then by Theorem 2.2, we have

$$\begin{aligned} & \text{Prob}\{z_k = z_l\omega^j, z_l = \omega^r\} \\ = \; & \text{Prob}\{z_k = \omega^{j+r}, z_l = \omega^r\} \\ = \; & \text{Prob}\left\{\text{Arg } \xi_k \in [\frac{j+r}{m}2\pi, \frac{j+r+1}{m}2\pi), \text{Arg } \xi_l \in [\frac{r}{m}2\pi, \frac{r+1}{m}2\pi)\right\} \\ = \; & \frac{1}{m^2} + \frac{1}{8\pi^2}\Bigg(2(\arccos(-\gamma\cos(\frac{j}{m}2\pi - \alpha)))^2 - (\arccos(-\gamma\cos(\frac{j-1}{m}2\pi - \alpha)))^2 \\ & -(\arccos(-\gamma\cos(\frac{j+1}{m}2\pi - \alpha)))^2\Bigg) \end{aligned}$$

for any $j, r \in \{0, 1, \ldots, m-1\}$. Therefore, for any given k and l we have

$$\begin{aligned} & \text{Prob}\{z_k\bar{z}_l = \omega^j\} \\ = \; & \sum_{r=0}^{m-1} \text{Prob}\{z_k = z_l\omega^j, z_l = \omega^r\} \\ = \; & \frac{1}{m} + \frac{m}{8\pi^2}(2(\arccos(-\gamma\cos(\frac{j}{m}2\pi - \alpha)))^2 - (\arccos(-\gamma\cos(\frac{j-1}{m}2\pi - \alpha)))^2 \\ & -(\arccos(-\gamma\cos(\frac{j+1}{m}2\pi - \alpha)))^2). \end{aligned} \tag{3}$$

It follows that

$$
\begin{aligned}
& \mathsf{E}[z_k \bar{z}_l] \\
= \; & \sum_{j=0}^{m-1} \omega^j \text{Prob}\{z_k \bar{z}_l = \omega^j\} \\
= \; & \frac{m}{8\pi^2} \sum_{j=0}^{m-1} \omega^j \Bigg(2(\arccos(-\gamma \cos(\frac{j}{m} 2\pi - \alpha)))^2 - (\arccos(-\gamma \cos(\frac{j-1}{m} 2\pi - \alpha)))^2 \\
& - (\arccos(-\gamma \cos(\frac{j+1}{m} 2\pi - \alpha)))^2 \Bigg) \\
= \; & \frac{m}{8\pi^2} \sum_{j=0}^{m-1} (2\omega^j - \omega^{j-1} - \omega^{j+1})(\arccos(-\gamma \cos(\frac{j}{m} 2\pi - \alpha)))^2 \\
= \; & \frac{m(2 - \omega^{-1} - \omega)}{8\pi^2} \sum_{j=0}^{m-1} \omega^j (\arccos(-\gamma \cos(\frac{j}{m} 2\pi - \alpha)))^2 \\
= \; & \frac{m(2 - \omega^{-1} - \omega)}{8\pi^2} \sum_{j=0}^{m-1} \omega^j (\arccos(-\text{Re } (\omega^{-j} Z_{kl})))^2. \qquad (4)
\end{aligned}
$$

By the linearity of mathematical expectation, we get

$$\mathsf{E}[z^{\mathrm{H}} Q z] = Q \bullet F_m(Z).$$

Since the solution z so generated is feasible to (P), we have

$$
\begin{aligned}
v(P) \; & \geq \; \mathsf{E}[z^{\mathrm{H}} Q z] \\
& = \; Q \bullet Z,
\end{aligned}
$$

for every feasible solution Z of (SP). This combining with $v(SP) \geq v(P)$ yields the desired result. □

In particular, if $m = 2$ then one can verify that problem (P) reduces to

$$
\begin{aligned}
\max \quad & x^{\mathrm{T}} Q x \\
\text{s.t.} \quad & x_k \in \{\pm 1\}, \; k = 1, \ldots, n,
\end{aligned}
$$

and problem (SP) reduces to

$$
\begin{aligned}
\max \quad & \tfrac{2}{\pi} Q \bullet \arcsin(X) \\
\text{s.t.} \quad & X_{kk} = 1, \; k = 1, \ldots, n, \\
& X \succeq 0,
\end{aligned}
$$

where $\arcsin(X) := [\arcsin(X_{kl})]_{n \times n}$. In that case, Theorem 3.2 specializes to Theorem 2.9 in Goemans and Williamson [8] or Theorem 1 in Zhang [15]. If $m = 3$, then (P) is

$$
\begin{aligned}
\max \quad & z^{\mathrm{H}} Q z \\
\text{s.t.} \quad & z_k \in \{1, \omega, \omega^2\}, \; k = 1, \ldots, n,
\end{aligned}
$$

with $\omega = e^{\boldsymbol{i}\frac{2\pi}{3}}$. In fact, Goemans and Williamson [9] model the max-3-cut problem as

$$\begin{array}{lll} \text{(M3C)} & \max & \sum_{1\le k<l\le n} w_{kl}(z_k - z_l)^{\mathrm{H}}(z_k - z_l) \\ & \text{s.t.} & z_k = \{1, \omega, \omega^2\},\ k = 1, \dots, n, \end{array}$$

and they consider the following complex SDP relaxation

$$\begin{array}{ll} \max & \sum_{1\le k<l\le n} w_{kl}(2 - 2\mathrm{Re}\ Z_{kl}) \\ \text{s.t.} & Z_{kk} = 1,\ k = 1, \dots, n \\ & \mathrm{Re}\ Z_{kl} \ge -1/2,\ \mathrm{Re}\ \omega Z_{kl} \ge -1/2,\ \mathrm{Re}\ \omega^2 Z_{kl} \ge -1/2,\ 1 \le k < l \le n \\ & Z \succeq 0. \end{array}$$

Let the optimal solution of the SDP relaxation be Z^*. Then, Theorem 3.2 asserts that the expected value of the randomized solution based on Z^* is

$$\sum_{1\le k<l\le n} w_{kl}(2 - 2\mathrm{Re}\ F_3(Z^*_{kl}))$$

where $F_3(z) = \frac{9}{8\pi^2}\left[(\arccos(-\mathrm{Re}\ z))^2 + \omega(\arccos(-\mathrm{Re}\ (\omega^2 z)))^2 + \omega^2(\arccos(-\mathrm{Re}\ (\omega z)))^2\right]$.

Since $(\arccos(x))^2$ is a convex function, it follows that

$$\begin{aligned} \mathrm{Re}\ F_3(Z^*_{kl}) &= \frac{9}{8\pi^2}\left[(\arccos(-\mathrm{Re}\ Z^*_{kl}))^2 - \frac{1}{2}\left((\arccos(-\mathrm{Re}\ (\omega^2 Z^*_{kl})))^2 + (\arccos(-\mathrm{Re}\ (\omega Z^*_{kl})))^2\right)\right] \\ &\le \frac{9}{8\pi^2}\left[(\arccos(-\mathrm{Re}\ Z^*_{kl}))^2 - \left(\arccos\left(-\frac{1}{2}\mathrm{Re}\ \left(\omega Z^*_{kl} + \omega^2 Z^*_{kl}\right)\right)\right)^2\right] \\ &= \frac{9}{8\pi^2}\left[(\arccos(-\mathrm{Re}\ Z^*_{kl}))^2 - (\arccos(\frac{1}{2}\mathrm{Re}\ Z^*_{kl}))^2\right]. \end{aligned}$$

Further noticing that

$$\min_{-\frac{1}{2}\le x<1} \frac{2 + \frac{9}{4\pi^2}\left[\left(\arccos(\frac{x}{2})\right)^2 - (\arccos(-x))^2\right]}{2 - 2x} = 0.8360...$$

the approximation ratio of Goemans and Williamson [9] thus follows from the fact that

$$\begin{aligned} & \sum_{1\le k<l\le n} w_{kl}(2 - 2\mathrm{Re}\ F_3(Z^*_{kl})) \\ \ge\ & \sum_{1\le k<l\le n} w_{kl}\left\{2 - 2\times\frac{9}{8\pi^2}\left[(\arccos(-\mathrm{Re}\ Z^*_{kl}))^2 - (\arccos(\frac{1}{2}\mathrm{Re}\ Z^*_{kl}))^2\right]\right\} \\ \ge\ & 0.836 \times \sum_{1\le k<l\le n} w_{kl}\,(2 - 2\mathrm{Re}\ Z^*_{kl}) \\ \ge\ & 0.836 \times v^*(M3C). \end{aligned}$$

The above analysis is due to Goemans and Williamson [9]. Therefore, in this sense Equation (3) is a generalization of Theorem 1 of [9] and our rounding procedure (2) is an extension of the procedure in Section 5.1 of [9].

3.2 Bounds on the Approximation Ratios

In this subsection, we investigate approximation algorithms for (P) with positive semidefinite Q via complex SDP relaxation.

Consider the following complex SDP relaxation for (P):

$$\begin{array}{lll} \text{(CSDP)} & \max & Q \bullet Z \\ & \text{s.t.} & Z_{kk} = 1,\ k = 1, \dots, n, \\ & & Z \succeq 0. \end{array}$$

Suppose that Z^* is an optimal solution of (CSDP). We draw a random vector $\xi \in \mathcal{N}_c(0, Z^*)$, and generate a feasible solution $z \in \mathbf{C}^n$ of (P) by applying the rounding procedure (2).

In what follows, we wish to establish an approximation ratio $\alpha \in (0, 1]$ for the approximation algorithm, i.e., an α such that

$$\mathsf{E}[Q \bullet zz^{\mathrm{H}}] \geq \alpha(Q \bullet Z^*),$$

for the randomized solution z.

To begin with, we need the following technical lemma, whose proof is given in the appendix of this paper.

Lemma 3.3 *Suppose that $Z \in \mathcal{H}^n$ is positive semidefinite. Then*

$$F_2(Z) \succeq \frac{1}{\pi}(Z + Z^T), \quad \textit{and } F_m(Z) \succeq \frac{m^2(1 - \cos\frac{2\pi}{m})}{8\pi} Z \textit{ for } m \geq 3.$$

Therefore, according to Equation (4) and Lemma 3.3, the expectation of the objective value of z can be estimated as

$$\begin{array}{rcl} \mathsf{E}[Q \bullet zz^{\mathrm{H}}] & = & Q \bullet F_m(Z^*) \\ & \geq & \alpha_m (Q \bullet Z^*), \end{array}$$

where

$$\alpha_m = \begin{cases} \frac{2}{\pi}, & \text{if } m = 2 \\ \frac{m^2(1-\cos\frac{2\pi}{m})}{8\pi}, & \text{if } m \geq 3. \end{cases}$$

Hence we arrive at the approximation ratio α_m for our randomized algorithm for solving (P) ($m \geq 2$). Summarizing, we have the following theorem:

Theorem 3.4 *Suppose that $Q \succeq 0$. Then there holds $\mathsf{E}[Q \bullet zz^H] \geq \alpha_m v(\mathrm{P})$, where z is obtained by the randomized algorithm and $v(\mathrm{P})$ is the optimal value of (P). In particular, $\alpha_3 \geq 0.5371$, $\alpha_4 \geq 0.6366$, $\alpha_5 \geq 0.6873$, $\alpha_{10} \geq 0.7599$, and $\alpha_{100} \geq 0.7851$.*

In the case of $m = 2$, (CSDP) is actually a real SDP problem. According to Lemma 3.3, one asserts that the real version of relaxation problem (CSDP) yields a $\frac{2}{\pi}$-approximation ratio, which is in accordance with the result of Nesterov [11].

Priori to our result in this subsection (Section 3.2), we learned through private communications that So, Zhang and Ye [12] used a very different technique based on Grothendieck's inequality and obtained the same approximation ratio result for the discrete complex quadratic optimization problem (P). We believe that both techniques are interesting and useful in their own rights.

3.3 Continuous Complex Quadratic Optimization

By taking the limit, i.e. $m \to \infty$, the quadratic optimization model (P) becomes

$$\begin{array}{lll} \text{(CP)} & \max & z^{\mathrm{H}} Q z \\ & \text{s.t.} & |z_k| = 1,\ k = 1, \ldots, n, \end{array}$$

where $Q \in \mathcal{H}_+^n$. In that case, the problem is equivalent to

$$\begin{array}{lll} \text{(SCP)} & \max & Q \bullet F(Z) \\ & \text{s.t.} & Z_{kk} = 1,\ k = 1, \ldots, n, \\ & & Z \succeq 0 \end{array}$$

with

$$\begin{aligned} F(z) &:= \lim_{m \to \infty} F_m(z) \\ &= \frac{1}{4\pi} \int_0^{2\pi} e^{\boldsymbol{i}\theta} \left(\arccos(-\gamma \cos(\theta - \alpha)) \right)^2 d\theta \end{aligned}$$

where $\gamma = |z| \leq 1$ and $\alpha = \text{Arg } z$.

The applications of Hermitian quadratic optimization models such as (P) and (CP) can be found, e.g. in Luo, Luo and Kisialiou [10] for applications in signal processing. Although in [10] the minimization version of the problem was considered, from the viewpoint of optimization both formulations are equivalent (see reduction below).

Proposition 3.5 *Problem* (CP) *is strongly NP-hard in general.*

Proof. The optimization problem in the form of

$$\begin{array}{ll} \max & |z^{\mathrm{T}} A z| \\ \text{s.t.} & z_k \in \mathbf{C},\ |z_k| \leq 1,\ k = 1, \ldots, n \end{array}$$

is called *complex programming*, and was shown in [13] to be NP-hard in general. (We thank André Tits for drawing our attention to complex programming.) Problem (CP) is related to complex programming, but they are not the same: the objective in (CP) takes the Hermitian form, and is assumed to be positive semidefinite. The proof for Proposition 3.5 to be presented below is due to Tom Luo of Minnesota University, who sketched this proof to us in a private communication.

As a first step we shall prove that the following problem

$$\begin{array}{ll} \min & z^{\mathrm{H}} Q z \\ \text{s.t.} & |z_k| = 1,\ k = 1, \ldots, n, \end{array}$$

is NP-hard in general, where $Q \in \mathcal{H}^n_+$.

To this end, we consider a reduction from the following strongly NP-complete matrix partition problem (see e.g. [7]); i.e., given a matrix $G = [G_1, ..., G_N] \in \Re^{M \times N}$, decide whether or not a subset of $\{1, ..., N\}$ exists, say I, such that

$$\sum_{k \in I} G_k = \frac{1}{2} \sum_{k=1}^{N} G_k.$$

Let the decision vector be

$$z = (z_0, z_1, \cdots, z_N, z_{N+1}, \cdots, z_{2N})^{\mathrm{T}} \in \mathbf{C}^{2N+1}.$$

Let $n = 2N + 1$, and

$$A := \begin{pmatrix} -e_N & I_N & I_N \\ -\frac{1}{2} G e_N & G & 0_N^{\mathrm{T}} \end{pmatrix} \in \Re^{(M+N) \times n},$$

where $e_N \in \Re^N$ is the vector of all ones. Let $Q := A^{\mathrm{T}} A$.

Next we show that a matrix partition exists is equivalent to the fact that there is $z \in \mathbf{C}^n$ with $|z_k| = 1$ for all k, such that $z^{\mathrm{H}} Q z = 0$. Clearly, $z^{\mathrm{H}} Q z = 0$ is equivalent to $Az = 0$; that is,

$$0 = -z_0 + z_k + z_{N+k},\ k = 1, ..., N \tag{5}$$

$$0 = -\frac{1}{2} \left(\sum_{k=1}^{N} G_k \right) z_0 + \sum_{k=1}^{N} G_k z_k. \tag{6}$$

Let $z_k / z_0 = e^{\boldsymbol{i}\theta_k}$ for $k = 1, ..., 2N$. Using (5) we have

$$\cos\theta_k + \cos\theta_{N+k} = 1 \tag{7}$$

$$\sin\theta_k + \sin\theta_{N+k} = 0 \tag{8}$$

where $k = 1, ..., N$. Equations (7) and (8) imply that $\theta_k \in \{-\pi/3, \pi/3\}$. This in particular means that $\cos\theta_k = \cos\theta_{N+k} = 1/2$ for $k = 1, ..., N$. Since

$$\mathrm{Re} \left(-\frac{1}{2} \left(\sum_{k=1}^{N} G_k \right) + \sum_{k=1}^{N} G_k z_k / z_0 \right) = -\frac{1}{2} \sum_{k=1}^{N} G_k + \sum_{k=1}^{N} G_k \cos\theta_k = 0$$

is always satisfied, (6) is true if and only if

$$\mathrm{Im}\left(-\frac{1}{2}\left(\sum_{k=1}^{N} G_k\right)+\sum_{k=1}^{N} G_k z_k/z_0\right)=\sum_{k=1}^{N} G_k \sin\theta_k=0,$$

which amounts to the existence of a matrix partition.

Let $\lambda_{\max}$ be the maximum eigenvalue of Q. By observing that

$$\begin{array}{ll} \min & z^{\mathrm{H}} Q z \\ \text{s.t.} & |z_k|=1,\ k=1,\ldots,n, \end{array}$$

is equivalent to

$$\begin{array}{ll} \max & z^{\mathrm{H}} (\lambda_{\max} I - Q) z \\ \text{s.t.} & |z_k|=1,\ k=1,\ldots,n, \end{array}$$

where $\lambda_{\max} I - Q \in \mathcal{H}_+^n$, the desired result follows. □

For a given $z \in \mathbf{C}$ with $z=\gamma e^{\boldsymbol{i}\alpha}$ and $|z|=\gamma \le 1$, we have

$$\begin{aligned} F(z) &= \frac{1}{4\pi}\int_0^{2\pi} e^{\boldsymbol{i}\theta}\left(\arccos(-\gamma\cos(\theta-\alpha))\right)^2 d\theta \\ &= \frac{1}{4\pi}e^{\boldsymbol{i}\alpha}\int_0^{2\pi} e^{\boldsymbol{i}\theta}\left(\arccos(-\gamma\cos\theta)\right)^2 d\theta \\ &= \frac{1}{4\pi}e^{\boldsymbol{i}\alpha}\left[\int_0^{\pi} e^{\boldsymbol{i}\theta}\left(\arccos(-\gamma\cos\theta)\right)^2 d\theta - \int_0^{\pi} e^{\boldsymbol{i}\theta}\left(\arccos(\gamma\cos\theta)\right)^2 d\theta\right] \\ &= \frac{1}{2}e^{\boldsymbol{i}\alpha}\int_0^{\pi} e^{\boldsymbol{i}\theta}\left(\frac{\pi}{2}-\arccos(\gamma\cos\theta)\right) d\theta \\ &= \frac{1}{2}e^{\boldsymbol{i}\alpha}\int_0^{\pi} e^{\boldsymbol{i}\theta}\arcsin(\gamma\cos\theta) d\theta \\ &= \frac{1}{2}e^{\boldsymbol{i}\alpha}\int_0^{\pi} e^{\boldsymbol{i}\theta}\left(\gamma\cos\theta+\sum_{k=1}^{\infty}\frac{(2k)!}{4^k(k!)^2(2k+1)}(\gamma\cos\theta)^{2k+1}\right) d\theta \\ &= \frac{\pi}{4}\gamma e^{\boldsymbol{i}\alpha}+\frac{\pi}{2}\sum_{k=1}^{\infty}\frac{((2k)!)^2}{2^{4k+1}(k!)^4(k+1)}\gamma^{2k+1}e^{\boldsymbol{i}\alpha} \\ &= \frac{\pi}{4}z+\frac{\pi}{2}\sum_{k=1}^{\infty}\frac{((2k)!)^2}{2^{4k+1}(k!)^4(k+1)}|z|^{2k}z, \end{aligned} \tag{9}$$

where the second last step follows from the fact that

$$\int_0^{\pi}\sin\theta(\cos\theta)^{2k+1}d\theta=0 \text{ and } \int_0^{\pi}(\cos\theta)^{2k+2}d\theta=\frac{(2k+1)(2k-1)\cdots 1}{(2k+2)(2k)\cdots 2}\pi,\ k=0,1,...$$

Clearly, if $Z \in \mathcal{H}_+^n$ then $Z^{\mathrm{T}} \in \mathcal{H}_+^n$. Furthermore, observe that the Hadamard product of any two positive semidefinite Hermitian matrices remains Hermitian positive semidefinite. Denote $A \circ B$ to be the Hadamard product of A and B, and denote $A^{(k)}$ to be $\overbrace{A \circ A \circ \cdots \circ A}^{k}$. It thus follows from (9) that

$$F(Z) = \frac{\pi}{4} Z + \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{((2k)!)^2}{2^{4k+1}(k!)^4(k+1)} (Z^{\mathrm{T}} \circ Z)^{(k)} \circ Z \succeq \frac{\pi}{4} Z.$$

Since $Q \succeq 0$, we have

$$Q \bullet F(Z) \geq \frac{\pi}{4} Q \bullet Z.$$

Consider the following complex SDP relaxation for (CP)

$$\begin{array}{lll} \text{(CSDP)} & \max & Q \bullet Z \\ & \text{s.t.} & Z_{kk} = 1, \; k = 1, \ldots, n, \\ & & Z \succeq 0. \end{array}$$

Let the optimal value of (CP) be $v^*(CP)$, the optimal value of (CSDP) be $v^*(CSDP)$, and Z^* be an optimal solution. Suppose that an randomized solution z is generated by independently setting $z_k = e^{\boldsymbol{i} \operatorname{Arg} \xi_k}$ for each $k = 1, \ldots, n$, and $\xi \in \mathcal{N}_c(0, Z^*)$. Let the expected value of the randomized solution z be $v(H(C))$. Then

$$v(H(C)) \geq \frac{\pi}{4} v^*(CSDP) \geq \frac{\pi}{4} v^*(CP) \approx 0.7854 \cdot v^*(CP).$$

Since (CP) can be viewed as the limit of (P) as $m \to \infty$, it is interesting to observe that the approximation ratio for (CP), $\frac{\pi}{4}$, is indeed the limit of $\alpha_m = \frac{m^2(1-\cos\frac{2\pi}{m})}{8\pi}$ as $m \to \infty$. It is also interesting to compare this ratio with that of its real counterpart:

$$\begin{array}{lll} \text{(RP)} & \max & x^{\mathrm{T}} Q x \\ & \text{s.t.} & x_k^2 = 1, \; k = 1, \ldots, n, \end{array}$$

where Q is a real positive semidefinite matrix. Nesterov [11] showed that in this case the randomization solution based on the SDP relaxation

$$\begin{array}{lll} \text{(RSDP)} & \max & Q \bullet X \\ & \text{s.t.} & X_{kk} = 1, \; k = 1, \ldots, n, \\ & & X \succeq 0, \end{array}$$

has the following approximation ratio

$$v(H(R)) \geq \frac{2}{\pi} v^*(RSDP) \geq \frac{2}{\pi} v^*(RP) \approx 0.6366 \cdot v^*(RP).$$

Therefore, the complex SDP relaxation for the complex quadratic optimization problem is more effective than the real SDP relaxation for its real counter-part, in the sense that the former has a slightly better approximation ratio.

Remark that similar as the analysis in Nesterov [11], Ye [14], and Zhang [15] for the real case, we can extend all the approximation results to the following more general setting

$$\begin{array}{ll} \max & z^{\mathrm{H}} Q z \\ \text{s.t.} & (|z_1|^2, |z_2|^2, \cdots, |z_n|^2)^{\mathrm{T}} \in \mathcal{F}, \end{array}$$

where $\mathcal{F}$ is a closed convex set in $\Re^n$. The corresponding complex and convex SDP relaxation is

$$\begin{array}{ll} \max & Q \bullet Z \\ \text{s.t.} & \text{diag } Z \in \mathcal{F} \\ & Z \succeq 0. \end{array}$$

In particular, if $\mathcal{F}$ is a hypercube and $Q \succ 0$, then the above $\frac{\pi}{4}$-approximation result also follows from the matrix cube theorem of Ben-Tal, Nemirovski, and Roos [3]. However, our technique appears to be very different in nature.

It is also interesting to remark that if we regard (CP) as an equivalent real quadratic problem

$$\begin{array}{ll} \max & (u^{\mathrm{T}}, v^{\mathrm{T}}) \begin{pmatrix} \text{Re } Q & -\text{Im } Q \\ \text{Im } Q & \text{Re } Q \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\ \text{s.t.} & u_k^2 + v_k^2 = 1,\ k = 1, ..., n, \end{array}$$

then the approximation ratio obtained that way would be $2/\pi$, instead of $\pi/4$. This shows that the complex SDP relaxation does have an advantage in this particular case.

3.4 Structured Continuous Complex Quadratic Optimization

In this subsection, we study a special case of (CP) with a sign structure on the object matrix, which is parallel to the original (real) max-cut model studied in [8]:

$$\begin{array}{lll} \text{(CPS)} & \max & z^{\mathrm{H}} Q z \\ & \text{s.t.} & |z_k| = 1,\ k = 1, \ldots, n, \end{array}$$

where we assume that $Q = [q_{jl}]_{n \times n} \in \mathcal{S}_+^n$ and $q_{jl} \le 0$ for all $1 \le j < l \le n$. Using (9) we know that the expected value of the randomized solution based on the complex SDP relaxation is

$$\begin{aligned} v(H(C)) &= 2 \sum_{j<l} q_{jl} \text{Re } F(Z_{jl}^*) + \sum_{j=1}^{n} q_{jj} \\ &= 2 \sum_{j<l} q_{jl} \left(\frac{\pi}{4} + \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{((2k)!)^2}{2^{4k+1} (k!)^4 (k+1)} |Z_{jl}^*|^{2k} \right) \text{Re } Z_{jl}^* + \sum_{j=1}^{n} q_{jj} \qquad (10) \end{aligned}$$

where Z^* is the optimal solution of the complex SDP relaxation. Define the following real function

$$g(y) := \frac{\pi}{4} + \frac{\pi}{2}\sum_{k=1}^{\infty} \frac{((2k)!)^2}{2^{4k+1}(k!)^4(k+1)} y^{2k}$$

on $y \in [0,1]$. We have $0 \le g(y) \le 1$ for all $y \in [0,1]$. Suppose that x is real, and $|x| \le y \le 1$. Then,

$$\min_{|x|\le y} \frac{1-g(y)x}{1-x} = \min_{|x|\le y} \left(g(y) + \frac{1-g(y)}{1-x} \right) = \frac{1+g(y)y}{1+y}.$$

One computes that

$$\min_{0\le y\le 1} \frac{1+g(y)y}{1+y} \approx 0.9349 =: \beta.$$

Therefore,

$$1 - g(y)x \ge \beta - \beta x,$$

for all $y \in [0,1]$ and $|x| \le y$, or equivalently,

$$g(y)x \le 1 - \beta + \beta x \tag{11}$$

for all $y \in [0,1]$ and $|x| \le y$. Using (11), we have

$$\left(\frac{\pi}{4} + \frac{\pi}{2}\sum_{k=1}^{\infty} \frac{((2k)!)^2}{2^{4k+1}(k!)^4(k+1)} |Z^*_{jl}|^{2k} \right) \text{Re } Z^*_{jl} \le 1 - \beta + \beta \text{Re } Z^*_{jl}. \tag{12}$$

Now we apply (12) in a componentwise fashion to (10), and obtain, thanks to the sign restriction, the following inequalities

$$\begin{aligned} v(H(C)) &= 2\sum_{j<l} q_{jl} \left(\frac{\pi}{4} + \frac{\pi}{2}\sum_{k=1}^{\infty} \frac{((2k)!)^2}{2^{4k+1}(k!)^4(k+1)} |Z^*_{jl}|^{2k} \right) \text{Re } Z^*_{jl} + \sum_{j=1}^{n} q_{jj} \\ &\ge 2\sum_{j<l} q_{jl}(1-\beta+\beta\text{Re } Z^*_{jl}) + \sum_{j=1}^{n} q_{jj} \\ &= (1-\beta)e^{\mathrm{T}}Qe + \beta Q \bullet Z^* \\ &\ge \beta v^*(CSDP) \\ &\ge \beta v^*(CPS). \end{aligned} \tag{13}$$

This yields an approximation ratio of 0.9349 for (CPS).

4 Concluding Remarks

In this paper we discussed complex quadratic maximization models, denoted as (P) and (CP) in the paper, in which the decision variables either take values as unit roots of the equation $z^m = 1$, or are assumed to have modulus 1. We established approximation ratios for randomization algorithms for these problems, based on the properties of the complex-valued normal distributions. In particular, the approximation ratio is $\frac{m^2(1-\cos\frac{2\pi}{m})}{8\pi}$ for (P), and is $\pi/4$ for (CP). If the off-diagonal elements of the objective matrix Q are non-positive, then the approximation ratio is improved to 0.9349. Our approach is based on a probability analysis of the complex-valued normally distributed random variables. The same results can also be obtained by different approaches. For example, recently, So, Zhang, and Ye [12] used Grothendieck's inequality and obtained the same $\frac{m^2(1-\cos\frac{2\pi}{m})}{8\pi}$ approximation bound for (P), and Ben-Tal, Nemirovski, and Roos [3] established a matrix cube theorem and obtained the $\pi/4$ approximation ratio for a model similar to (CP). Moreover, Ben-Tal, Nemirovski, and Roos [3] also suggested that the $\pi/4$ approximation ratio is a tight bound. However, it remains unknown whether or not $\alpha_m = \frac{m^2(1-\cos\frac{2\pi}{m})}{8\pi}$ is a tight bound for (P). Related to our models, Charikar and Wirth [5] discussed quadratic maximization models (the real case) in which Q is not assumed to be positive semidefinite; instead, the diagonals of Q are assumed to be all zeros. They proposed a randomized algorithm for such quadratic maximization model and established an $\Omega(1/\log n)$-approximation ratio. We plan to extend our analysis to such models in the future.

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A Proof of Lemma 3.3

Consider

$$\begin{aligned} F_m(z) &= \frac{m(2-\omega-\omega^{-1})}{8\pi^2}\sum_{j=0}^{m-1}\omega^j(\arccos(-\mathrm{Re}\ (\omega^{-j}z)))^2 \\ &= \frac{m(1-\cos\frac{2\pi}{m})}{4\pi^2}e^{\boldsymbol{i}\alpha}\sum_{j=0}^{m-1}e^{\boldsymbol{i}\theta_j}(\arccos(-\gamma\cos\theta_j))^2, \end{aligned}$$

where $z=\gamma e^{\boldsymbol{i}\alpha}$, $\omega = e^{\boldsymbol{i}\frac{2\pi}{m}}$ and $\theta_j = \frac{j}{m}2\pi - \alpha$ for $j=0,\dots,m-1$.

Since $\arccos(-x) = \frac{\pi}{2} - \arcsin(-x)$, we have

$$\begin{aligned} F_m(z) &= \frac{m(1-\cos\frac{2\pi}{m})}{4\pi^2} e^{\boldsymbol{i}\alpha} \sum_{j=0}^{m-1} e^{\boldsymbol{i}\theta_j} (\frac{\pi^2}{4} + \pi \arcsin(\gamma\cos\theta_j) + (\arcsin(\gamma\cos\theta_j))^2) \\ &= \frac{m(1-\cos\frac{2\pi}{m})}{4\pi^2} e^{\boldsymbol{i}\alpha} \sum_{j=0}^{m-1} (\pi e^{\boldsymbol{i}\theta_j} \arcsin(\gamma\cos\theta_j) + e^{\boldsymbol{i}\theta_j} (\arcsin(\gamma\cos\theta_j))^2) \\ &= \frac{m(1-\cos\frac{2\pi}{m})}{4\pi^2} e^{\boldsymbol{i}\alpha} \sum_{j=0}^{m-1} (\pi e^{\boldsymbol{i}\theta_j} \gamma\cos\theta_j + \pi e^{\boldsymbol{i}\theta_j} (\arcsin(\gamma\cos\theta_j) - \gamma\cos\theta_j) \\ &\quad + e^{\boldsymbol{i}\theta_j} (\arcsin(\gamma\cos\theta_j))^2). \end{aligned}$$

Set

$$\begin{aligned} I_1 &= \gamma e^{\boldsymbol{i}\alpha} \sum_{j=0}^{m-1} e^{\boldsymbol{i}\theta_j} \cos\theta_j \\ I_2 &= e^{\boldsymbol{i}\alpha} \sum_{j=0}^{m-1} e^{\boldsymbol{i}\theta_j} (\arcsin(\gamma\cos\theta_j) - \gamma\cos\theta_j) \\ I_3 &= e^{\boldsymbol{i}\alpha} \sum_{j=0}^{m-1} e^{\boldsymbol{i}\theta_j} (\arcsin(\gamma\cos\theta_j))^2. \end{aligned}$$

Thus, we shall have

$$F_m(z) = \frac{m(1-\cos\frac{2\pi}{m})}{4\pi} \left(I_1 + I_2 + I_3/\pi\right). \tag{14}$$

Let us now treat these items one by one. First, we note that

$$\begin{aligned} I_1 &= \gamma e^{\boldsymbol{i}\alpha} \sum_{j=0}^{m-1} e^{\boldsymbol{i}\theta_j} \cos\theta_j \\ &= \frac{\gamma e^{\boldsymbol{i}\alpha}}{2} \sum_{j=0}^{m-1} e^{\boldsymbol{i}\theta_j} (e^{\boldsymbol{i}\theta_j} + e^{-\boldsymbol{i}\theta_j}) \\ &= \frac{\gamma e^{\boldsymbol{i}\alpha}}{2} \left(m + \sum_{j=0}^{m-1} e^{\boldsymbol{i}\frac{4\pi}{m}j} e^{-2\boldsymbol{i}\alpha} \right) \\ &= \begin{cases} \frac{\gamma e^{\boldsymbol{i}\alpha}}{2} m = mz/2, & \text{if } m \geq 3 \\ \gamma e^{\boldsymbol{i}\alpha} + \gamma e^{-\boldsymbol{i}\alpha} = z + \bar{z}, & \text{if } m = 2. \end{cases} \end{aligned}$$

Let us denote $a_n = \frac{(2n)!}{2^{2n}(n!)^2(2n+1)}$, $n = 0, 1, \ldots$. Then we have the Taylor expansion $\arcsin(t) =$

$\sum_{n=0}^{\infty} a_n t^{2n+1}$, and so

$$
\begin{aligned}
I_2 &= e^{\boldsymbol{i}\alpha} \sum_{j=0}^{m-1} e^{\boldsymbol{i}\theta_j} \sum_{n=1}^{\infty} a_n \gamma^{2n+1} (\cos\theta_j)^{2n+1} \\
&= \sum_{n=1}^{\infty} \frac{a_n}{2^{2n+1}} \gamma^{2n+1} e^{\boldsymbol{i}\alpha} \sum_{j=0}^{m-1} e^{\boldsymbol{i}\theta_j} \left(e^{-\boldsymbol{i}\theta_j} + e^{\boldsymbol{i}\theta_j} \right)^{2n+1} \\
&= \sum_{n=1}^{\infty} \frac{a_n}{2^{2n+1}} \gamma^{2n+1} e^{\boldsymbol{i}\alpha} \sum_{j=0}^{m-1} \sum_{k=0}^{2n+1} \begin{pmatrix} 2n+1 \\ k \end{pmatrix} e^{\boldsymbol{i}\theta_j (2n+2-2k)} \\
&= \sum_{n=1}^{\infty} \frac{a_n}{2^{2n+1}} \gamma^{2n+1} e^{\boldsymbol{i}\alpha} \sum_{k=0}^{2n+1} \begin{pmatrix} 2n+1 \\ k \end{pmatrix} \left[\sum_{j=0}^{m-1} e^{\boldsymbol{i}\frac{2\pi}{m}(2n+2-2k)j} \right] e^{-\boldsymbol{i}\alpha(2n+2-2k)}.
\end{aligned}
$$

Let us denote

$$
b_k = \sum_{j=0}^{m-1} e^{\boldsymbol{i}\frac{2\pi}{m}kj}
$$

where k is an integer number. Obviously, b_k is either 0 or m. In particular, if m is even and k is odd, then $b_k = 0$. We further obtain that

$$
\begin{aligned}
I_2 &= \sum_{n=1}^{\infty} \frac{a_n}{2^{2n+1}} \gamma^{2n+1} e^{\boldsymbol{i}\alpha} \sum_{k=0}^{2n+1} \begin{pmatrix} 2n+1 \\ k \end{pmatrix} b_{2n+2-2k} e^{-\boldsymbol{i}\alpha(2n+2-2k)} \\
&= \sum_{n=1}^{\infty} \frac{a_n}{2^{2n+1}} \sum_{k=0}^{2n+1} \begin{pmatrix} 2n+1 \\ k \end{pmatrix} b_{2n+2-2k} \bar{z}^{2n+1-k} z^k.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
I_3 &= e^{\boldsymbol{i}\alpha} \sum_{j=0}^{m-1} e^{\boldsymbol{i}\theta_j} (\arcsin(\gamma\cos\theta_j))^2 \\
&= e^{\boldsymbol{i}\alpha} \sum_{j=0}^{m-1} \sum_{s=0,t=0}^{\infty} a_s a_t \gamma^{2s+2t+2} (\cos\theta_j)^{2s+2t+2} \\
&= e^{\boldsymbol{i}\alpha} \sum_{s=0,t=0}^{\infty} \frac{a_s a_t}{2^{2s+2t+2}} \gamma^{2s+2t+2} \sum_{k=0}^{2s+2t+2} \begin{pmatrix} 2s+2t+2 \\ k \end{pmatrix} \sum_{j=0}^{m-1} e^{\boldsymbol{i}\theta_j (2s+2t+3-2k)} \\
&= e^{\boldsymbol{i}\alpha} \sum_{s=0,t=0}^{\infty} \frac{a_s a_t}{2^{2s+2t+2}} \gamma^{2s+2t+2} \sum_{k=0}^{2s+2t+2} \begin{pmatrix} 2s+2t+2 \\ k \end{pmatrix} b_{2s+2t+3-2k} e^{-\boldsymbol{i}\alpha(2s+2t+3-2k)} \\
&= \sum_{s=0,t=0}^{\infty} \frac{a_s a_t}{2^{2s+2t+2}} \sum_{k=0}^{2s+2t+2} \begin{pmatrix} 2s+2t+2 \\ k \end{pmatrix} b_{2s+2t+3-2k} \bar{z}^{2s+2t+2-k} z^k.
\end{aligned}
$$

If m is even, then $I_3 = 0$ since $b_{2s+2t+3-2k} = 0$ in each term.

Observing that the Hadamard product of any two positive semidefinite Hermitian matrices remains Hermitian positive semidefinite, it follows from (14) that

$$\begin{aligned} & F_m(Z) \\ = \; & \frac{m^2(1-\cos\frac{2\pi}{m})}{8\pi} Z + \frac{m^2(1-\cos\frac{2\pi}{m})}{4\pi} \sum_{n=1}^{\infty} \left[\frac{a_n}{2^{2n+1}} \sum_{k=0}^{2n+1} \left(\begin{array}{c} 2n+1 \\ k \end{array} \right) b_{2n+2-2k} \right. \\ & \left. Z^{(k)} \circ (Z^{\mathrm{T}})^{(2n+1-k)} \right] + \frac{m^2(1-\cos\frac{2\pi}{m})}{4\pi^2} \sum_{s=0,t=0}^{\infty} \left[\frac{a_s a_t}{2^{2s+2t+2}} \sum_{k=0}^{2s+2t+2} \left(\begin{array}{c} 2s+2t+2 \\ k \end{array} \right) \right. \\ & \left. b_{2s+2t+3-2k} Z^{(k)} \circ (Z^{\mathrm{T}})^{(2s+2t+2-k)} \right], \end{aligned}$$

for $m \geq 3$ while a similar expansion of $F_2(Z)$ can be obtained easily. Since $Z \succeq 0$ and $Z^{\mathrm{T}} \succeq 0$, hence we get

$$F_m(Z) \succeq \frac{m^2(1-\cos\frac{2\pi}{m})}{8\pi} Z,$$

for $m \geq 3$ and $F_2(Z) \succeq \frac{1}{\pi}(Z + Z^{\mathrm{T}})$. This completes the proof.

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