Improved approximation algorithm for the facility location problem with service installation costs

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Abstract

In this paper, we consider the uncapacitated facility location problem with service installation costs. Our main result is an LP rounding 1.808-approximation algorithm under the assumption that the installation cost depends only on the service which improves the previously approximation ratio 2.391 from [11].

Keywords: Facility location problem, approximation algorithm, linear programming relaxation

1. Introduction

In the uncapacitated facility location problem (UFL), we are given a set of clients or demands, D, and a set of facilities F. We want to open a subset of the facilities such that all the clients are served by the open facilities and the total costs of opening facilities and serving clients is minimized. The first constant approximation algorithm for metric UFL is given by [12] which based on LP rounding. This ratio is improved subsequently by different approaches such as LP-rounding [5, 13], primal-dual schema [9], scaling and greedy augmentation [6, 7], dual fitting and factor-revealing LP [8] etc. Noting that the 1.52-approximation algorithm of [10] does not close

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the gap with the lower bound [4], Byrka [3] obtains a 1.50-approximation algorithm based on LP rounding and dual fitting approaches, which is the currently best guarantee. For other variations of UFL, we refer to [1, 2, 11, 14, 15, 16, 17] and references therein.

In the uncapacitated facility location problem with service installation costs (UFLSC) which first introduced by [11], each client $j \in D$ requests a specific service $g(j) \in S$. To satisfy client j we have to assign it to an open facility on which service $g(j)$ is installed. If we install service l on an open facility i we incur a service installation cost of s_{il}. We want to open a set of facilities, install service at the open facilities, and assign each client j to an open facility i such that service $g(j)$ is installed at i. The cost of a solution is the sum of the facility opening costs, the service installation costs and the client assignment costs, and the goal is to find a solution with minimum total cost. We assume that the assignment costs c_{ij} form a metric.

Our main result is an LP rounding 1.808-approximation algorithm under the assumption that $s_{il} = s_l$, i.e., the installation cost depends only on the service l and not on the location at which it is installed. This is currently the best known guarantee. The previously known approximation ratio for this special case is 2.391 ([11]). Our algorithm is based on the techniques of [3, 5, 11]. Same as [3], we use $D^C_{\max}(j)$ (see Section 2) as g-close (cf. [12]) which results in more tight analysis for expected connection cost than [11]. Of course, the lower bound 1.463 for UFL [7] still holds for UFLSC.

The linear programming relaxation (from [11]) is as follows:

$$\begin{aligned}
\min \quad & \sum_{i\in F, j\in D} c_{ij}x_{ij} + \sum_{i\in F} f_i y_i + \sum_{i\in F, l\in S} s_l z_{il} & \\
\text{s.t.} \quad & \sum_{i\in F} x_{ij} = 1, & \text{for all } j \in D \\
& x_{ij} \le z_{ig(j)}, & \text{for all } i \in F, j \in D \\
& x_{ij} \le y_i, & \text{for all } i \in F, j \in D \\
& x_{ij}, y_i, z_{il} \ge 0, & \text{for all } i \in F, j \in D, l \in S.
\end{aligned} \tag{1.1}$$

Variable y_i indicates if facility i is open, z_{il} indicates if service type l is installed at the i, and x_{ij} indicates if client j is connected to facility i. The first constraint states that each client must be connected to a facility, the second and the third constraints say that if client j is assigned to facility i, then service $g(j)$ must be installed on i and i must be open. An integral $\{0, 1\}$ solution corresponds exactly to a solution of UFLSC.

2. Our algorithm

Let (x^*, y^*, z^*) be optimal solution of (1.1). The corresponding objective value is $C^* + F^* + S^*$, where $C^* = \sum\limits_{i\in F, j\in D} c_{ij}x^*_{ij}$, $F^* = \sum\limits_{i\in F} f_i y^*_i$, and $S^* = \sum\limits_{i\in F, l\in S} s_l z^*_{il}$. Let $C^*_j = \sum\limits_{i\in F} c_{ij}x^*_{ij}$. Let G^l be the set of clients requesting service l.

We discuss some properties of the solution of (1.1). The following lemma follows from the approach of [5, 13].

Lemma 2.1 *Suppose (x, y, z) is a feasible solution to the linear program (1.1) for a given instance of UFLSC I. Then we can find, in polynomial time, an equivalent instance $\tilde{I}$ and a feasible solution $(\tilde{x}, \tilde{y}, \tilde{z})$ such that both solutions have the same fractional facility opening costs, service installation costs and client assignment costs. The new instance $\tilde{I}$ differs only by replacing each facility location by at most $(|S| - 1)(|D| - 1) + 1$ copies of the same location and the new feasible solution $(\tilde{x}, \tilde{y}, \tilde{z})$ has three additional properties:*

1. $\tilde{x}_{ij} = \tilde{z}_{ig(j)}$ for all $i \in F$, $j \in D$ such that $\tilde{x}_{ij} > 0$.

2. $\tilde{z}_{il} = \tilde{y}_i$ for all $i \in F$, $l \in S$ such that $\tilde{z}_{il} > 0$.

3. For any demand point j there is a permutation π^j of facility points $\tilde{F}$ and facility point $\pi^j_{s(j)}$ such that $c_{\pi_1 j} \le c_{\pi_2 j} \le \cdots \le c_{\pi_n j}$ (we omit superscript j from π^j) and $\sum\limits_{i=1}^{s(j)} \tilde{y}_{\pi_i} = 1$.

Proof : We start with $\tilde{F} = F$ and $(\tilde{x}, \tilde{y}, \tilde{z}) = (x, y, z)$. On each iteration, we will change an instance of the problem by adding new copies for some facilities, deleting the old ones and changing the current feasible solution.

1. Pick any facility i which violates the first property, i.e. there is a client j such that $0 < \tilde{x}_{ij} < \tilde{z}_{ig(j)}$. Let j_0 be a client with smallest value of $\tilde{x}_{ij}$ among all clients with $0 < \tilde{x}_{ij} < \tilde{z}_{ig(j)}$. Instead of facility i create two new facilities i_1 and i_2 with the same location and opening cost and set $\tilde{z}_{i_1 g(j)} = \tilde{x}_{ij_0}$, $\tilde{z}_{i_2 g(j)} = \tilde{z}_{ig(j)} - \tilde{x}_{ij_0}$, $\tilde{x}_{i_1 j} = \tilde{x}_{ij_0}$, and $\tilde{x}_{i_2 j} = \tilde{x}_{ij} - \tilde{x}_{ij_0}$ for all j such that $0 < \tilde{x}_{ij} < \tilde{z}_{ig(j)}$. Set $\tilde{z}_{i_1 g(j)} = \tilde{z}_{ig(j)}$, $\tilde{z}_{i_2 g(j)} = 0$, $\tilde{x}_{i_1 j} = \tilde{x}_{ij}$, and $\tilde{x}_{i_2 j} = 0$ for all j such that $\tilde{x}_{ij} = 0$ or $\tilde{z}_{ig(j)}$. Repeat the process until the solution satisfies property 1 of the lemma. During the process, each facility can be copied at most $|D| - 1$ times.

2. Noting $\tilde{z}_{il} \le \tilde{y}_i$, we can prove the second property in a similar way.

3. The third property is proved identically as [13].

□

Without loss of generality, we assume that the optimal fractional solution (x^*, y^*, z^*) has the three property of Lemma 2.1. Modify (x^*, y^*, z^*) by scaling the y and z by a parameter $\gamma > 1$ (which will be specified later) to obtain a suboptimal fractional solution $(x^*, \gamma y^*, \gamma z^*)$.

Then we want to change the x to minimize the total cost. For each client $j \in D$, denote $F_j = \{i \in F | x^*_{ij} > 0\}$. We choose an ordering of facilities in F_j with nondecreasing distances to client j. Connect client j to the first facilities in the ordering so that for any two facilities i' and i'' such that i'' is later in the ordering if $x_{i'j} < y_{i'}$ than $x_{i''j} = 0$.

Adjust the above solution to get a complete solution (i.e. there are no $i \in F, j \in D$ such that $0 < x_{ij} < z_{ig(j)}$ or $0 < x_{ij} < y_i$). Let $(\bar{x}, \bar{y}, \bar{z})$ denote the obtained complete solution.

For a client $j \in D$ we say that a facility i is one of his close facilities if it fractionally serves j in $(\bar{x}, \bar{y}, \bar{z})$. If $\bar{x}_{ij} = 0$, but facility i was serving client j in solution (x^*, y^*, z^*), then we say, that i is a distant facility of client j. For each $j \in D$, denote $N_j = \{i \in F | \bar{x}_{ij} > 0\}$ and $M_j = \{i \in F | \bar{x}_{ij} = 0 \text{ and } x^*_{ij} > 0\}$. For every j, we have that

his average distance to a close facility is $D^C_{\text{av}}(j) = \sum_{i \in F} c_{ij}\bar{x}_{ij}$,

his average distance to a distant facility is $D^D_{\text{av}}(j) = \dfrac{\gamma}{\gamma - 1} \sum_{i \in M_j} c_{ij} x^*_{ij}$,

his maximal distance to a close facility is at most the average distance to a distant facility, $D^C_{\max}(k) \leq D^D_{\text{av}}(k)$.

One can prove that

$$C^*_j - D^C_{\text{av}}(j) = (\gamma - 1)(D^D_{\text{av}}(j) - C^*_j). \tag{2.1}$$

$$\begin{aligned} D^D_{\text{av}}(j) - C^*_j &= \frac{\gamma}{\gamma - 1} \sum_{i \in M_j} c_{ij} x^*_{ij} - \sum_{i \in F_j} c_{ij} x^*_{ij} \\ &= \frac{1}{\gamma - 1} \sum_{i \in M_j} c_{ij} x^*_{ij} - \sum_{i \in N_j} c_{ij} x^*_{ij} \\ &\leq \frac{1}{\gamma - 1} C^*_j. \end{aligned} \tag{2.2}$$

Let

$$h(\gamma) = \frac{1}{e} + \frac{1}{e^\gamma} - (\gamma - 1)(1 - \frac{1}{e} + \frac{1}{e^\gamma}).$$

Denote $\bar{h}(\gamma) = \max\{h(\gamma), 0\}$. Given j, $k \in D$, we say that j and k are dependent if $N_j \bigcap N_k = \emptyset$.

Now we are ready to present the LP rounding algorithm as follows.

Algorithm 2.2

Step 1. Solve the LP relaxation to obtain a solution (x^*, y^*, z^*).

Step 2. Scale up the value of the y *and* z *by a constant* $1 < \gamma < 2$, *then change the value of the* x *so as to use the closest possible fractionally open facilities.*

Step 3. Split facilities to obtain a complete solution $(\bar{x}, \bar{y}, \bar{z})$.

Step 4. Compute a greedy clustering for the solution $(\bar{x}, \bar{y}, \bar{z})$. *For every service type* l, *we consider the clients in* G_l *and cluster the facilities on which service* l *is installed around some cluster centers: pick* $j \in G_l$ *with smallest* $\bar{h}(\gamma)D_{\text{av}}^D(j) + D_{\max}^C(j) + \left(1 + \frac{2}{e^\gamma} - \bar{h}(\gamma)\right) C_j^*$ *as the cluster center. Form a cluster around* j *consisting of the facilities in* N_j. *We remove every client* $k \in G_l$ *(including* j*) that is dependent with* j. *Recurse on the remaining set of clients until no client in* G_l *is left. Let* D_l *be the set of cluster centers.*

Step 5. Let $\tilde{D} = \bigcup_l D_l$. *Consider clients in* $\tilde{D}$ *in order of increasing* $D_{\max}^C(j) + D_{\text{av}}^C(j)$ *and choose a maximal independent subset* $\tilde{D}'$.

Step 6. For each client $j \in \tilde{D}'$, *open one of his close facilities randomly with probabilities* $\bar{x}_{ij}$.

Step 7. For each facility $i \in F$ *that is not a close facility of any cluster center* $j \in \tilde{D}'$, *open it independently with probability* $\bar{y}_i$.

Step 8. For any facility $i \in F$, *if* i *is opened, we install on it all services that are installed on it in the fractional solution, i.e., all* l *such that* $z_{il}^* > 0$.

Step 9. For each client $j \in \tilde{D} \backslash \tilde{D}'$, *there is some* $k \in \tilde{D}'$ *with* $D_{\max}^C(k) + D_{\text{av}}^C(k) \le D_{\max}^C(j) + D_{\text{av}}^C(j)$ *such that* k *and* j *are dependent. Call* k *the neighbor of* j *and denote it* $\text{nbr}(j)$. *If no facility from* F_j *is open, we install service* $g(j)$ *on the facility opened in Step 6 from* $N_{\text{nbr}(j)}$.

Step 10. Connect each client j *to the closest open facility at which service* $g(j)$ *is installed.*

3. Analysis of the algorithm

Lemma 3.1 *Given* n *independent events* $a_1, a_2, \ldots, a_n$, *that occur with probabilities* $p_1, p_2, \ldots, p_n$ *respectively, the event* $\bar{a}_1 \bigcap \bar{a}_2 \bigcap \ldots \bigcap \bar{a}_n$ *(i.e. none of* a_i*) occurs with probability at most* $e^{-\sum_{i=1}^{n} p_i}$, *where* e *denote the base of the natural logarithm.*

The following lemma is exactly Lemma 4.4 of [11]. We present the proof for completeness.

Lemma 3.2 *The expected cost of opening facilities is γF^*. The expected cost of installing services is at most $(\gamma + \frac{1}{e^\gamma})S^*$.*

Proof : Each facility i is opened with probability γy^*. The expected facility opening cost of the solution is

$$E[F_{SOL}] = \sum_{i \in F} f_i \bar{y}_i = \gamma \sum_{i \in F} f_i y_i^* = \gamma F^*.$$

There are two cases which we install services at some open facilities.

Case 1. The cost of installing services in Step 8 is bounded by

$$\begin{aligned}\sum_{i \in F} \Pr\{i \text{ is opened}\} \sum_{l \in \{l | z_{il}^* > 0\}} s_l &= \sum_{i \in F} \gamma y_i^* \sum_{l \in \{l | z_{il}^* > 0\}} s_l \\ &= \gamma \sum_{i \in F, l \in S} s_l z_{il}^* = \gamma S^*.\end{aligned}$$

Case 2. Consider client $j \in \tilde{D} \backslash \tilde{D}'$ with $g(j) = l$. The probability that service l is installed in Step 9 due to client j, is the probability that no facility from F_j is open after Step 6 and 7, which is at most $\frac{1}{e^\gamma}$. So the cost of installing services in Step 9 is at most

$$\frac{1}{e^\gamma} \sum_{j \in \tilde{D} \backslash \tilde{D}'} s_{g(j)} \leq \frac{1}{e^\gamma} \sum_{i \in F, l \in S} s_l z_{il}^*.$$

Combining the above two cases, we have

$$E[S_{SOL}] \leq (\gamma + \frac{1}{e^\gamma})S^*.$$

□

Lemma 3.3 *The expected connection cost is at most*

$$\left(1 + 2(1 + \frac{1}{e^\gamma})\frac{1}{e^\gamma} + \frac{1}{\gamma - 1} \max\left\{\left(\frac{1}{e} + (1 + \bar{h}(\gamma))\frac{1}{e^\gamma} - (\gamma - 1)(1 - \frac{1}{e})\right), 0\right\}\right) C^*.$$

Proof : For each client $j \in D$, we consider the following three possibilities.

Case 1. If $j \in \tilde{D}'$, one of his close facilities is open and the expected distance to this open facility is $D_{\mathrm{av}}^C(j)$.

Case 2. If $j \in \tilde{D} \backslash \tilde{D}'$, there are three sub-possibilities.

Case 2.1. One of close facilities of j is open. The expected distance to the closest open facility is at most $D_{\mathrm{av}}^C(j)$. From Lemma 3.1, with probability $p_c \geq 1 - \frac{1}{e}$, at least one close facility is open.

Case 2.2. None of the close facilities of j is open, but at least one of his distant facilities is open. Let p_d denote the probability of this event. The expected distance to the closest facility is then at most $D^D_{\text{av}}(j)$.

Case 2.3. Neither any close nor any distant facility of j is open, then he connects to the facility serving $k = \text{nbr}(j)$. Such event happens with probability $p_s \leq \frac{1}{e^\gamma}$. The expected distance from j to the facility serving k is at most $D^C_{\max}(j) + D^C_{\max}(k) + D^C_{\text{av}}(k) \leq 2D^C_{\max}(j) + D^C_{\text{av}}(j)$.

Summing the above three sub-possibilities, the expected total connection cost of j is bounded by

$$\begin{aligned}
& p_c D^C_{\text{av}}(j) + p_d D^D_{\text{av}}(j) + p_s(2D^C_{\max}(j) + D^C_{\text{av}}(j)) \\
\leq\ & (p_d + 2p_s) D^D_{\text{av}}(j) + (p_c + p_s) D^C_{\text{av}}(j) \\
=\ & (1 + 2p_s)C^*_j + (p_d + 2p_s)(D^D_{\text{av}}(j) - C^*_j) + (p_c + p_s)(D^C_{\text{av}}(j) - C^*_j) \\
=\ & (1 + 2p_s)C^*_j + (p_d + 2p_s - (\gamma - 1)(p_c + p_s)) \cdot (D^D_{\text{av}}(j) - C^*_j) \\
\leq\ & (1 + \frac{2}{e^\gamma})C^*_j + \left(\frac{1}{e} + \frac{1}{e^\gamma} - (\gamma - 1)(1 - \frac{1}{e} + \frac{1}{e^\gamma})\right) \cdot (D^D_{\text{av}}(j) - C^*_j) \\
\leq\ & (1 + \frac{2}{e^\gamma})C^*_j + \bar{h}(\gamma)(D^D_{\text{av}}(j) - C^*_j) \\
=\ & \bar{h}(\gamma) D^D_{\text{av}}(j) + \left(1 + \frac{2}{e^\gamma} - \bar{h}(\gamma)\right) C^*_j.
\end{aligned}$$

It is easy to see that

$$\bar{h}(\gamma) D^D_{\text{av}}(j) + \left(1 + \frac{2}{e^\gamma} - \bar{h}(\gamma)\right) C^*_j \geq (1 + \frac{2}{e^\gamma})C^*_j \geq C^*_j \geq D^C_{\text{av}}(j). \tag{3.1}$$

Case 3. If $j \in D \backslash \tilde{D}$, there must be a client $k' \in D_{g(j)}$ such that j was removed from $G_{g(j)}$ because a cluster was builded around k' in Step 4. Again, we consider three sub-possibilities as follows.

Case 3.1. One of close facilities of j is open. The expected distance to the closest open facility is at most $D^C_{\text{av}}(j)$. From Lemma 3.1, with probability $p_c \geq 1 - \frac{1}{e}$, at least one close facility is open.

Case 3.2. None of the close facilities of j is open, but at least one of his distant facilities is open. Let p_d denote the probability of this event. The expected distance to the closest facility is then at most $D^D_{\text{av}}(j)$.

Case 3.3. Neither any close nor any distant facility of j is open, then he connects to the facility serving k'. Such event happens with probability $p_s \leq \frac{1}{e^\gamma}$. Combining Case 1 and 2, together with (3.1), we have that the expected distance of k' is at most

$$\bar{h}(\gamma)D_{\text{av}}^{D}(k') + \left(1 + \frac{2}{e^{\gamma}} - \bar{h}(\gamma)\right) C_{k'}^{*}.$$

The expected distance from j to the facility serving k' is at most

$$\begin{aligned}
& D_{\max}^{C}(j) + D_{\max}^{C}(k') + \left(\bar{h}(\gamma)D_{\text{av}}^{D}(k') + \left(1 + \frac{2}{e^{\gamma}} - \bar{h}(\gamma)\right) C_{k'}^{*}\right) \\
\leq \ & \bar{h}(\gamma)D_{\text{av}}^{D}(j) + 2D_{\max}^{C}(j) + \left(1 + \frac{2}{e^{\gamma}} - \bar{h}(\gamma)\right) C_{j}^{*}.
\end{aligned}$$

Summing all the above three cases, together with (2.1)-(2.2) and (3.1), we bound the expected total connection cost

$$\begin{aligned}
E[C_{SOL}] &\leq \sum_{j\in D}\left(p_c D_{\text{av}}^{C}(j) + p_d D_{\text{av}}^{D}(j) + p_s\left(\bar{h}(\gamma)D_{\text{av}}^{D}(j) + 2D_{\max}^{C}(j) + \left(1 + \frac{2}{e^{\gamma}} - \bar{h}(\gamma)\right) C_{j}^{*}\right)\right) \\
&\leq \left(1 + \frac{2}{e^{\gamma}} - \bar{h}(\gamma)\right) p_s C^{*} + \sum_{j\in D}\left((p_d + (2 + \bar{h}(\gamma))p_s)D_{\text{av}}^{D}(j) + p_c D_{\text{av}}^{C}(j)\right) \\
&= \left(1 + 2(1 + \frac{1}{e^{\gamma}})p_s\right) C^{*} + \sum_{j\in D}\left((p_d + (2 + \bar{h}(\gamma))p_s)(D_{\text{av}}^{D}(j) - C_{j}^{*}) + p_c(D_{\text{av}}^{C}(j) - C_{j}^{*})\right) \\
&= \left(1 + 2(1 + \frac{1}{e^{\gamma}})p_s\right) C^{*} + \sum_{j\in D}(p_d + (2 + \bar{h}(\gamma))p_s - (\gamma-1)p_c)\cdot(D_{\text{av}}^{D}(j) - C_{j}^{*}) \\
&\leq \left(1 + 2(1 + \frac{1}{e^{\gamma}})\frac{1}{e^{\gamma}}\right) C^{*} + \sum_{j\in D}\left(\frac{1}{e} + (1 + \bar{h}(\gamma))\frac{1}{e^{\gamma}} - (\gamma-1)(1 - \frac{1}{e})\right)\cdot(D_{\text{av}}^{D}(j) - C_{j}^{*}) \\
&\leq \left(1 + 2(1 + \frac{1}{e^{\gamma}})\frac{1}{e^{\gamma}} + \frac{1}{\gamma-1}\max\left\{\left(\frac{1}{e} + (1 + \bar{h}(\gamma))\frac{1}{e^{\gamma}} - (\gamma-1)(1 - \frac{1}{e})\right), 0\right\}\right) C^{*}.
\end{aligned}$$

□

Theorem 3.4 *Taking* $\gamma = 1.6067$*, Algorithm 2.2 produces a solution with expected cost at most* $1.808 \cdot OPT$*.*

Proof : Combing Lemma 3.2-3.3, we conclude the theorem.

□

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