Optioned Portfolio Selection: Models and Analysis *

Jianfeng LIANG † Shuzhong ZHANG ‡ Duan LI §

January 2005

Abstract

We study in this paper the portfolio selection problem with a stock index and European style options on the index. A refined mean-variance methodology is adopted in the study. Single-stage and two-stage investment models are studied and solved. In the later case a scenario tree and stochastic programming formulation are used. Explicit forms of the optimal portfolio and its corresponding efficient frontier are derived, which reveal rich structures of the optimal payoffs. Finally, illustrative numerical examples are presented.

Keywords: options, portfolio selection, scenario tree, stochastic programming.

1 Introduction

Options have played an important role in financial markets. One reason for wide applications of options is due to versatile payoff structures of options, which can be used to form investment portfolios with desirable risk profiles [8, 14]. The performance evaluator of portfolios with options has been investigated extensively in the literature; see [1, 3, 4, 5, 6]. The optioned portfolio selection has attracted much attention both in theory and in practice.

Stochastic programming has been widely applied to financial planning. Many discrete assets allocation problems are formulated as stochastic programming models based on scenario tree structure; see [2, 10, 16]. Stochastic linear programming models for

*Research was partially supported by Hong Kong RGC Earmarked Grants CUHK4174/03E and CUHK4180/03E.

†Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Shatin, Hong Kong.

‡Corresponding author. Email: zhang@se.cuhk.edu.hk. Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Shatin, Hong Kong.

§Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Shatin, Hong Kong.

optioned portfolio selection have been discussed in the literature. Mostly, the models are proposed to maximize the expected portfolio return under certain necessary constraints. Pelsser and Vorst [17] presented a linear programming model with shortfall constraints, which were expressed by a set of chance constraints. Dert and Oldenkamp [7] proposed a linear programming model with a requirement of a guaranteed return. Casino effect was illustrated, and was controlled by introducing shortfall constraints. A two-stage stochastic linear programming model was discussed by Berkelaar, Dert, Odenkamp, and Zhang [2], and it was numerically solved by a primal-dual decomposition-based interior point method.

The mean-variance formulation by Markowitz [12, 13] is a fundamental basis for portfolio selection theory in a single stage setting. The Markowitz's mean-variance model has been recently extended to multi-period setting by Li and Ng [11] and to continuous-time setting by Zhou and Li [19]. A generalized mean-variance model was proposed by Morard and Naciri [15] to optimize the hedging rates. The hedging was implemented with the covered call writing strategy. The empirical results showed that the use of covered calls improved the performance of stock portfolios, while there was no analytical formula for the optimal portfolio derived. Isakov and Marard [9] pointed out that in the case of incomplete hedging, the mean-variance formula could be applied for the optioned portfolio selection problems, since the hedged return does not necessarily have a non-symmetric distribution. We thus believe that the mean-variance criteria is an acceptable choice for the optioned portfolio selection problem.

Our optioned portfolio consists of one index, a set of European options on this index, and a risk-free asset. In our case, the scenario tree is generated using the distribution of the index. We apply the mean-variance formulation to investigate our investment problem, in order to make a further analysis of the payoff based on the explicit expressions of the optimal portfolio.

The paper is organized as follows. In Section 2, we introduce a single-stage mean-variance model for the optioned portfolio selection problem. We present the formulations of the model and its solutions, following by discussing some interesting properties of the optimal function. In Section 3, we extend the model to a two-stage case. We formulate a model and discuss its solutions. We shall see that the payoff of the second stage subtree inherits similar properties as for the single-stage case. Particular properties of the two-stage case will be discussed. Throughout the paper, numerical examples with real life data are used to illustrate and validate our results.

2 The single-stage optioned mean-variance model

2.1 The model and its formulation

We first consider a single-stage investment problem. The available assets are: one index, a class of m European call options on this underlying index, and a risk-free asset. The options have the same expiration date, and their strike prices are $K_1 < K_2 < \cdots < K_m$.

The decision horizon is the same as the options' expiration, and r is the risk-free return rate in this period. Given initial wealth B and expected return R, the object is to construct a portfolio with minimum final payoff variance. The decision variables are X and x_f, where X is the number of shares of the index and options being held, $X \in \Re^{m+1}$, and x_f is the amount invested in the risk-free asset.

We formulate the model based on a single-stage scenario tree structure. There are n scenarios in the tree with probability on the ith scenario being $p_i,\ i = 1, 2, \cdots, n$, where $\sum_{i=1}^n p_i = 1$. Other data include the price vector of risky assets (including index and options) in the beginning, u, and the unit payoff vector of risky assets at the ith scenario, v_i. We denote $\bar{v} = \sum_{i=1}^n p_i v_i$ as the average unit payoff vector, and $A = \sum_{i=1}^n p_i (v_i - \bar{v})(v_i - \bar{v})'$ as the covariance matrix of risky assets.

Assumption 2.1 *Assume that the scenario tree is well generated in the following sense. First, given intervals $I_0 = (0,\quad K_1)$, $I_i = (K_i,\quad K_{i+1})$, $i = 1, 2, \cdots, m-1$, and $I_m = (K_m,\quad +\infty)$, there are at least $m+2$ scenarios, and at least one scenario in each of the above intervals. This structure makes sure the estimated covariance matrix A is positive definite. Second, this structure presents no arbitrage opportunity in the tree.*

Besides, we introduce the following notations:

$$\begin{aligned}
\widetilde{A} &= A + \bar{v}\bar{v}'; \\
\alpha &= u'A^{-1}u; \\
\beta &= \bar{v}'A^{-1}u; \\
\gamma &= \bar{v}'A^{-1}\bar{v}; \\
\delta &= r^2\alpha - 2r\beta + \gamma = (\bar{v} - r \cdot u)'A^{-1}(\bar{v} - r \cdot u); \\
\rho &= \frac{r}{\delta + 1}.
\end{aligned}$$

The single-stage mean-variance model is of the following form:

$$\begin{array}{ll}
\min & \frac{1}{2}\mathrm{Var}\ (W) \\
\text{s.t.} & u'X + x_f = B \\
 & \mathsf{E}(W) = R,
\end{array}$$

where W is the final wealth. Based on the scenario tree structure, we have

$$\begin{aligned}
\mathsf{E}(W) &= \sum_{i=1}^n p_i v_i' X + r x_f = \bar{v}'X + r x_f \\
\mathrm{Var}\ (W) &= \mathsf{E}\left[(v_i'X + r x_f - R)^2\right] = \begin{pmatrix} X \\ x_f \end{pmatrix}' \begin{pmatrix} \widetilde{A} & r\bar{v} \\ r\bar{v}' & r^2 \end{pmatrix} \begin{pmatrix} X \\ x_f \end{pmatrix} - R^2.
\end{aligned}$$

Thus the equivalent deterministic form of the single-stage mean-variance model is

$$(M_1) \quad \min \quad \frac{1}{2}\begin{pmatrix} X \\ x_f \end{pmatrix}'\begin{pmatrix} \widetilde{A} & r\bar{v} \\ r\bar{v}' & r^2 \end{pmatrix}\begin{pmatrix} X \\ x_f \end{pmatrix} - \frac{1}{2}R^2 \tag{1}$$

$$\text{s.t.} \quad u'X + x_f = B \tag{2}$$

$$\bar{v}'X + rx_f = R. \tag{3}$$

Under Assumption 2.1, (M_1) is a strictly convex quadratic optimization problem, and its solution is given specifically in the following theorem.

Theorem 2.1 *The single-stage mean-variance model (M_1) has the following unique primal-dual solution*

$$\begin{aligned} X &= -\frac{\lambda}{r}A^{-1}(\bar{v} - ru), \\ x_f &= \frac{1}{r}\left[\frac{\lambda}{r}(\gamma - r\beta + 1) + \mu\right], \\ \lambda &= \frac{r}{\delta + 1}(rB - \mu) \equiv \rho(rB - \mu), \\ \mu &= r\frac{R - \rho B}{r - \rho}, \end{aligned}$$

where λ and μ are the Lagrangian multipliers related to the constraints (2) and (3), respectively, and the associated risk is

$$\text{Var}\,(W) = \frac{\rho}{r - \rho}(R - rB)^2.$$

Proof: The Lagrangian function for the optimization problem(M_1) is

$$\begin{aligned} L &= \frac{1}{2}\begin{pmatrix} X \\ x_f \end{pmatrix}'\begin{pmatrix} \widetilde{A} & r\bar{v} \\ r\bar{v}' & r^2 \end{pmatrix}\begin{pmatrix} X \\ x_f \end{pmatrix} - \frac{1}{2}R^2 \\ &\quad -\lambda(u'X + x_f - B) - \mu(\bar{v}'X + rx_f - R). \end{aligned} \tag{4}$$

Applying the KKT optimality conditions leads to

$$\frac{\partial L}{\partial X} = 0 \quad : \quad \widetilde{A}X + r\bar{v}x_f - \lambda u - \mu\bar{v} = 0 \tag{5}$$

$$\frac{\partial L}{\partial x_f} = 0 \quad : \quad r\bar{v}'X + r^2x_f - \lambda - r\mu = 0 \tag{6}$$

$$\frac{\partial L}{\partial \lambda} = 0 \quad : \quad u'X + x_f = B \tag{7}$$

$$\frac{\partial L}{\partial \mu} = 0 \quad : \quad \bar{v}'X + rx_f = R. \tag{8}$$

Using (6), we get

$$x_f = \frac{1}{r}\left[-\bar{v}'X + \frac{\lambda}{r} + \mu\right].$$

Substituting the above equation into (5) yields

$$AX + \frac{\lambda}{r}(\bar{v} - ru) = 0.$$

Thus, with $\beta = \bar{v}'A^{-1}u$, and $\gamma = \bar{v}'A^{-1}\bar{v}$, we have

$$\begin{aligned} X &= -\frac{\lambda}{r}A^{-1}(\bar{v} - ru) \\ x_f &= \frac{1}{r}\left[\frac{\lambda}{r}(\gamma - r\beta + 1) + \mu\right]. \end{aligned}$$

Therefore, (7) becomes

$$\frac{\lambda}{r^2}(\delta + 1) + \frac{\mu}{r} = B$$

leading to

$$\lambda = \frac{r}{\delta + 1}(rB - \mu) = \rho(rB - \mu),$$

where we use the definitions that $\delta = r^2\alpha - 2r\beta + \gamma$ and $\alpha = u'A^{-1}u$. Finally, (8) yields

$$R = -\frac{\lambda}{r}(\gamma - r\beta) + \frac{\lambda}{r}(\gamma - r\beta + 1) + \mu = \frac{\lambda}{r} + \mu,$$

which further implies

$$\mu = r\frac{R - \rho B}{r - \rho}.$$

Using the optimal solutions, the corresponding variance follows:

$$\begin{aligned} \text{Var }(W) &= \mathsf{E}\left[(v_i'X + rx_f - R)^2\right] \\ &= X'AX \\ &= \left(\frac{\lambda}{r}\right)^2 (\bar{v} - ru)'A^{-1}(\bar{v} - ru) \\ &= \left(\frac{\lambda}{r}\right)^2 (r^2\alpha - 2r\beta + \gamma) \\ &= \left(\frac{\lambda}{r}\right)^2 \delta \\ &= \delta\left[\frac{\rho(rB - \mu)}{r}\right]^2 \\ &= \delta\left[\frac{\rho}{r}(rB - r\frac{R - \rho B}{r - \rho})\right]^2 \\ &= \frac{(R - rB)^2}{\delta} \\ &= \frac{\rho}{r - \rho}(R - rB)^2. \end{aligned}$$

□

An obvious consequence of Theorem 2.1 is following. If R is set to be rB, then the optimal variance reaches its minimum of value zero. The associated solutions are: $(X^*,\ x_f^*) = (0,\ B)$, $\lambda^* = 0$, and $\mu^* = rB$.

2.2 Properties of the optimal portfolio

Denote

$$\psi := \frac{\rho r}{r-\rho} \text{ and } \theta := \frac{\psi}{r} A^{-1}(\bar{v} - ru).$$

The optimal solutions of (M_1) can be written in the following simpler form:

$$\begin{aligned} \lambda &= \psi(rB - R) \\ X &= \theta(R - rB) \\ x_f &= B - u'\theta(R - rB) \equiv B + a(R - rB), \end{aligned}$$

where $\theta = (\theta_0,\ \theta_1, \cdots, \theta_m)'$, and $a := -u'\theta$.

Lemma 2.1 *The optimal payoff curve is piecewise linear with respect to the index value. At any breakpoint where the slope of the curve changes, the index value must equal to one of the strike prices of the options. Furthermore, the slopes of the line segments are steeper for a larger target R value for all $R > rB$.*

Proof: Here the scenarios are represented by possible index values at the investment horizon, where scenario S means that the index value is S in this scenario. Denote the payoff at scenario S as $P(S)$. Let the m European call options be with strike prices $K_1 < K_2 < \cdots < K_m$. For $K_j < S < K_{j+1}$,

$$P(S) = x_0 S + \sum_{i=1}^{j} x_i (S - K_i) + r x_f.$$

Suppose that there are two scenarios S_1 and S_2 between K_j and K_{j+1}, i.e.

$$K_j < S_1 < S_2 < K_{j+1}.$$

Then we have

$$P(S_2) - P(S_1) = (S_2 - S_1)x_0 + \sum_{i=1}^{j} x_i (S_2 - S_1) = (S_2 - S_1) \sum_{i=0}^{j} x_i.$$

Since

$$x_i = \theta_i (R - rB), \quad i = 0, 1, \cdots, m,$$

the slope of the payoff function between K_j and K_{j+1} is

$$\frac{P(S_2) - P(S_1)}{S_2 - S_1} = (R - rB) \sum_{i=0}^{j} \theta_i.$$

Therefore, for a fixed R this slope is constant, and the payoff is linear between two neighboring strike prices. Also because θ_i is independent of R, for any $R \geq rB$, the value of $\sum_{i=0}^{j} \theta_i$ is fixed and so the payoff function is linear in R. Moreover, for a larger R value with $R > rB$, the slopes of line segments between any two neighboring strike prices will get steeper. $\square$

In fact, we observe that a scenario $S = K_j$ is a local maximum point for the payoff function iff $\sum_{i=0}^{j-1} \theta_i > 0$, and $\sum_{i=0}^{j} \theta_i < 0$, and similarly, it is a local minimum point iff $\sum_{i=0}^{j-1} \theta_i < 0$, and $\sum_{i=0}^{j} \theta_i > 0$, as one can observe from an example to be discussed later in this section.

Proposition 2.1 *There are scenarios where the payoffs of the optimal portfolio are constantly rB, regardless the variable value R.*

Proof: Since

$$P(S) = x_0 S + \sum_{i=1}^{m} x_i (S - K_i)_+ + r x_f = rB$$

is equivalent to

$$(R - rB) \left(\theta_0 S + \sum_{i=1}^{m} \theta_i (S - K_i)_+ + ra \right) = 0,$$

where $a := -u'\theta$. Hence, the scenarios in question correspond to the roots of the equation

$$\theta_0 S + \sum_{i=1}^{m} \theta_i (S - K_i)_+ + ra = 0.$$

$\square$

Proposition 2.2 *In any scenario S, for all $R > rB$, if $P(S) > rB$, then the payoff of a higher R dominates that of a lower R; else if $P(S) < rB$, then the payoff of a higher R is dominated by that of a lower R.*

Proof: For convenience, let us denote $K_0 = 0$. For $R > rB$, in scenario S, we have

$$P(S) = rB + (R - rB) \left(\sum_{i=0}^{m} \theta_i (S - K_i)_+ + ra \right).$$

If $P(S) > rB$, then

$$(R - rB) \left(\sum_{i=0}^{m} \theta_i (S - K_i)_+ + ra \right) > 0$$

and so

$$\sum_{i=0}^{m} \theta_i (S - K_i)_+ + ra > 0.$$

So for any $R > rB$, $P(S)$ will always be larger than rB, and it is increasing in R. Similarly, if $P(S) < rB$, then it is decreasing in R. $\square$

The above propositions show a clear structure of the optimal payoff curve. For the same data with different target expected payoff R, the optimal payoff curves form a group of piecewise line segments with a similar structure, all of which intersect at some fixed points with the payoff rB, and whose slopes getting steeper with larger target value R. The results are clearly illustrated in the following example.

Example 1 We consider the results of a single-stage portfolio selection problem based on the market data of options on the S&P 500 index, which are listed on the CBOE. The prices are drawn from the CBOE web page in the morning of Oct. 13, 2003, shown in Table 1. The horizon is equal to the expiration date of the options, which is Nov. 21, 2003. The investment horizon is 39 days. In this example, we simply use the mid prices as the initial prices for both longing and shorting.

The scenario tree is generated under the assumption that the index value at the horizon is lognormally distributed with an expected annualized growth rate $\mu = 18.81\%$ and annualized standard deviation of $\sigma = 18.45\%$, which are listed in the Standard & Poor's company's web page. The risk free return rate for the investment horizon is $r = 1 + 0.14\%$. The initial wealth $B = \$10,000$. We assume three different target expected payoffs $R_1 = rB = \$10,014$, $R_2 = \$10,020$, and $R_3 = \$10,025$.

The information of the optimal solutions is shown in Table 2. The third column shows the θ of the optimal portfolio, which is independent of the target R value. We have proved in Lemma 2.1 that the slope of the $(i+1)th$ line segment of the payoff curve is $(R - rB)\sum_{j=0}^{i} \theta_j$. So from the values given in the fourth column, we can tell if a breakpoint is a local maximum or minimum point of the payoff curve, as shown in the fifth column.

Figure 1 shows the optimal payoff curves for R_1, R_2 and R_3. First, the payoff curves of R_2 and R_3 are piecewise linear in the index value. Both curves have the same group of breakpoints, and the index values at the breakpoints are exactly those of the strike prices of the options. Second, the slopes of the line segments of the curve for R_3 are always steeper than those for R_2. Third, above the line of rB, the curve of R_3 dominates that of R_2; and below the line of rB, the curve of R_3 is dominated by that of R_2. Finally, we see that both curves always intersect on the line of rB. So from the figure, the results proved in previous propositions are clearly verified.

In the next section, we investigate a two-stage optioned mean-variance model. Based on a two-stage scenario tree, we shall formulate a model and discuss its solutions. We shall then see that the payoff of the second stage subtree inherits quite a number of properties from the single-stage model.

	Expiration	Bid	Mid	Ask
S&P 500		1038.06	1038.06	1038.06
Call 1005	November	46.30	47.30	48.30
Call 1010	November	42.50	43.50	44.50
Call 1015	November	38.90	39.90	40.90
Call 1020	November	35.40	36.40	37.40
Call 1025	November	32.00	33.00	34.00
Call 1030	November	28.80	29.80	30.80
Call 1035	November	25.80	26.65	27.50
Call 1040	November	22.90	23.90	24.90
Call 1045	November	20.20	21.20	22.20
Call 1050	November	18.00	18.60	19.20
Call 1060	November	13.60	14.35	15.10
Call 1065	November	11.60	12.35	13.10
Call 1070	November	10.00	10.75	11.50
Call 1075	November	8.60	9.10	9.60

Table 1: The data of the S&P 500 index and the options of Example 1

	the variable	θ_i^*	$\sum_{j=0}^{i} \theta_j^*$	local max/min of the breakpoint on the strike price
S&P 500	x_0	0.0334	0.0334	
Call 1005	x_1	-0.4431	-0.4097	max
Call 1010	x_2	0.7104	0.3007	min
Call 1015	x_3	-0.2639	0.0368	neither
Call 1020	x_4	-0.4307	-0.3938	max
Call 1025	x_5	1.2481	0.8543	min
Call 1030	x_6	-2.3537	-1.4994	max
Call 1035	x_7	2.9072	1.4078	min
Call 1040	x_8	-1.8149	-0.4071	max
Call 1045	x_9	-0.1096	-0.5167	neither
Call 1050	x_{10}	0.9225	0.4059	min
Call 1060	x_{11}	-1.2045	-0.7986	max
Call 1065	x_{12}	1.4448	0.6462	min
Call 1070	x_{13}	-0.9166	-0.2704	max
Call 1075	x_{14}	0.2796	0.0092	min

Table 2: The optimal solutions of Example 1

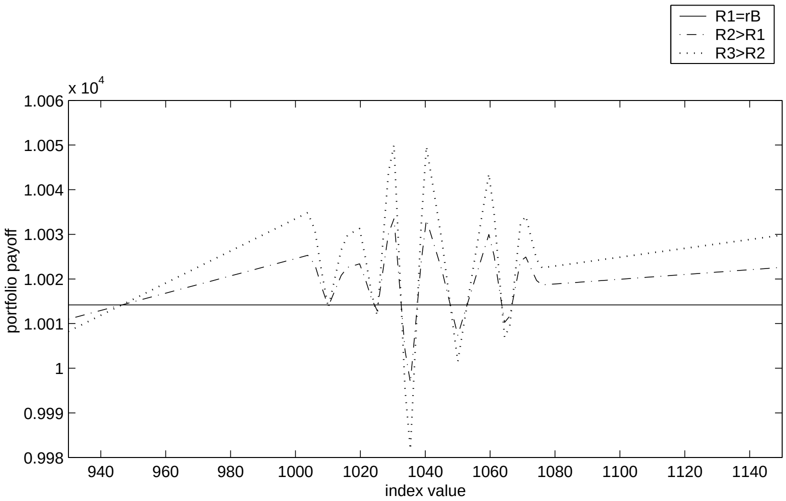

Figure 1: The optimal payoff curves of Example 1

3 The two-stage optioned mean-variance model

3.1 The two-stage model and its formulation

We now introduce the two-stage problem. An investment portfolio is supposed to be constructed at the beginning, and will be revised at the end of the first stage. We denote B as the initial wealth, and R as the target expected payoff in the end. As before, the assets to be considered are still European call options on a certain underlying index, the index itself and a risk-free asset. The options expire at different times. Suppose that there are m_1 options expiring at the end of the first stage, with strike prices $Q_1 < Q_2 < \cdots < Q_{m_1}$; and there are m_2 options expiring at the end of the second stage, with strike prices $K_1 < K_2 < \cdots < K_{m_2}$. In the two-stage scenario tree, there are n_1 scenarios in the first stage with probability p_i, and n_2 scenarios in each second stage subtree with conditional probability q_j. The risk free return rates in the two stages are denoted by r_1 and r_2, respectively. Naturally, $r = r_1 \cdot r_2$ is the risk-free rate for the entire period. Other notations are as follows:

X_0: the number of holding shares of risky assets in the first stage ($X_0 \in \Re^{m_1+m_2+1}$);

X_i: the number of holding shares of risky assets in the ith subtree of the second stage ($X_i \in \Re^{m_2+1}$, $i = 1, 2, \cdots, n_1$);

x_{if}: the amount invested in the risk-free asset at each decision point, $i = 0, 1, \cdots, n_1$;

u_0: the price vector in the beginning ($u_0 \in \Re^{m_1+m_2+1}$);

u_i: the price vector at the beginning of the second stage ($u_i \in \Re^{m_2+1}$);

v_i: the unit payoff vector at scenario i in the first stage ($v_i \in \Re^{m_1+m_2+1}$, $i = 1, 2, \cdots, n_1$);

v_i^j: the unit payoff vector at the jth scenario in the ith subtree of the second stage ($v_i^j \in \Re^{m_2+1}$, $i = 1, 2, \cdots, n_1$, $j = 1, 2, \cdots, n_2$);

$\bar{v}_i$: the expected unit payoff vector of the second stage risky assets in ith subtree;

A_i: the conditional covariance matrix of the second-stage risky assets ($A_i \in \Re^{(m_2+1)\times(m_2+1)}$);

A: the covariance matrix of the first-stage risky assets ($A \in \Re^{(m_1+m_2+1)\times(m_1+m_2+1)}$).

To formulate the two-stage mean-variance model clearly, we introduce the following notations:

$$
\begin{aligned}
\widetilde{A_i} &= p_i(A_i + \bar{v}_i\bar{v}_i');\\
\widetilde{v_i} &= p_i \cdot \bar{v}_i;\\
\widetilde{r_i} &= p_i \cdot r_2;\\
\alpha_i &= u_i' A_i^{-1} u_i;\\
\beta_i &= \bar{v}_i' A_i^{-1} u_i;\\
\gamma_i &= \bar{v}_i' A_i^{-1} \bar{v}_i;\\
\delta_i &= (r_2)^2\alpha_i - 2r_2\beta_i + \gamma_i = (\bar{v}_i - r_2u_i)' A_i^{-1}(\bar{v}_i - r_2u_i);\\
p_0 &= \sum_{i=1}^{n_1} \frac{p_i}{\delta_i + 1};\\
\tilde{r} &= p_0 r = \sum_{i=1}^{n_1} \frac{p_i r}{\delta_i + 1} = \sum_{i=1}^{n_1} \frac{p_i r_1 r_2}{\delta_i + 1} = \sum_{i=1}^{n_1} \frac{r_1 \widetilde{r_i}}{\delta_i + 1};\\
\bar{v}_0 &= \sum_{i=1}^{n_1} \frac{p_i/(\delta_i + 1)}{p_0} v_i;\\
\widetilde{v_0} &= p_0 \bar{v}_0;\\
A_0 &= \frac{1}{p_0}\sum_{i=1}^{n_1} \frac{p_i}{\delta_i + 1} v_i v_i' - \bar{v}_0\bar{v}_0'.
\end{aligned}
$$

If we take $\frac{p_i/(\delta_i+1)}{p_0}$ as the modified probability of the ith scenario in the first stage, then $\bar{v}_0$ and A_0 as defined are the first-stage expected unit payoff and the covariance matrix based on the modified probabilities. Next, we further introduce the following notations:

$$
\widetilde{A_0} = p_0(A_0 + \bar{v}_0\bar{v}_0');
$$

$$
\begin{aligned}
\alpha_0 &= u_0' A_0^{-1} u_0; \\
\beta_0 &= \bar{v}_0' A_0^{-1} u_0; \\
\gamma_0 &= \bar{v}_0' A_0^{-1} \bar{v}_0; \\
\delta_0 &= (r_1)^2 \alpha_0 - 2r_1\beta_0 + \gamma_0 = (\bar{v}_0 - r_1 u_0)' A_0^{-1} (\bar{v}_0 - r_1 u_0); \\
\rho &= \frac{\tilde{r}}{\delta_0 + 1} \equiv \frac{p_0 r}{\delta_0 + 1}.
\end{aligned}
$$

Similar to Assumption 2.1, we assume that the two-stage scenario tree does not admit arbitrage opportunities and that the covariance matrices at both stages are positive definite. In that case, it follows that $A_0 \succ 0$. To see this let us suppose for the sake of obtaining a contradiction that A_0 is not positive definite. Then there should exist $X_0 \neq 0$, such that $X_0' A_0 X_0 = 0$, which means that in the modified probability, the portfolio X_0 is risk-free, i.e., in all scenarios at the first stage, the portfolio X_0 will have the same payoff, which is independent of the probabilities. So with this X_0, we must also have $X_0' A X_0 = 0$, where A is the covariance matrix with the actual probabilities. This leads to a contradiction since $A \succ 0$, and so we must have $A_0 \succ 0$.

The general two-stage mean-variance model is

$$
\begin{array}{ll}
\min & \frac{1}{2}\mathrm{Var}\,(W_T) \\
\text{s.t.} & u_0' X_0 + x_{0f} = B \\
 & v_i' X_0 + r_1 x_{0f} = u_i' X_i + x_{if}, \ i = 1, 2, \cdots, n_1 \\
 & \mathsf{E}(W_T) = R,
\end{array}
$$

where W_T is the wealth at the end of the second stage. Now we derive the expressions for $\mathsf{E}(W_T)$ and $\mathrm{Var}\,(W_T)$. Denoting $\hat{v}$ as the final unit payoff, we have

$$
\begin{aligned}
\mathsf{E}(W_T) &= \mathsf{E}[\hat{v}' X_T + r_2 x_{Tf}] = \mathsf{E}[\mathsf{E}[\hat{v}' X_T + r_2 x_{Tf} | S_i]] \\
&= \mathsf{E}[\bar{v}_i' X_i + r_2 x_{if}] = \sum_{i=1}^{n_1} p_i \left(\bar{v}_i' X_i + r_2 x_{if} \right) = \sum_{i=1}^{n_1} \left(\widetilde{v_i}' X_i + \widetilde{r}_i x_{if} \right).
\end{aligned}
$$

By the definition

$$
A_i = \mathsf{E}[(\hat{v}_i - \bar{v}_i)(\hat{v}_i - \bar{v}_i)' | S_i] = \mathsf{E}[\hat{v}_i (\hat{v}_i)' | S_i] - \bar{v}_i \bar{v}_i',
$$

we have

$$
\mathsf{E}[\hat{v}_i (\hat{v}_i)' | S_i] = A_i + \bar{v}_i \bar{v}_i'.
$$

Then, the final variance is

$$
\begin{aligned}
\mathrm{Var}\,(W_T) &= \mathsf{E}[(\hat{v}' X_T + r_2 x_{Tf} - R)^2] \\
&= \mathsf{E}[(\hat{v}' X_T + r_2 x_{Tf})^2] - R^2 \\
&= \mathsf{E}[\mathsf{E}[(\hat{v}_i' X_i + r_2 x_{if})^2 | S_i]] - R^2 \\
&= \mathsf{E}[X_i' E[\hat{v}_i (\hat{v}_i)' | S_i] X_i + 2 r_2 x_{if} \bar{v}_i' X_i + (r_2 x_{if})^2] - R^2 \\
&= \sum_{i=1}^{n_1} p_i [X_i' (A_i + \bar{v}_i \bar{v}_i') X_i + 2 r_2 x_{if} \bar{v}_i' X_i + (r_2)^2 x_{if}^2] - R^2
\end{aligned}
$$

$$= \sum_{i=1}^{n_1} \begin{pmatrix} X_i \\ x_{if} \end{pmatrix}' \begin{pmatrix} \widetilde{A_i} & r_2\widetilde{v_i} \\ r_2\widetilde{v_i}' & r_2\widetilde{r_i} \end{pmatrix} \begin{pmatrix} X_i \\ x_{if} \end{pmatrix} - R^2.$$

This yields a deterministic form of the two-stage mean-variance model as follows:

$$(M_2) \quad \min \quad \sum_{i=1}^{n_1} \frac{1}{2} \begin{pmatrix} X_i \\ x_{if} \end{pmatrix}' \begin{pmatrix} \widetilde{A_i} & r_2\widetilde{v_i} \\ r_2\widetilde{v_i}' & r_2\widetilde{r_i} \end{pmatrix} \begin{pmatrix} X_i \\ x_{if} \end{pmatrix} - \frac{1}{2}R^2 \tag{9}$$

$$\text{s.t.} \quad u_0' X_0 + x_{0f} = B \tag{10}$$

$$v_i' X_0 + r_1 x_{0f} = u_i' X_i + x_{if}, \quad i = 1, 2, \cdots, n_1 \tag{11}$$

$$\sum_{i=1}^{n_1} \left(\widetilde{v_i}' X_i + \widetilde{r_i} x_{if} \right) = R. \tag{12}$$

Lemma 3.1 *We have*
(1) $\alpha_i > 0$, $\gamma_i > 0$, $\quad i = 1, 2, \cdots, n_1$;
(2) $\delta_i \geq 0$, *and* $\exists\, i$, *such that* $\delta_i > 0$, $\quad i = 1, 2, \cdots, n_1$;
(3) $p_0 \in (0, 1)$, $\rho \in (0, r)$.

Proof: The first two parts are obvious from the definitions. Then

$$0 < p_0 = \sum_{i=1}^{n_1} \frac{p_i}{\delta_i + 1} < \sum_{i=1}^{n_1} p_i = 1,$$

so $p_0 \in (0, \ 1)$.

Since $A_0 \succ 0$, and $\bar{v}_0 \neq r_1 u_0$, we have $\delta_0 > 0$. Thus

$$0 < \rho = \frac{p_0 r}{\delta_0 + 1} < r.$$

□

Therefore, our model is a strictly convex quadratic optimization problem. It can be solved by using the KKT conditions, as shown in the following theorem.

Theorem 3.1 *The two-stage mean-variance problem* (M_2) *has the following unique primal-dual solution,*

$$X_i = -\tfrac{\lambda_i}{\widetilde{r_i}} A_i^{-1} (\bar{v}_i - r_2 u_i), \qquad x_{if} = \tfrac{1}{r_2} \left[\tfrac{\lambda_i}{\widetilde{r_i}} (\gamma_i - r_2 \beta_i + 1) + \mu \right]$$

$$X_0 = -\tfrac{\lambda_0}{r_2 \tilde{r}} A_0^{-1} (\bar{v}_0 - r_1 u_0), \qquad x_{0f} = \tfrac{1}{r} \left[\tfrac{\lambda_0}{\tilde{r}} (\gamma_0 - r_1 \beta_0 + 1) + \mu \right]$$

$$\lambda_i = \tfrac{\widetilde{r_i}}{\delta_i + 1} \left[r_2 (v_i' X_0 + r_1 x_{0f}) - \mu \right], \quad \lambda_0 = \tfrac{\tilde{r}}{\delta_0 + 1} (rB - \mu) \equiv \rho (rB - \mu)$$

$$\mu = r \tfrac{R - \rho B}{r - \rho},$$

where λ_0, λ_i and μ are the Lagrangian multipliers of the model, and the associated risk is

$$\text{Var} \ = \frac{\rho}{r-\rho}(R - rB)^2.$$

Some comments are in order here. When R is set to be rB, the variance reaches its minimum value of zero, and the associated solutions are:

$$(X_0^*,\ x_{0f}^*) = (0,\ B),\ (X_i^*,\ x_{if}^*) = (0,\ r_1 B),\ \lambda^* = 0, \mu^* = rB.$$

Proof of Theorem 3.1: The Lagrangian function of (M_2) is

$$\begin{aligned} L \ &= \ \sum_{i=1}^{n_1} \frac{1}{2} \begin{pmatrix} X_i \\ x_{if} \end{pmatrix}' \begin{pmatrix} \widetilde{A_i} & r_2 \widetilde{v_i} \\ r_2 \widetilde{v_i}' & r_2 \widetilde{r_i} \end{pmatrix} \begin{pmatrix} X_i \\ x_{if} \end{pmatrix} - \frac{1}{2} R^2 \\ & \quad -\lambda_0 (u_0' X_0 + x_{0f} - B) - \sum_{i=1}^{n_1} \lambda_i (u_i' X_i + x_{if} - v_i' X_0 - r_1 x_{0f}) \\ & \quad -\mu \left(\sum_{i=1}^{n_1} \left(\widetilde{v_i}' X_i + \widetilde{r_i} x_{if} \right) - R \right). \end{aligned} \tag{13}$$

Therefore, the KKT optimality conditions are given as

$$\frac{\partial L}{\partial X_0} = 0 \quad : \quad -\lambda_0 u_0 + \sum_{i=1}^{n_1} \lambda_i v_i = 0 \tag{14}$$

$$\frac{\partial L}{\partial x_{0f}} = 0 \quad : \quad -\lambda_0 + \sum_{i=1}^{n_1} \lambda_i r_1 = 0 \tag{15}$$

$$\frac{\partial L}{\partial X_i} = 0 \quad : \quad \widetilde{A_i} X_i + r_2 \widetilde{v_i} x_{if} - \lambda_i u_i - \mu \widetilde{v_i} = 0 \tag{16}$$

$$\frac{\partial L}{\partial x_{if}} = 0 \quad : \quad r_2 \widetilde{v_i}' X_i + r_2 \widetilde{r_i} x_{if} - \lambda_i - \mu \widetilde{r_i} = 0 \tag{17}$$

$$\frac{\partial L}{\partial \lambda_0} = 0 \quad : \quad u_0' X_0 + x_{0f} = B \tag{18}$$

$$\frac{\partial L}{\partial \lambda_i} = 0 \quad : \quad u_i' X_i + x_{if} - v_i' X_0 - r_1 x_{0f} = 0 \tag{19}$$

$$\frac{\partial L}{\partial \mu} = 0 \quad : \quad \sum_{i=1}^{n_1} \left(\widetilde{v_i}' X_i + \widetilde{r_i} x_{if} \right) = R. \tag{20}$$

Solving the second stage variables in (17) gives

$$x_{if} = \frac{1}{r_2} \left[-\bar{v}_i' X_i + \frac{\lambda_i}{\widetilde{r_i}} + \mu \right].$$

Substituting the above equation into (16) yields

$$p_i A_i X_i + \frac{\lambda_i}{r_2} (\bar{v}_i - r_2 u_i) = 0.$$

Solving X_i in the above equation and substituting it back to the equation for x_{if} give the following,

$$\begin{aligned} X_i &= -\frac{\lambda_i}{\widetilde{r}_i} A_i^{-1}(\bar{v}_i - r_2 u_i) \\ x_{if} &= \frac{1}{r_2}\left[\frac{\lambda_i}{\widetilde{r}_i}(\gamma_i - r_2\beta_i + 1) + \mu\right]. \end{aligned}$$

Therefore, (19) becomes

$$v_i' X_0 + r_1 x_{0f} = \frac{\lambda_i}{r_2} \cdot \frac{\delta_i + 1}{\widetilde{r}_i} + \frac{\mu}{r_2}$$

and so

$$\lambda_i = \frac{\widetilde{r}_i}{\delta_i + 1}\left[r_2(v_i' X_0 + r_1 x_{0f}) - \mu\right].$$

Now we shall deal with the first stage variables. Condition (15) reads

$$-\lambda_0 + \sum_{i=1}^{n_1} r_1 \frac{\widetilde{r}_i}{\delta_i + 1}[r_2(v_i' X_0 + r_1 x_{0f}) - \mu] = 0$$

and so

$$-\lambda_0 + r r_2 \widetilde{v_0}' X_0 + r\tilde{r} x_{0f} - \mu\tilde{r} = 0$$

and

$$x_{0f} = \frac{1}{r}[-r_2\bar{v}_0' X_0 + \frac{\lambda_0}{\tilde{r}} + \mu].$$

Similarly, substituting λ_i and x_{0f} in (14) gives

$$-\lambda_0 u_0 + \sum_{i=1}^{n_1} v_i \frac{\widetilde{r}_i}{\delta_i + 1}[r_2(v_i' X_0 + r_1 x_{of}) - \mu] = 0;$$

that is,

$$-\lambda_0 u_0 + (r_2)^2 \widetilde{A_0} X_0 + r r_2 \widetilde{v_0} x_{0f} - \mu r_2 \widetilde{v_0} = 0,$$

which further yields

$$(r_2)^2 p_0 A_0 X_0 + \frac{\lambda_0}{r_1}(\bar{v}_0 - r_1 u_0) = 0,$$

and so

$$X_0 = -\frac{\lambda_0}{r_2\tilde{r}} A_0^{-1}(\bar{v}_0 - r_1 u_0).$$

Thus, we obtain

$$x_{0f} = \frac{1}{r}\left[\frac{\lambda_0}{\tilde{r}}(\gamma_0 - r_1\beta_0 + 1) + \mu\right].$$

In light of the above developed relations, equation (18) can be now rewritten as

$$B = -\frac{\lambda_0}{r_2\tilde{r}}(\beta_0 - r_1\alpha_0) + \frac{\lambda_0}{r\tilde{r}}(\gamma_0 - r_1\beta_0 + 1) + \frac{\mu}{r}$$

or, equivalently,

$$B = \frac{\lambda_0}{r} \times \frac{\delta_0 + 1}{\tilde{r}} + \frac{\mu}{r}$$

which gives

$$\lambda_0 = \frac{\tilde{r}}{\delta_0 + 1}(rB - \mu) = \rho(rB - \mu).$$

Using the expressions for X_i and x_{if}, we then have

$$\begin{aligned} R_i &= \widetilde{v_i}' X_i + \tilde{r}_i x_{if} \\ &= -\frac{\lambda_i}{r_2}(\gamma_i - r_2\beta_i) + \frac{\lambda_i}{r_2}(\gamma_i - r_2\beta_i + 1) + p_i\mu \\ &= \frac{\lambda_i}{r_2} + p_i\mu. \end{aligned}$$

We also have the following from the expressions of X_0 and x_{0f},

$$r_2\widetilde{v_0}' X_0 + \tilde{r}x_{0f} = \frac{\lambda_0}{r} + p_0\mu.$$

Therefore,

$$\begin{aligned} R &= \sum_{i=1}^{n_1} R_i \\ &= \sum_{i=1}^{n_1} \left(\widetilde{v_i}' X_i + \widetilde{r}_i x_{if}\right) \\ &= \sum_{i=1}^{n_1} \left(\frac{\lambda_i}{r_2} + p_i\mu\right) \\ &= \left\{\sum_{i=1}^{n_1} \frac{p_i}{\delta_i + 1}[r_2(v_i' X_0 + r_1 x_{0f}) - \mu]\right\} + \mu \\ &= r_2\widetilde{v_0}' X_0 + \tilde{r}x_{0f} - p_0\mu + \mu \\ &= \frac{\lambda_0}{r} + \mu \\ &= \frac{\rho}{r}(rB - \mu) + \mu, \end{aligned}$$

i.e.,

$$R = \rho B + \mu(r - \rho)/r$$

and so

$$\mu = r\frac{R - \rho B}{r - \rho}.$$

We now are ready to calculate the optimal variance. Let

$$\mathsf{E}(W_i^2) = X_i' A_i X_i + (\frac{R_i}{p_i})^2.$$

Using the expressions for R_i and X_i, we have

$$\begin{aligned}
\mathsf{E}(W_i^2) &= \left(\frac{\lambda_i}{\widetilde{r}_i}\right)^2 \delta_i + \left(\frac{\lambda_i}{\widetilde{r}_i} + \mu\right)^2 \\
&= \left(\frac{\lambda_i}{\widetilde{r}_i}\right)^2 (\delta_i + 1) + 2\frac{\lambda_i}{\widetilde{r}_i}\mu + \mu^2 \\
&= \frac{[r_2(v_i' X_0 + r_1 x_{0f}) - \mu]^2}{\delta_i + 1} + 2\frac{[r_2(v_i' X_0 + r_1 x_{0f}) - \mu]}{\delta_i + 1}\mu + \mu^2 \\
&= \frac{[r_2(v_i' X_0 + r_1 x_{0f})]^2}{\delta_i + 1} + \mu^2 \frac{\delta_i}{\delta_i + 1}.
\end{aligned}$$

Since Var $(W) = \sum_{i=1}^{n_1} p_i \mathsf{E}(W_i^2) - R^2$, it follows that

$$\begin{aligned}
\text{Var }(W) + R^2 &= (r_2)^2 \sum_{i=1}^{n_1} \frac{p_i}{\delta_i + 1}\left[(v_i' X_0)^2 + 2r_1 x_{0f} v_i' X_0 + (r_1 x_{0f})^2\right] + \mu^2 \sum_{i=1}^{n_1} \frac{p_i \delta_i}{\delta_i + 1} \\
&= (r_2)^2 \left[X_0' \widetilde{A_0} X_0 + 2 r_1 x_{0f} \widetilde{v_0}' X_0 + p_0 (r_1 x_{0f})^2\right] + \mu^2 (1 - p_0) \\
&= (r_2)^2 p_0 \left[X_0' A_0 X_0 + (\overline{v}_0' X_0 + r_1 x_{0f})^2\right] + \mu^2 (1 - p_0) \\
&= (r_2)^2 p_0 \left[\left(\frac{\lambda_0}{r_2 \tilde{r}}\right)^2 \delta_0 + \left(\frac{\lambda_0}{r_2 \tilde{r}} + \frac{\mu}{r_2}\right)^2\right] + \mu^2 (1 - p_0) \\
&= p_0 \left[\left(\frac{\lambda_0}{\tilde{r}}\right)^2 (\delta_0 + 1) + 2\frac{\lambda_0}{\tilde{r}}\mu + \mu^2\right] + \mu^2 (1 - p_0) \\
&= p_0 \left[\frac{(rB - \mu)^2}{\delta_0 + 1} + 2\frac{rB - \mu}{\delta_0 + 1}\mu + \mu^2\right] + \mu^2 (1 - p_0) \\
&= p_0 \frac{(rB)^2 + \delta_0 \mu^2}{\delta_0 + 1} + \mu^2 (1 - p_0) \\
&= r\rho B^2 + \mu^2 \left(1 - \frac{p_0}{\delta_0 + 1}\right) \\
&= r\rho B^2 + \mu^2 \left(1 - \frac{\rho}{r}\right) \\
&= r\rho B^2 + \frac{r}{r - \rho}(R - \rho B)^2.
\end{aligned}$$

Summarizing, we have

$$\text{Var }(W) = \frac{\rho}{r - \rho}(R - rB)^2.$$

□

3.2 Properties of the optimal portfolio

Based on the explicit forms of the optimal solutions, and also the special payoff structures of the options, we are now in a position to analyze the payoff of the corresponding portfolio. First, let us denote:

$$
\begin{aligned}
\psi &:= \frac{\rho r}{r-\rho} \\
\theta &:= \psi \cdot \frac{1}{r_2 \tilde{r}} A_0^{-1}(\bar{v}_0 - r_1 u_0) \\
\tau_i &:= k_i \frac{\psi}{r_1}[\gamma_0 - r_1\beta_0 + 1 - v_i' A_0^{-1}(\bar{v}_0 - r_1 u_0)] \\
\phi_i &:= \tau_i \cdot \frac{1}{\tilde{r}_i} A_i^{-1}(\bar{v}_i - r_2 u_i).
\end{aligned}
$$

Thus, the optimal solutions can be simplified as:

$$
\begin{array}{lll}
\lambda_0 = \psi(rB - R), & X_0 = (R - rB)\theta, & x_{0f} = B - u_0'\theta(R - rB), \\
\lambda_i = \tau_i(rB - R), & X_i = (R - rB)\phi_i, & x_{if} = v_i' X_0 + r_1 x_{0f} - u_i' X_i,
\end{array}
$$

where $\psi,\ \theta,\ \ \tau_i,\ \phi_i$ are all constants that are independent of R and B. So for a given scenario in the middle stage, the optimal second-stage solution X_i is linear in $(R - rB)$.

Proposition 3.1 *Given a fixed scenario in the middle stage, the optimal final payoff of this subtree is piecewise linear with breakpoints being the strike prices of two-stage options. The slopes of the payoff curve becomes steeper as the target R value gets larger.*

Proof: Suppose that for the given scenario S_i at the end of the first stage, the second stage optimal solution is

$$X_i = \phi_i(R - rB),$$

where $\phi_i = (\phi_{i0}, \phi_{i1}, \cdots, \phi_{im_2})'$. Suppose that there are two final scenarios S_{i1} and S_{i2} following S_i with $K_t < S_{i1} < S_{i2} < K_{t+1}$. Similar to the single-stage case, we have

$$\frac{P(S_{i1}) - P(S_{i2})}{S_{i1} - S_{i2}} = (R - rB)\sum_{j=0}^{t} \phi_{ij}.$$

Now we see if R is fixed, then between each pair of the neighboring strike prices, the slope of the payoff curve is constant, and so the payoff curve is piecewise linear, with the break points being the strike prices. Furthermore, if R changes, then the larger the R value, the steeper the slope of the curve. □

Proposition 3.2 *For a given scenario in the middle stage, for the final payoff of this subtree, the payoff curves with different R values intersect only at some fixed final scenarios, where the payoff is always rB.*

Proof: Suppose that from a given middle stage scenario S_i, there is a related final stage scenario S_{ij}. The final payoff is

$$\begin{aligned} P(S_i,\ S_{ij}) &= (S_{ij}\iota - K)'_+ X_i + r_2 x_{if} \\ &= (S_{ij}\iota - K)'_+ X_i + r_2[v_i' X_0 + r_1 x_{0f} - u_i' X_i] \\ &= [(S_{ij}\iota - K)_+ - r_2 u_i]' X_i + r_2[v_i - r_1 u_0]' X_0 + rB \\ &= [(S_{ij}\iota - K)_+ - r_2 u_i]' \phi_i (R - rB) + r_2[v_i - r_1 u_0]' \theta (R - rB) + rB \\ &= rB + (R - rB)\{[(S_{ij}\iota - K)_+ - r_2 u_i]' \phi_i + [r_2 v_i - r u_0]' \theta\}. \end{aligned}$$

For given $R > rB$, if $P(S_i,\ S_{ij}) = rB$, then

$$[(S_{ij}\iota - K)_+ - r_2 u_i]' \phi_i + [r_2 v_i - r u_0]' \theta = 0,$$

and $P(S_i,\ S_{ij})$ will remain rB for any other R value. If $P(S_i,\ S_{ij}) > rB$, then

$$[(S_{ij}\iota - K)_+ - r_2 u_i]' \phi_i + [r_2 v_i - r u_0]' \theta > 0.$$

Thus, the payoff is increasing in the R value. Similarly, if $P(S_i,\ S_{ij}) < rB$, then it will always be less than rB, and will be decreasing in R. □

Suppose that $R_1 > R_2$. For a fixed middle-stage scenario, if there are some final scenarios attaining the payoff rB, then the final payoff curves will all intersect in these scenarios. For other scenarios, the payoff curve of R_1 will dominate that of R_2 if $R_2 > rB$, and be dominated if $R_2 < rB$, as shown in the next example.

Now we shall concentrate on a final-stage scenario S. Note that there may be different paths to get to this scenario. Even though the eventual index value is the same, different middle-stage scenarios may lead to quite different payoffs.

Proposition 3.3 *Consider a final-stage scenario S. Suppose that there are different middle-stage scenarios that can reach S. Then the optimal payoffs from different paths will follow a fixed order. Furthermore, the differences are proportional to $R - rB$.*

Proof: Suppose there are two middle-stage scenarios S_i and S_j, such that both second stage subtrees contain the final-stage scenario with index value S. The optimal payoffs are

$$\begin{aligned} P(S_i,\ S) &= rB + (R - rB)\{[(S\iota - K)_+ - r_2 u_i]' \phi_i + [r_2 v_i - r u_0]' \theta\} \\ P(S_j,\ S) &= rB + (R - rB)\{[(S\iota - K)_+ - r_2 u_j]' \phi_j + [r_2 v_j - r u_0]' \theta\}. \end{aligned}$$

Therefore,

$$\begin{aligned} \Delta P_{ij}(S) &= P(S_i,\ S) - P(S_j,\ S) \\ &= (R - rB)\{(S\iota - K)'_+(\phi_i - \phi_j) - r_2(u_i'\phi_i - u_j'\phi_j) + r_2(v_i - v_j)'\theta\}. \end{aligned}$$

So for given S_i, S_j and R, $\Delta P_{ij}(S)$ is proportional to $R - rB$. If $\Delta P_{ij}(S) > 0$, then so is true for any $R > rB$. That is, the payoff dominance relationship is invariant with regard to R provided that $R > rB$.

If $\Delta P_{ij}(S) = 0$, then

$$(S\iota - K)'_+(\phi_i - \phi_j) - r_2(u_i'\phi_i - u_j'\phi_j) + r_2(v_i - v_j)'\theta = 0.$$

In that case, for any R, the two subtrees from S_i and S_j will always share the same optimal payoff at this final-stage scenario S. $\square$

Example 2 We solve a two-stage portfolio selection problem with options on the S&P 500 index, which are listed on the CBOE. The prices are drawn from the CBOE web page in the morning of Oct. 13, 2003, shown in Table 3. The portfolios can be constructed in the beginning of the investment and can also be reorganized on Oct. 31, 2003. The investment horizon is Nov. 21, 2003, on which the options can be exercised. The whole investment horizon is 39 days, with the first stage 18 days, and the second stage 21 days. In this example, we simply use the mid prices as the initial prices for both buying and selling. The options prices at the end of first stage are calculated using the Black-Scholes formula.

The scenario tree is generated under the assumption that the index value at the horizon is lognormally distributed with an expected annualized growth rate $\mu = 18.81\%$ and an annualized standard deviation of $\sigma = 18.45\%$, which are listed in the Standard & Poor's company's web page. The risk-free return rates for both stages are $r_1 = r_2 = 1+0.07\%$. The initial wealth is \$10,000, and we assume two different target expected payoffs $R_1 = rB = \$10,014$, and $R_2 = \$10,050$. Figure 2 shows the optimal payoff surfaces for R_1 and R_2. The flat surface is for the risk-free expected payoff R_1, and the other is for R_2. We see that they cross at some points, where the payoffs equal the risk-free return for any R values. Figure 3 shows the final payoff of one of the second stage subtree, in which we tried another $R_3 = \$10,025$. We see that the final payoff of the subtree has the same properties as those of the single-stage model in Section 2. Figure 4 is the mean-variance efficient frontier of the two-stage problem.

	Expiration	Bid	Mid	Ask
S&P 500		1038.06	1038.06	1038.06
Call 1010	November	42.50	43.50	44.50
Call 1020	November	35.40	36.40	37.40
Call 1050	November	18.00	18.60	19.20
Call 1065	November	11.60	12.35	13.10

Table 3: The data of the S&P 500 index and the options of Example 2

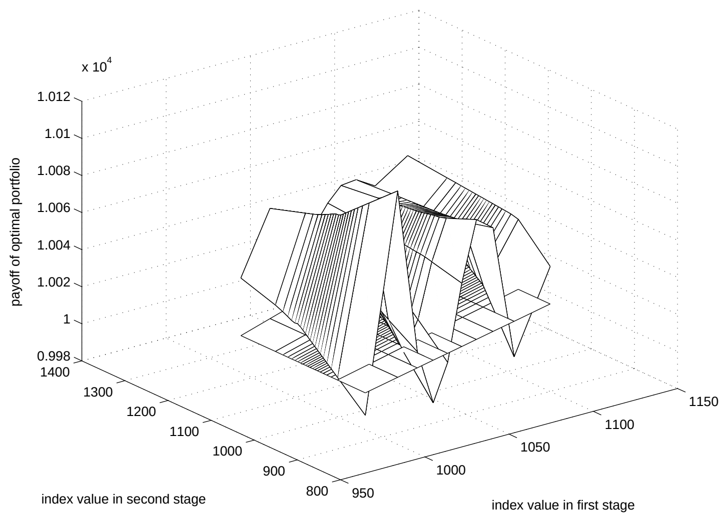

Figure 2: The optimal payoff surfaces of Example 2

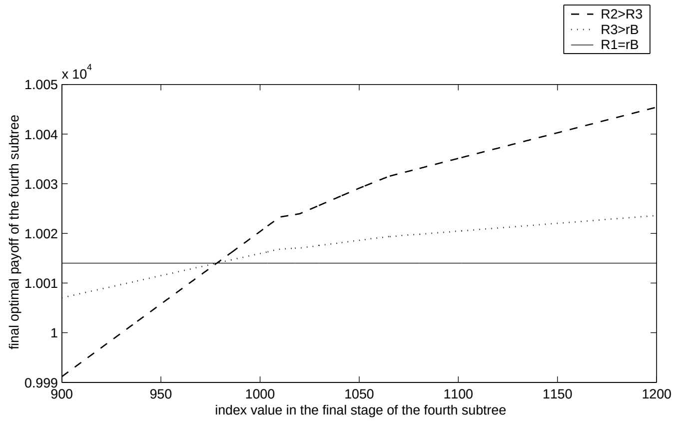

Figure 3: The optimal final payoff curves of a subtree

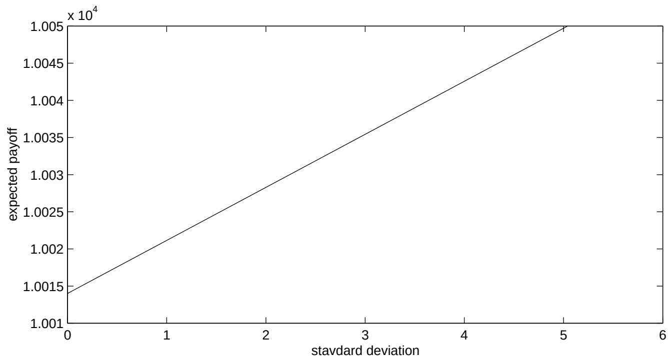

Figure 4: The efficient frontier of Example 2

References

[1] Albrecht, P., R. Maurer, and M. Timpel (1995): A shortfall approach to the evaluation of risk and return of positions with options, *Proceedings 5th AFIR International Colloquium, Brussels* 3, 1003-1023.

[2] Berkelaar, A., C. Dert, B. Odenkamp, and S. Z. Zhang (2002): A primal-dual decomposition-based interior point approach to two-stage stochastic linear progeamming, *Operations Research* 50, 904-915.

[3] Bookstaber, R. and R. G. Clarke (1981): Options can alter portfolio return distributions, *Journals of Portfolio Management* 7(3), 63-70.

[4] Bookstaber, R. and R. G. Clarke (1984): Option portfolio strategies: measurement and evaluation, *Journal of Business* 57, 469-492.

[5] Bookstaber, R. and R. G. Clarke (1985): Problems in evaluating the performence of portfolios with options, *Financial Analysts Journal* 41(1), 48-62.

[6] Brooks, R., H. Levy, and J. Yoder (1987): Using stochastic dominance to evaluate the performance of portfolios with options, *Financial Analysts Journal* 43(2), 79-82.

[7] Dert, C. and B. Oldenkamp (2000): Optiaml guaranteed return portfolio and the casino effect, *Operations Research* 48, 768-775.

[8] Hull, J. C. (1999): Options, Futures, & Other Derivatives, Prentice Hall, Upper Saddle River, NJ07458.

[9] Isakov, D. and B. Morard (2001): Improving portfoio performance with option strategies: evidence from Switzerland, *European Financial Management* 7(1), 73-91.

[10] Koskosidis, Y. A. and A. M. Duarte (1997): A scenario-based approach to active asset allocation, *The Journal of Portfolio Management* 23(2), 74-85.

[11] Li, D., and W. L. Ng (2000): Optimal dynamic portfolio selection: multi-period mean-variance formulation, *Mathematical Finance* 10, 387-406.

[12] Markowitz, H. M. (1952): Portfolio Selection, *Journal of finance* 7, 77-91.

[13] Markowitz, H. M. (1959): Portfolio Selection: Efficient Diversification of Investment, Wiley, New York.

[14] Mcmillan, L. G. (2002): Options as a Strategic Investment, New York Institute of Finance.

[15] Morard, B. and A. Naciri (1990): Options and investment strategies. *Journal of Future Markets* 10, 505-517.

[16] Odenkamp, B. (1999): Derivatives in Portfolio Management. Ph,D. thesis, Erasmus University Rotterdam, Thesis Publishers, Amsterdam, The Netherlands.

[17] Pelsser, A. and T. Vorst (1995): Optiomal optioned portfolios with confidence limits on shortfall constraints, *Advances in Quantitative Analysis of Finance and Accounting* 3A, 205-220.

[18] Pounds, H. M. (1978): Covered call oprion writing: strategies and rules, *Journal of Portfolio Management* 4(2), 31-42.

[19] Zhou, X. Y. and D. Li (2000): Continuous-time mean-variance portfolio selection: a stochastic LQ framework, *Applied Mathematics Optimization* 42, 19-33.