A Semidefinite Relaxation Scheme for Multivariate Quartic Polynomial Optimization With Quadratic Constraints

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Abstract

We present a general semidefinite relaxation scheme for general n-variate quartic polynomial optimization under homogeneous quadratic constraints. Unlike the existing sum-of-squares (SOS) approach which relaxes the quartic optimization problems to a sequence of (typically large) linear semidefinite programs (SDP), our relaxation scheme leads to a (possibly nonconvex) quadratic optimization problem with linear constraints over the semidefinite matrix cone in $\mathbb{R}^{n\times n}$. It is shown that each α-factor approximate solution of the relaxed quadratic SDP can be used to generate in randomized polynomial time an $O(\alpha)$-factor approximate solution for the original quartic optimization problem, where the constant in $O(\cdot)$ depends only on problem dimension. In the case where only one positive definite quadratic constraint is present in the quartic optimization problem, we present a polynomial time approximation algorithm which can provide a guaranteed relative approximation ratio of $(1 - O(n^{-2}))$.

1 Introduction

In this paper we consider the optimization of a multivariate fourth order (quartic) homogenous polynomial under quadratic constraints. The problem can take either the maximization form

$$\boxed{\begin{array}{ll} \text{maximize} & f(x) = \displaystyle\sum_{1\le i,j,k,\ell\le n} a_{ijk\ell}x_ix_jx_kx_\ell \\ \text{subject to} & x^{\mathrm{T}}A_ix \le 1,\ i=1,...,m, \end{array}} \tag{1.1}$$

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or the minimization form

$$\boxed{\begin{array}{ll} \text{minimize} & f(x) = \sum\limits_{1\le i,j,k,\ell\le n} a_{ijk\ell} x_i x_j x_k x_\ell \\ \text{subject to} & x^{\mathrm{T}} A_i x \ge 1,\ i = 1, ..., m, \end{array}} \tag{1.2}$$

where A_i's are positive semidefinite matrices in $\mathbb{R}^{n\times n}$, $i = 1, ..., m$. Let $f_{\max}$ and $f_{\min}$ denote the optimal values of (1.1) and (1.2) respectively. Throughout this paper, we assume $\sum_{i=1}^m A_i \succ 0$ so that the overall feasible region is compact and the optimal values $f_{\max}$ and $f_{\min}$ are attained.

Quartic optimization problems arise in various engineering applications such as independent component analysis [7], blind channel equalization in digital communication [10] and sensor localization [19]. In particular, the latter problem takes the form of

$$\begin{array}{ll} \text{minimize} & \sum\limits_{i,j\in\mathcal{S}} \left(\|\mathbf{x}_i - \mathbf{x}_j\|^2 - d_{ij}^2\right)^2 + \sum\limits_{i\in\mathcal{S}, j\in\mathcal{A}} \left(\|\mathbf{x}_i - \mathbf{s}_j\|^2 - d_{ij}^2\right)^2 \\ \text{subject to} & \mathbf{x}_i \in \mathbb{R}^3,\ i \in \mathcal{S} \end{array} \tag{1.3}$$

where $\mathcal{A}$ and $\mathcal{S}$ denote the set of anchor nodes and sensor nodes respectively, d_{ij}'s are (possibly noisy) distance measurements, $\mathbf{s}_j$'s denote the known positions of anchor nodes, while $\mathbf{x}_i$'s represent the positions of sensor nodes to be estimated. By homogenizing the objective function, we immediately obtain a quartic minimization problem in the form (1.2) with a single quadratic constraint. The sensor localization problem (1.3) is known to be NP-hard[1]. Therefore, from the complexity standpoint, the nonconvex quartic polynomial optimization problems (1.1)–(1.2) are NP-hard. This motivates us to consider polynomial time relaxation procedures that can deliver provably high quality approximate solutions for the quartic optimization problems (1.1)–(1.2).

As a special case of the general polynomial optimization problem, the quartic optimization problems (1.1)–(1.2) can be relaxed using the standard Sum of Squares (SOS) procedure of semidefinite programming relaxation. Specifically, by representing each nonnegative polynomial as a sum of squares of some other polynomials [9, 16] of a given degree, it is possible to relax each polynomial inequality as a convex linear matrix inequality (LMI). In this way, as the polynomial degree in SOS representation increases, the nonconvex quartic optimization problems (1.1)–(1.2) can be approximated by a hierarchy of semidefinite programs (SDP) with increasing size. While this SOS relaxation scheme can achieve, at least theoretically, the global optimality, the size of the resulting SDPs in the hierarchy grows exponentially fast in the problem dimension. Moreover, if we use only a finite number of levels in the SOS hierarchy before the eventual optimality is attained, we have no good error estimate available to determine the quality of the resulting approximate solution. These and other factors have severely limited the application scope of SOS polynomial optimization procedure in practice. Indeed, so far the most effective use of SDP relaxation has

[1] In fact, even determining if the optimal objective value of (1.3) is zero or not is NP-hard.

been for the quadratic optimization problems whereby only the first level relaxation in the SOS hierarchy is used. Even though such SDP relaxation cannot always achieve global optimality, it does lead to provably high quality approximate solution for certain type of quadratic optimization problems. The latter includes various graph problems such as the Max-Cut problem [6] as well as some homogeneous nonconvex quadratic optimization problems [11–14, 20].

The goal of this paper is to extend the existing strong SDP approximation results for quadratic problems to the quartic optimization problems (1.1)–(1.2). In contrast to the SOS approach which represents each quartic polynomial as a *linear* function of a matrix variable of size at least $n^2 \times n^2$, we propose to represent each quartic polynomial as a (possibly nonconvex) *quadratic* function in terms of a matrix variable of size $n \times n$ only. In this way, we ensure an acceptable problem size in the relaxation process. More specifically, we adopt, as in the case of SDP relaxation for quadratic optimization, the matrix lifting transformation $X = xx^{\mathrm{T}}$. Under this transformation, each quartic polynomial of x is mapped to a quadratic function in X, although not necessarily uniquely. For example, the monomial $x_1x_2x_3x_4$ can be relaxed to either $X_{12}X_{34}$, $X_{13}X_{24}$ or $X_{14}X_{23}$. Which of these relaxations should be used? How will the choice of relaxation affect the quality of SDP relaxation? We will address this nonuniqueness problem and analyze the approximation quality of the resulting SDP relaxation in this paper.

The main contributions of this paper are as follows. First, we propose to use a symmetric matrix-lifting mapping to relax each quartic polynomial to a quadratic function. For instance, the monomial $x_1x_2x_3x_4$ will be relaxed symmetrically to $(X_{12}X_{34} + X_{13}X_{24} + X_{14}X_{23})/3$. As a result, the quartic optimization problems (1.1)–(1.2) are relaxed to quadratic optimization problems under LMI constraints. Unfortunately (and not surprisingly), the resulting quadratic SDP problem remains NP-hard. Nonetheless, we prove that the ratio of optimal values of between the quartic optimization problems and their quadratic SDP relaxations are finitely bounded and independent of problem data. More importantly, we show that each constant-factor approximate solution of the relaxed quadratic SDP problem can be used to generate, in randomized polynomial time, a constant-factor approximate solution to the original quartic optimization problem. In this way, we effectively reduced the quartic optimization problem to a quadratic optimization problem under LMI constraints. It is worth noting that, the latter problem, in some cases, can be well approximated in polynomial time. For instance, for the single Euclidean-ball constrained quartic optimization problem, we provide polynomial time approximation algorithms for the relaxed quadratic SDP problem, which when combined with an appropriate probabilistic rounding procedure (described in Section 4), can deliver a provably high quality approximate solution $\hat{x}$ for the original quartic optimization problem (1.1)–(1.2). In particular, when there is only one positive definite quadratic constraint in (1.1)–(1.2) ($m = 1$ and $A_1 \succ 0$), then it is possible to find an approximate solution of (1.1)–(1.2) in polynomial time with a guaranteed *relative* approximation ratio of $(1 - O(1/n^2))$, where the constant in $O(\cdot)$ is independent of problem data. When the quartic objective function satisfies a certain nonnegativity assumption (cf. (5.3)), the quality bound improves to a multiplicative constant factor $O(n^{-2})$-approximation (rather than the relative approximation ratio; see Section 2 for definitions). Finally, we suggest that the symmetric mapping (3.3) can be used in polynomial optimization of even higher orders (see Section 6).

2 Preliminaries

In this section, we first characterize the complexity of quartic optimization problems (1.1)–(1.2) and then review two different measures of approximation ratio in the context of continuous optimization. We also present a simple example to motivate the symmetric relaxation mapping for quartic optimization problems (1.1)–(1.2).

2.1 Complexity and approximation

Before discussing approximation algorithms, let us first note that the problems under consideration, i.e. (1.1)–(1.2), are in general NP-hard. In particular, the NP-hardness of quartic maximization problem (1.1) follows directly from Nesterov [15], where it was shown that maximizing a cubic polynomial over a unit ball is NP-hard. In addition, the sensor network localization problem (1.3), which can be cast as a special case of the quartic minimization problem (1.2), was also known to be NP-hard. This implies that the quartic minimization problem (1.2) is NP-hard.

The above NP-hardness results make finding a global optimal solution for (1.1)–(1.2) in polynomial time an elusive goal. We therefore turn our attention to polynomial time algorithms to find approximate solutions of (1.1)–(1.2). The quality of approximate solutions can be measured in two ways. First, if $f_{\min}$ (or $f_{\max}$) are nonnegative, then a feasible solution $\hat{x}$ is said to be an **α-factor approximation** of $f_{\min}$ if

$$f_{\min} \le f(\hat{x}) \le \alpha f_{\min}$$

with α independent of problem data. The constant α is called the **approximation ratio**. If an α-factor approximate solution $\hat{x}$ is found, then $f_{\min} = 0$ iff $f(\hat{x}) = 0$. Thus, the existence of a polynomial time constant-factor approximation algorithm for the quartic minimization problem (1.2) would imply that one can determine if $f_{\min} = 0$ in polynomial time. The latter is itself an NP-hard problem since determining if the global optimal value of a sensor network localization problem is zero or not is NP-hard (see the footnote on page 2). Thus, there cannot be a polynomial time constant-factor approximation algorithm for the general quartic minimization problem (1.2) unless P=NP.

Since a constant α-factor approximation of $f_{\min}$ is unlikely to exist for (1.2), we consider a weaker notion of $(1-\epsilon)$**-relative approximation** of $f_{\min}$ which is defined by

$$f(\hat{x}) - f_{\min} \le (1-\epsilon)(\bar{f} - f_{\min})$$

with ϵ independent of problem data, and where $\bar{f}$ is a reference value of f of a candidate feasible point for (1.2). Notice that $(\bar{f} - f_{\min})$ is the range of f over the feasible region of (1.2). So an $(1-\epsilon)$-relative approximation of $f_{\min}$ means that $f(\hat{x})$ belongs to the ϵ fraction of lowest function values. It is worth pointing out that, unlike a constant-factor approximation of $f_{\min}$, an $(1-\epsilon)$-relative approximation of $f_{\min}$ does not imply the equivalence "$f_{\min} = 0$ iff $f(\hat{x}) = 0$". Instead, we only have the implication "$f(\hat{x}) = 0 \Rightarrow f_{\min} = 0$". The reverse implication does not hold in general as we can only claim $0 \le f(\hat{x}) \le (1-\epsilon)\bar{f}$ when $f_{\min} = 0$.

Similarly we can define the notions of α-factor approximation, and $(1-\epsilon)$-relative approximation for the quartic maximization problem (1.1).

2.2 Symmetric relaxation: a motivating example

Example. Consider the following quartic optimization problem in $\mathbb{R}^2$:

$$\begin{array}{ll} \text{minimize} & f(x) = (x_1x_2)^2 \\ \text{subject to} & x_1^2 \ge 1, \quad x_2^2 \ge 1. \end{array} \tag{2.1}$$

Clearly, we have $f_{\min} = 1$. Also, under the matrix lifting transformation $X = xx^{\mathrm{T}}$, we can relax (2.1) to

$$\begin{array}{ll} \text{minimize} & g(X) = (X_{12})^2 \\ \text{subject to} & X_{11} \ge 1, \quad X_{22} \ge 1, \quad X \succeq 0. \end{array} \tag{2.2}$$

where the notation $X \succeq 0$ signifies that the matrix X is positive semidefinite, and the notation X_{ij} denotes the (i,j)-th entry of X. It can be easily checked that the minimum value of (2.2) $g_{\min} = 0$ since $X = I$ is a feasible solution. This example shows that the approximation ratio satisfies

$$\frac{f_{\min}}{g_{\min}} = \infty. \tag{2.3}$$

Does this example imply that SDP relaxation of (2.1) (or (1.2) in general) cannot guarantee a bounded worst-case performance approximation ratio that is independent of problem data?

Interestingly, the answer to this question is negative. Indeed, we show in this paper (Section 4) that, under a suitable relaxation, the worst-case SDP approximation ratio for (1.2) is at most $O(\max\{m^2, mn\})$ which is finite and independent of problem data (i.e., the coefficients $a_{ij\ell m}$'s and the matrices A_i's). The key observation that has made possible this finite bound is that the mapping of $f(x)$ to $g(X)$ is *not unique* under the transformation $X = xx^{\mathrm{T}}$. For example, we can relax the quartic objective function

$$f(x) = (x_1x_2)^2 = x_1^2x_2^2$$

to

$$h(x) = \frac{1}{3}(X_{11}X_{22} + 2X_{12}^2).$$

This relaxation also satisfies $h(X) = f(x)$ when $X = xx^{\mathrm{T}}$. In this way, the quartic optimization problem (2.1) can be relaxed to the following quadratic optimization problem:

$$\begin{array}{ll} \text{minimize} & h(X) = \frac{1}{3}(X_{11}X_{22} + 2X_{12}^2) \\ \text{subject to} & X_{11} \geq 1, \quad X_{22} \geq 1, \quad X \succeq 0. \end{array} \tag{2.4}$$

Clearly, the minimum value of (2.4) is $h_{\min} = 1/3$, implying

$$\frac{f_{\min}}{h_{\min}} = \frac{1}{\frac{1}{3}} = 3,$$

which, unlike the relaxation bound (2.3), is actually finite. It turns out that this finite bound is a consequence of a general bound (Theorem 4.2, see Section 4) on the approximation ratio for the quartic minimization problem (1.2) and its SDP relaxation when the symmetric mapping (3.3) is used.

3 Relaxation to a Quadratic SDP

We propose a new SDP relaxation for (1.1) and (1.2) which is different from the existing SOS relaxation. To motivate this SDP relaxation, let us recall that the standard approach to relax a quadratic polynomial to a linear function is well known and uniquely defined: define $X = xx^{\mathrm{T}} \succeq 0$ which implies

$$x_ix_j \quad \mapsto \quad X_{ij}, \qquad \forall\, 1 \leq i, j \leq n.$$

In this way, a quadratic polynomial $x^{\mathrm{T}}Ax$ is relaxed to $\operatorname{tr}(AX)$. However, the same relaxation becomes ambiguous for a quartic polynomial $f(x)$ since each quartic monomial term can be relaxed to a quadratic term in various ways; see Section 2.2. This begs the natural question: what is the right relaxation to use? The answer to this question depends on what our objectives are for the SDP relaxation. Notice that a SDP relaxation of the quartic optimization problem (1.1)–(1.2) will result in a linearly constrained quadratic maximization problem over the semidefinite matrix cone. Naturally we desire that the resulting relaxation is a close approximation of the original NP-hard problem (1.1)–(1.2), and is efficiently solvable. Below we motivate one specific way to relax (1.1)–(1.2) so that the resulting SDP relaxation can provide a *constant-factor approximation* of (1.1)–(1.2).

Suppose $g(X)$ is a quadratic function to be used as a relaxation of the quartic function $f(x)$. Then the quadratic function $g(X)$ should satisfy the consistency property

$$g(X) = f(x), \quad \text{whenever } X = xx^{\mathrm{T}}. \tag{3.1}$$

There are many quadratic functions $g(X)$ satisfying this property. Which one should we pick? The answer depends on the approximate rounding procedure to be used to generate a feasible solution for (1.1). Let $X \succeq 0$ denote the optimal solution of the following SDP relaxation of the quartic maximization problem (1.1):

$$\begin{array}{ll} \text{maximize} & g(X) \\ \text{subject to} & \mathrm{tr}(A_i X) \le 1, \quad i = 1, 2, ..., m, \\ & X \succeq 0. \end{array}$$

If X is rank-one, then $X = xx^{\mathrm{T}}$ for some $x \in \mathbb{R}^n$ which is feasible for (1.1). If the rank of X is greater than 1, a standard way to generate a feasible solution for the original problem (1.1) is to use a probabilistic rounding procedure whereby a sequence of random samples are drawn from the Gaussian distribution $N(0, X)$. Each sample vector can be appropriately scaled to attain feasibility. The best (scaled) sample vector $\hat{x}$ (achieving the largest objective value $f(\hat{x})$) is then chosen as the final approximate solution for (1.1). To ensure the derived approximate solution $\hat{x}$ has an objective value that is close to $f_{\max}$, we must make sure $\mathrm{E}[f(x)]$ is close to $g(X)$ when x is drawn from the Gaussian distribution $N(0, X)$. It turns out that $\mathrm{E}[f(x)]$ is a quadratic function of X. This motivates us to choose

$$g(X) = c\,\mathrm{E}[f(x)], \quad \text{for some } c > 0. \tag{3.2}$$

If (3.2) holds, we say $g(X)$ is compatible with $f(x)$. Is there a positive constant c satisfying both (3.1) and (3.2)?

It turns out the answer to the above question is positive. Let us recall some useful results connecting the fourth order statistics with the second order statistics of a multivariate Gaussian distribution (cf. e.g., [1]).

Lemma 3.1 *Suppose $x \in \mathbb{R}^n$ is a random vector drawn a Gaussian distribution $N(0, X)$ where X is a $n \times n$ symmetric positive semidefinite matrix. Then for any $1 \le i \ne j \ne k \ne \ell \le n$, we have*

$$\begin{aligned} \mathrm{E}[x_i^4] &= 3X_{ii}^2 \\ \mathrm{E}[x_i^3 x_j] &= 3X_{ii}X_{ij} \\ \mathrm{E}[x_i^2 x_j^2] &= X_{ii}X_{jj} + 2X_{ij}^2 \\ \mathrm{E}[x_i^2 x_j x_k] &= X_{ii}X_{jk} + 2X_{ij}X_{ik} \\ \mathrm{E}[x_i x_j x_k x_\ell] &= X_{ij}X_{k\ell} + X_{ik}X_{j\ell} + X_{i\ell}X_{jk}. \end{aligned}$$

Based on Lemma 3.1, we propose to relax each quartic term symmetrically as

$$x_i x_j x_k x_\ell \;\mapsto\; \frac{1}{3}\left(X_{ij}X_{k\ell} + X_{ik}X_{j\ell} + X_{i\ell}X_{jk}\right), \quad \forall\, 1 \le i, j, \ell, m \le n. \tag{3.3}$$

It can be easily checked that the consistency property (3.1) is satisfied. Moreover, the desired compatibility property (3.2) is satisfied readily with $c = 1/3$. Under the above symmetric mapping, the quartic polynomial maximization problem (1.1) is relaxed to

$$\boxed{\begin{array}{ll} \text{maximize} & g(X) = \dfrac{1}{3}\displaystyle\sum_{1\le i,j,k,\ell\le n} a_{ijk\ell}\left(X_{ij}X_{k\ell} + X_{ik}X_{j\ell} + X_{i\ell}X_{jk}\right) \\ \text{subject to} & \operatorname{tr}(A_i X) \le 1,\; i = 1, ..., m \\ & X \succeq 0, \end{array}} \tag{3.4}$$

and the quartic polynomial minimization problem (1.2) can be relaxed as

$$\boxed{\begin{array}{ll} \text{minimize} & g(X) = \dfrac{1}{3}\displaystyle\sum_{1\le i,j,k,\ell\le n} a_{ijk\ell}\left(X_{ij}X_{k\ell} + X_{ik}X_{j\ell} + X_{i\ell}X_{jk}\right) \\ \text{subject to} & \operatorname{tr}(A_i X) \ge 1,\; i = 1, ..., m \\ & X \succeq 0. \end{array}} \tag{3.5}$$

Consider now a special case for (1.1) whereby $m = 1$ and $A_1 \succ 0$. Without loss of generality, we assume $A_1 = I$. Then the relaxation is

$$\begin{array}{ll} \text{maximize} & \dfrac{1}{3}\displaystyle\sum_{1\le i,j,k,\ell\le n} a_{ijk\ell}\left(X_{ij}X_{k\ell} + X_{ik}X_{j\ell} + X_{i\ell}X_{jk}\right) \\ \text{subject to} & \operatorname{tr}(X) \le 1 \\ & X \succeq 0. \end{array} \tag{3.6}$$

Clearly, if X is rank-1, then $X = xx^{\mathrm{T}}$ for some $x \in \mathbb{R}^n$. It can be easily checked that x is a feasible solution of (1.1)–(1.2) and achieves an objective value of $f(x)$. If X is not rank-1, then (3.6) is a relaxation of (1.1)–(1.2).

In general, problem (3.6) is still NP-hard to solve. To see this,

$$a_{ijk\ell} = \begin{cases} c_{ik}, & \text{if } i = j \text{ and } k = \ell; \\ 0, & \text{otherwise}, \end{cases}$$

where $C = \left[c_{ik}\right]_{n\times n} > 0$ is a positive (entry-wise) symmetric matrix. Then we have

$$\frac{1}{3}\sum_{1\le i,j,k,\ell\le n} a_{ijk\ell}\left(X_{ij}X_{k\ell} + X_{ik}X_{j\ell} + X_{i\ell}X_{jk}\right) = \frac{1}{3}\sum_{1\le i,k\le n} c_{ik}\left(X_{ii}X_{kk} + 2X_{ik}^2\right) \le \sum_{1\le i,k\le n} c_{ik}X_{ii}X_{kk},$$

where the last step follows from $X_{ik}^2 \le X_{ii}X_{kk}$ (due to $X \succeq 0$). We claim, in this case, solving (3.6) is in fact equivalent to solving the standard simplex-constrained quadratic program

$$\begin{array}{ll} \text{maximize} & z^{\mathrm{T}}Cz \\ \text{subject to} & e^{\mathrm{T}}z \le 1 \\ & z \ge 0. \end{array} \tag{3.7}$$

In particular, if z^* is an optimal solution of (3.7), then $X^* = z^*(z^*)^{\mathrm{T}}$ is an optimal solution of (3.6) because the above upper bound holds with equality at $X^* = z^*(z^*)^{\mathrm{T}}$. Moreover, due to the positivity of c_{ik}, the optimal solution of (3.6) must be rank one. Thus, the relaxed problem (3.6) is equivalent to the simplex-constrained quadratic program (3.7). The latter is known to be NP-hard. Finally, restricting C to be entry-wise positive does not change the complexity of problem (3.7), as one may replace C by $\tau E + C$ with E the all-one matrix and τ a sufficiently large positive constant. Such transformation does not change the optimal solutions. Since (3.6) is a special case of (3.4), both (3.4) and (3.5) are NP-hard, which is unfortunate.

Naturally, one may ask why we have chosen to relax the NP-hard quartic optimization problems (1.1)–(1.2) to some other NP-hard problems (i.e., the quadratic SDPs (3.4) and (3.5)). From a complexity standpoint, such a relaxation may seem completely unreasonable at first. However, we will argue in next section that the quadratic SDP relaxations (3.4) and (3.5) are quite useful from an approximation standpoint. In fact, we will show that any α-factor approximation of the quadratic SDP relaxations (3.4) and (3.5)) can be used to generate a constant-factor approximation of the original quartic optimization problems (1.1)–(1.2) through a polynomial time probabilistic rounding procedure. Thus, the proposed SDP relaxation effectively reduces the quartic optimization problems (1.1)–(1.2) to the quadratic SDPs (3.4) and (3.5). The latter, as we will show in Section 5, can be well-approximated in polynomial time in some special cases, resulting in overall polynomial time approximation of some special quartic optimization problem (1.1)–(1.2).

4 Approximation Ratios

Our goal is to design polynomial time approximation algorithms for (1.1) and (1.2). The approach that we take consists of two steps. First, we argue that there is a finite and data-independent approximation bound between the optimal values of (1.1) (respectively (1.2)) and its SDP relaxation (3.4) (respectively (3.5)). Second, since the SDP relaxation (3.4) (respectively (3.5)) is NP-hard, we provide a polynomial time approximation algorithm for this nonconvex problem. It turns out that the approximate solution for the relaxed SDP problem (3.4) (respectively (3.5)) provides a constant factor (data-independent) approximation to the original quartic maximization problem (1.1) (respectively (1.2)). Step 1 will be presented in this section, while Step 2 will be described in Section 5.

Let $\hat{X} \succeq 0$ be an α-factor approximate solution of (3.4) in the sense that $g(\hat{X}) \geq \alpha g_{\max}$, where $\alpha > 0$ is a constant and $g_{\max}$ denotes the optimal value the SDP relaxation (3.4). Our main result is as follows.

Theorem 4.1 *Let $\hat{X}$ be an α-factor approximate solution of* (3.4). *Suppose we randomly generate a sample x from Gaussian distribution $N(0,\hat{X})$. Let $\hat{x} = x/\max_{1\leq i\leq m} x^{\mathrm{T}} A_i x$. Then $\hat{x}$ is a feasible solution of* (1.1) *and the probability that*

$$f_{\max} \geq f(\hat{x}) \geq \frac{3\alpha}{4\left(\ln \frac{2m}{\theta}\right)^2} f_{\max}$$

is at least $\theta/2$ with $\theta := 1.14 \times 10^{-7}$, where $f_{\max}$ denotes the optimal value of (1.1).

Theorem 4.1 implies that if we randomly generate L Gaussian samples from the distribution $N(0,\hat{X})$, scale them according to $\hat{x} = x/\max_{1\leq i\leq m} x^{\mathrm{T}} A_i x$, and pick among them the best candidate $\hat{x}$ that gives the largest objective value $f(\hat{x})$, then with probability $1-(1-\theta/2)^L$ (which approaches 1 exponentially fast), the function value $f(\hat{x})$ will be a β-factor approximation of $f_{\max}$, where $\beta = \frac{3\alpha}{4\left(\ln \frac{2m}{\theta}\right)^2}$.

The proof of Theorem 4.1 relies on two probability estimates regarding, respectively, homogeneous quadratic and quartic functions evaluated at Gaussian random vectors. The probability estimate for the quadratic function is known (Lemam 4.1 below) while the other is an estimate for the probability $\mathrm{Prob}(f(x) \geq \mathrm{E}(f(x)))$, where x is drawn according to a Gaussian distribution and f is an arbitrary homogeneous quartic polynomial. To facilitate the proof of Theorem 4.1, we first state the lemma and the probability estimate for $\mathrm{Prob}(f(x) \geq \mathrm{E}(f(x)))$ without proof. The proof of the latter estimate (which is quite involved) is deferred to the end of this section.

The following estimate is due to a result in So, Ye, and Zhang [20, Section 2].

Lemma 4.1 *Let $A \in \mathcal{S}_+^{n\times n}$, $\hat{X} \in \mathcal{S}_+^{n\times n}$. Suppose $\xi \in \mathbb{R}^n$ is a random vector generated from the Gaussian distribution $N(0,\hat{X})$. Then, for any $\gamma > 0$,*

$$\mathrm{Prob}\left\{\xi^{\mathrm{T}} A\xi > \gamma \mathrm{E}[\xi^{\mathrm{T}} A\xi]\right\} \leq \exp\left[\frac{1}{2}\left(1-\gamma+\ln\gamma\right)\right].$$

For convenience later, letting $\gamma = 1/\rho^2$ we have

$$\mathrm{Prob}\left\{\rho^2\xi^{\mathrm{T}} A\xi > \mathrm{E}[\xi^{\mathrm{T}} A\xi]\right\} \leq \exp\left[\frac{1}{2}\left(1-1/\rho^2-2\ln\rho\right)\right]. \tag{4.1}$$

Proposition 4.1 *Let $f(x)$ be an arbitrary homogeneous quartic polynomial and suppose x is drawn from a zero mean Gaussian distribution in $\mathbb{R}^n$. Then*

$$\mathrm{Prob}\left\{f(x) \geq \mathrm{E}[f(x)]\right\} \geq 1.14 \times 10^{-7},$$

and

$$\text{Prob}\{f(x) \le \text{E}[f(x)]\} \ge 1.14 \times 10^{-7}.$$

We point out that the constants in Theorem 4.1 have not been optimized. The important point is that these constants exist and they are all universal (i.e., data-independent). To gain some understanding of the size of the probabilities in Proposition 4.1, we randomly generated 1000 instances for some selected dimensions $5 \le n \le 70$. For each given random instance f we then generate 1000 independent Gaussian samples from $N(0, I)$, which are used to estimate $\text{E}[f(x)]$, $\text{Prob}\{f(x) \ge \text{E}[f(x)]\}$ and $\text{Prob}\{f(x) \le \text{E}[f(x)]\}$ for that f. For each given dimension n, among the 1000 instances, we compute the minimum values of $\text{Prob}\{f(x) \ge \text{E}[f(x)]\}$ and $\text{Prob}\{f(x) \le \text{E}[f(x)]\}$ respectively. The results are shown in the table below[2], where the second column is the minimum value of the probability $\text{Prob}\{f(x) \ge \text{E}[f(x)]\}$ among these 1000 random cases, while the third column is the minimum value of the probability $\text{Prob}\{f(x) \le \text{E}[f(x)]\}$ among these 1000 random cases.

n	$\text{Prob}\{f(x) \ge \text{E}[f(x)]\}$	$\text{Prob}\{f(x) \le \text{E}[f(x)]\}$
5	0.273	0.263
10	0.359	0.348
20	0.426	0.423
30	0.448	0.450
40	0.442	0.447
50	0.440	0.445
60	0.442	0.454
70	0.453	0.449

As we can see, the two columns of probabilities are fairly large and almost equal (they should be the same in theory due to symmetry). Thus, the estimated constants in Proposition 4.1 and Theorem 4.1 are likely quite loose.

Proof of Theorem 4.1: Suppose we randomly generate a sample x from Gaussian distribution $N(0, \hat{X})$. Then, Proposition 4.1 asserts that

$$\text{Prob}\{f(\rho x) = \rho^4 f(x) < \rho^4 \text{E}[f(x)]\} < 1 - \theta,$$

and Lemma 4.1 states that

$$\text{Prob}\left\{\rho^2 \xi^{\text{T}} A_i \xi > \text{E}[\xi^{\text{T}} A_i \xi]\right\} \le \exp\left[\frac{1}{2}\left(1 - 1/\rho^2 - 2\ln\rho\right)\right]. \tag{4.2}$$

[2]We thank Zhening Li for his assistance in producing this table.

Therefore, by letting $\rho := 1/\sqrt{2\ln(2m/\theta)}$, we have

$$\begin{aligned} & \text{Prob}\{f(\rho x) \geq \rho^4 \mathrm{E}[f(x)],\ \rho^2 x^{\mathrm{T}} A_i x \leq \mathrm{E}[x^{\mathrm{T}} A_i x],\ i = 1, ..., m\} \\ \geq\ & 1 - (1-\theta) - m \exp\left[\frac{1}{2}\left(1 - 1/\rho^2 - 2\ln\rho\right)\right] \\ \geq\ & \theta - m \exp\left[\frac{1}{2}\left(1 - 1/\rho^2 - 2\ln\rho\right)\right] \\ \geq\ & \theta/2, \end{aligned}$$

where the first step is due to the union bound, and the last step follows from the definition of ρ. Once the above event does occur (the probability of occurrence is at least $\theta/2$), then

$$f(\hat{x}) \geq f(\rho x) \geq \frac{1}{4\left(\ln \frac{2m}{\theta}\right)^2} \mathrm{E}[f(x)], \tag{4.3}$$

while,

$$\begin{aligned} \mathrm{E}[f(x)] &= \sum_{i,j,k,\ell} a_{ijk\ell} \mathrm{E}[x_i x_j x_k x_\ell] \\ &= \sum_{i,j,k,\ell} a_{ijk\ell} \left(\hat{X}_{ij}\hat{X}_{k\ell} + \hat{X}_{ik}\hat{X}_{j\ell} + \hat{X}_{i\ell}\hat{X}_{jk}\right) \\ &= 3g(\hat{X}) \geq 3\alpha\, g_{\max} \end{aligned}$$

where the last step follows from the fact that $\hat{X}$ is an α-factor approximate solution of (3.4). Since (3.4) is a relaxation of (1.1), we have $g_{\max} \geq f_{\max}$, which further implies

$$\mathrm{E}[f(x)] \geq 3\alpha f_{\max}. \tag{4.4}$$

Combining (4.3) and (4.4) gives

$$f(\hat{x}) \geq \frac{3\alpha}{4\left(\ln \frac{2m}{\theta}\right)^2} f_{\max},$$

which completes the proof of the theorem. **Q.E.D.**

It remains to establish the probability estimate in Proposition 4.1. To this end, we need the following result which bounds the probability of a random variable being above its mean, provided that the fourth order central moment of the random variable can be controlled by the square of its variance. This result was first used in He *et al.* [12].

Lemma 4.2 *Let ξ be a random variable with bounded fourth order moment. Suppose*

$$\mathrm{E}[(\xi - \mathrm{E}(\xi))^4] \leq \tau\, \mathrm{Var}^2(\xi), \quad \textit{for some } \tau > 0.$$

Then

$$\text{Prob}\,\{\xi \geq \mathrm{E}(\xi)\} \geq 0.25\tau^{-1} \ \textit{and}\ \ \text{Prob}\,\{\xi \leq \mathrm{E}(\xi)\} \geq 0.25\tau^{-1}.$$

To prove the desired probability bound in Proposition 4.1, we need to specialize the random variable ξ in Lemma 4.2 to a homogeneous quartic polynomial evaluated at a zero mean Gaussian random vector. This motivates the following fourth order central moment estimate.

Proposition 4.2 *Consider a general homogeneous quartic polynomial in* $\mathbb{R}^n$ *given by*

$$f(x) = \sum_{i=1}^n a_i x_i^4 + \sum_{1\le i<j\le n} b_{ij} x_i^2 x_j^2 + \sum_{1\le i\ne j\le n} c_{ij} x_i^3 x_j + \sum_{i\ne j,k} d_{ijk} x_i^2 x_j x_k + \sum_{1\le i<j<k<\ell\le n} e_{ijk\ell} x_i x_j x_k x_\ell$$

and let $x \sim N(0, I)$ *be a zero-mean Gaussian random vector with covariance equal to* I *(identity). Then*

$$\begin{aligned} \mathrm{Var}(f) &\ge 60\|a\|^2 + 2\|b\|^2 + \sum_{i=1}^n \left(\sum_{j:\, j>i} b_{ij}\right)^2 + \sum_{i=1}^n \left(\sum_{j:\, j<i} b_{ji}\right)^2 \\ &\quad +1.5\|c\|^2 + 1.5\|d\|^2 + 0.5 \sum_{j,k:\, j<k} \left(\sum_{i:\, i\ne k,j} d_{ijk}\right)^2 + \|e\|^2, \end{aligned}$$

where $a = (\cdots, a_i, \cdots)$, $b = (\cdots, b_{ij}, \cdots)_{i<j}$, $c = (\cdots, c_{ij}, \cdots)_{i\ne j}$, $d = (\cdots, d_{ijk}, \cdots)_{\substack{i\ne j,k\\ j<k}}$, *and* $d = (\cdots, d_{ijk\ell}, \cdots)_{i<j<k<\ell}$. *Moreover,*

$$\mathrm{E}\left[(f(x) - \mathrm{E}[f(x)])^4\right] \le 2176632\, \mathrm{Var}^2(f(x))$$

Proof. The first few moments of a standard Gaussian distribution are well known. For ease of reference, we list them as follows. If $x \sim N(0, 1)$, then,

$$\begin{aligned} &\mathrm{E}[x^2] = 1,\ \mathrm{E}[x^4] = 3,\ \mathrm{E}[x^6] = 15,\ \mathrm{E}[x^8] = 105, \\ &\mathrm{E}[x^{10}] = 945,\ \mathrm{E}[x^{12}] = 10395,\ \mathrm{E}[x^{14}] = 135135,\ \mathrm{E}[x^{16}] = 2027025. \end{aligned}$$

By direct computation, we have

$$\begin{aligned} \mathrm{Var}(f(x)) &= 96\|a\|^2 + 24 \sum_{1\le i<j\le n} a_i b_{ij} + 2 \sum_{1\le i<j<k\le n} b_{ij} b_{ik} + 2 \sum_{1\le i<j<k\le n} b_{ik} b_{jk} + 8\|b\|^2 + 15\|c\|^2 \\ &\quad +3 \sum_{\substack{i\ne j,k\\ j<k}} c_{jk} d_{ijk} + 3 \sum_{\substack{i\ne j,k\\ j<k}} c_{kj} d_{ijk} + 9 \sum_{i\ne j} c_{ij} c_{ji} + 3\|d\|^2 + \sum_{\substack{i,\ell\ne j,k,\\ i\ne \ell,\ j<k}} d_{ijk} d_{\ell jk} + \|e\|^2 \\ &= 96\|a\|^2 + 24 \sum_{1\le i<j\le n} a_i b_{ij} + \sum_{i=1}^n \left(\sum_{j:\, j>i} b_{ij}\right)^2 + \sum_{i=1}^n \left(\sum_{j:\, j<i} b_{ji}\right)^2 + 6\|b\|^2 + 15\|c\|^2 \\ &\quad +3 \sum_{\substack{i\ne j,k\\ j<k}} c_{jk} d_{ijk} + 3 \sum_{\substack{i\ne j,k\\ j<k}} c_{kj} d_{ijk} + 9 \sum_{i\ne j} c_{ij} c_{ji} + 2\|d\|^2 \\ &\quad + \sum_{j,k:\, j<k} \left(\sum_{i:\, i\ne k,j} d_{ijk}\right)^2 + \|e\|^2. \end{aligned} \tag{4.5}$$

Using the following simple inequalities

$$24a_ib_{ij} \geq -36a_i^2 - 4b_{ij}^2, \quad c_{ji}c_{ij} \geq -0.5(c_{ij}^2 + c_{ji}^2),$$

and the inequality that for each fixed pair of (j,k) with $j < k$,

$$\begin{aligned} 3\sum_{i:\, i\neq j,k} (c_{jk} + c_{kj})d_{ijk} &= 3(c_{jk}+c_{kj}) \sum_{i:\, i\neq j,k} d_{ijk} \geq -4.5(c_{jk}+c_{kj})^2 - 0.5\left(\sum_{i:\, i\neq j,k} d_{ijk}\right)^2 \\ &\geq -9(c_{jk}^2 + c_{kj}^2) - 0.5\left(\sum_{i:\, i\neq j,k} d_{ijk}\right)^2, \end{aligned}$$

we can lower bound the cross terms in (4.5) to obtain

$$\begin{aligned} \mathrm{Var}(f) &\geq 60\|a\|^2 + 2\|b\|^2 + \sum_{i=1}^n \left(\sum_{j:\, j>i} b_{ij}\right)^2 + \sum_{i=1}^n \left(\sum_{j:\, j<i} b_{ji}\right)^2 \\ &\quad +1.5\|c\|^2 + 1.5\|d\|^2 + 0.5\sum_{j,k:\, j<k}\left(\sum_{i:\, i\neq k,j} d_{ijk}\right)^2 + \|e\|^2. \end{aligned} \tag{4.6}$$

Next, we notice that

$$\begin{aligned} f(x) - \mathrm{E}[f(x)] &= \underbrace{\sum_{i=1}^n a_i(x_i^4 - 3)}_{\text{Term I}} + \underbrace{\sum_{1\leq i<j\leq n} b_{ij}(x_i^2x_j^2 - 1)}_{\text{Term II}} + \underbrace{\sum_{1\leq i\neq j\leq n} c_{ij}x_i^3x_j}_{\text{Term III}} \\ &\quad + \underbrace{\sum_{\substack{i\neq j,k \\ j<k}} d_{ijk}x_i^2x_jx_k}_{\text{Term IV}} + \underbrace{\sum_{1\leq i<j<k<\ell\leq n} e_{ijk\ell}x_ix_jx_kx_\ell}_{\text{Term V}}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\mathrm{E}[(f(x) - \mathrm{E}[f(x)])^4] \\ = \;& \mathrm{E}[(\text{Term I} + \text{Term II} + \text{Term III} + \text{Term IV} + \text{Term V})^4] \\ \leq \;& 125\left(\mathrm{E}[(\text{Term I})^4] + \mathrm{E}[(\text{Term II})^4] + \mathrm{E}[(\text{Term III})^4] + \mathrm{E}[(\text{Term IV})^4] + \mathrm{E}[(\text{Term V})^4]\right). \end{aligned} \tag{4.7}$$

Thus, it is sufficient to show that there exists some universal constant $t > 0$ such that

$$\mathrm{E}[(\text{Term Z})^4] \leq t\,\mathrm{Var}^2(f(x)), \quad \text{for Z = I, II, III, IV and V.}$$

Simple calculations yield

$$
\begin{aligned}
\mathrm{E}[(\text{Term I})^4] &= \mathrm{E}\left[\left(\sum_{i=1}^n a_i(x_i^4-3)\right)^4\right] \\
&= \mathrm{E}\left[\sum_{i=1}^n a_i^4(x_i^4-3)^4\right] + 6\mathrm{E}\left[\sum_{i<j} a_i^2 a_j^2 (x_i^4-3)^2(x_j^4-3)^2\right] \\
&= 2026980\sum_{i=1}^n a_i^4 + 55296\sum_{i<j} a_i^2 a_j^2 \\
&\le 2026980\|a\|^4.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\mathrm{E}[(\text{Term II})^4] &= \mathrm{E}\left[\left(\sum_{1\le i<j\le n} b_{ij}(x_i^2x_j^2-1)\right)^4\right] \le 11076\|b\|^4, \\
\mathrm{E}[(\text{Term III})^4] &= \mathrm{E}\left[\left(\sum_{1\le i\ne j\le n} c_{ij}x_i^3x_j\right)^4\right] \le 31185\|c\|^4, \\
\mathrm{E}[(\text{Term IV})^4] &= \mathrm{E}\left[\left(\sum_{\substack{i\ne j,k\\ j<k}} d_{ijk}x_i^2x_jx_k\right)^4\right] \le 315\|d\|^4, \\
\mathrm{E}[(\text{Term V})^4] &= \mathrm{E}\left[\left(\sum_{1\le i<j<k<\ell\le n} e_{ijk\ell}x_ix_jx_kx_\ell\right)^4\right] \le 81\|e\|^4.
\end{aligned}
$$

Using (4.6), we have

$$
\mathrm{E}[(\text{Term I})^4 + (\text{Term II})^4 + (\text{Term III})^4 + (\text{Term IV})^4 + (\text{Term V})^4] \le 17414\mathrm{Var}(f)^2. \qquad (4.8)
$$

Furthermore, combining (4.7) and (4.8) yields

$$
\mathrm{E}[(f(x)-\mathrm{E}[f(x)])^4] \le 2176632\,\mathrm{Var}(f)^2,
$$

which completes the proof of Proposition 4.2. **Q.E.D.**

Combining Lemma 4.2 and Proposition 4.2 immediately establishes the desired probability estimate in Proposition 4.1.

Analogously we can analyze the performance ratio for the quartic minimization problem (1.2). First we note the following lemma from [11]:

Lemma 4.3 *Let* $A \in \mathcal{S}_+^{n\times n}$*,* $X \in \mathcal{S}_+^{n\times n}$*. Suppose* $\xi \in \mathbb{R}^n$ *is a random vector generated from the real-valued Gaussian distribution* $N(0, X)$*. Then, for any* $\gamma > 0$*,*

$$\text{Prob}\left\{\xi^{\mathrm{T}} A\xi < \gamma \mathrm{E}[\xi^{\mathrm{T}} A\xi]\right\} \le \max\left\{\sqrt{\gamma}, \frac{2(r-1)\gamma}{\pi - 2}\right\},$$

where $r := \min\{\text{ rank }(A), \text{ rank }(X)\}$*. In our context, it is convenient to use the form:*

$$\text{Prob}\left\{\rho^2 \xi^{\mathrm{T}} A\xi < \mathrm{E}[\xi^{\mathrm{T}} A\xi]\right\} \le \max\left\{\frac{1}{\rho}, \frac{2(r-1)}{(\pi - 2)\rho^2}\right\}. \tag{4.9}$$

Theorem 4.2 *Suppose that the optimal value of* (3.5) *is nonnegative. Let* $\hat{X}$ *be an* α*-factor approximate solution of* (3.5)*. Suppose we randomly generate a sample* x *from Gaussian distribution* $N(0, \hat{X})$*. Let* $\hat{x} = x / \min_{1\le i\le m} x^{\mathrm{T}} A_i x$*. Then* $\hat{x}$ *is a feasible solution of* (1.2) *and the probability that*

$$f_{\min} \le f(\hat{x}) \le 12\alpha \max\left\{\frac{m^2}{\theta^2}, \frac{m(n-1)}{\theta(\pi - 2)}\right\} f_{\min}$$

is at least $\theta/2$ *with* $\theta := 1.14 \times 10^{-7}$*, where* $f_{\min}$ *denotes the optimal value of* (1.2)*.*

The proof of Theorem 4.2 is almost identical to the proof of Theorem 4.1. All that is needed is to invoke union bound, just like the proof of Theorem 4.1, except that we replace Lemma 4.1 by Lemma 4.3 in the argument. We omit the details here.

5 Approximation of Relaxed Problems

Our goal is to design polynomial time approximation algorithms for the quartic optimization problems (1.1)–(1.2) with provable worst-case constant approximation ratios. Theorems 4.1 and 4.2 suggest that this task depends on our ability to approximately solve the relaxed problems (3.4) and (3.5), which by themselves are also hard optimization problems.

In some special cases, however, it is possible to design efficient approximation algorithms for the relaxed quadratic SDP problems (3.4) and (3.5). As an example, let us consider the ball-constrained quartic maximization problem

$$\begin{array}{ll} \text{maximize} & f(x) = \displaystyle\sum_{1\le i,j,k,\ell\le n} a_{ijk\ell} x_i x_j x_k x_\ell \\ \text{subject to} & x^{\mathrm{T}} A_1 x \le 1. \end{array} \tag{5.1}$$

The corresponding quadratic SDP relaxation is given by (cf. (3.6)):

$$\begin{array}{ll} \text{maximize} & g(X) = \dfrac{1}{3} \displaystyle\sum_{1\le i,j,k,\ell\le n} a_{ijk\ell} \left(X_{ij} X_{k\ell} + X_{ik} X_{j\ell} + X_{i\ell} X_{jk}\right) \\ \text{subject to} & \mathrm{tr}(A_1 X) \le 1, \quad X \succeq 0. \end{array} \tag{5.2}$$

We will show next that, under a suitable assumption, $f_{\max}$ can be approximated within a multiplicative constant factor of $O(1/n^2)$ in polynomial time (Theorem 5.1). In the absence of this assumption, we will show that $f_{\max}$ can be approximated to within $(1-O(1/n^2))$-relative accuracy in polynomial time (Theorem 5.2).

Theorem 5.1 *Let $A_1 = I$. Assume that the condition*

$$g(I) = \sum_{i=1}^{n} a_{iiii} + \frac{1}{3} \sum_{1\le i<j\le n} (a_{iijj} + a_{ijij} + a_{ijji} + a_{jjii} + a_{jiji} + a_{jiij}) \ge 0 \tag{5.3}$$

holds, and that the optimal value $g_{\max}$ of (5.2) *is positive. Then the optimal value of the ball-constrained quartic maximization problem* (5.1) *can be approximated within a multiplicative factor of $O(1/n^2)$ in randomized polynomial time, where the constant in $O(\cdot)$ notation is universal, i.e., independent of problem data.*

Proof. The nonconcavity of $g(X)$, plus the simplex-like constraint, makes the problem (5.2) difficult to solve (in fact NP-hard, see discussion at the end of Section 3). To overcome this difficulty, we recall that if the simplex-like constraint in (5.2) is replaced by a *single* quadratic constraint, then the relaxed problem (5.2) becomes the so-called trust region subproblem and is polynomial time solvable. This motivates us to approximate the simplex-like feasible region by a second order cone constraint. Such an approximation was given in [3]:

$$\left\{X \in \mathcal{S}^{n\times n} \mid \sqrt{n-1}\,\|X\|_F \le \operatorname{tr}(X)\right\} \subseteq \mathcal{S}_+^{n\times n} \subseteq \left\{X \in \mathcal{S}^{n\times n} \mid \|X\|_F \le \operatorname{tr}(X)\right\}, \tag{5.4}$$

where $\mathcal{S}^{n\times n}$ and $\mathcal{S}_+^{n\times n}$ denote the cone of $n\times n$ symmetric matrices and the cone of symmetric positive semidefinite matrices respectively.

According to (5.4), a *restriction* of (5.2) is

$$\begin{array}{ll} \text{maximize} & g(X) = \dfrac{1}{3} \displaystyle\sum_{1\le i,j,k,\ell\le n} a_{ijk\ell}\,(X_{ij}X_{k\ell} + X_{ik}X_{j\ell} + X_{i\ell}X_{jk}) \\ \text{subject to} & \operatorname{tr}(X) \le 1, \quad \sqrt{n-1}\,\|X\|_F \le \operatorname{tr}(X) \end{array} \tag{5.5}$$

and a *relaxation* of (5.2) is given by

$$\begin{array}{ll} \text{maximize} & g(X) = \dfrac{1}{3} \displaystyle\sum_{1\le i,j,k,\ell\le n} a_{ijk\ell}\,(X_{ij}X_{k\ell} + X_{ik}X_{j\ell} + X_{i\ell}X_{jk}) \\ \text{subject to} & \operatorname{tr}(X) \le 1, \quad \|X\|_F \le \operatorname{tr}(X). \end{array} \tag{5.6}$$

Since $g_{\max} > 0$ (by assumption), any optimal solution of (5.5) or (5.6) must satisfy the linear constraint $\operatorname{tr}(X) \le 1$ with equality. Thus, the above problems (5.5)–(5.6) can be equivalently written as

$$\begin{array}{ll} \text{maximize} & g(X) = \dfrac{1}{3} \displaystyle\sum_{1 \le i,j,k,\ell \le n} a_{ijk\ell} \left(X_{ij} X_{k\ell} + X_{ik} X_{j\ell} + X_{i\ell} X_{jk} \right) \\ \text{subject to} & \operatorname{tr}(X) = 1, \quad \sqrt{n-1}\, \|X\|_F \le 1 \end{array} \tag{5.7}$$

and

$$\begin{array}{ll} \text{maximize} & g(X) = \dfrac{1}{3} \displaystyle\sum_{1 \le i,j,k,\ell \le n} a_{ijk\ell} \left(X_{ij} X_{k\ell} + X_{ik} X_{j\ell} + X_{i\ell} X_{jk} \right) \\ \text{subject to} & \operatorname{tr}(X) = 1, \quad \|X\|_F \le 1 \end{array} \tag{5.8}$$

respectively. Furthermore, we notice that the linear equality constraint $\operatorname{tr}(X) = 1$ in both problems (5.7)–(5.8) can be eliminated by variable reduction, and the resulting problems will have only a single quadratic constraint. Thus, the problems (5.7)–(5.8) are in the form of a trust-region subproblem which is solvable in polynomial-time (using the well-known S-Procedure in control theory).

Now let us consider

$$\begin{array}{rll} v(\lambda) = & \text{maximize} & \dfrac{1}{3} \displaystyle\sum_{1 \le i,j,k,\ell \le n} a_{ijk\ell} \left(X_{ij} X_{k\ell} + X_{ik} X_{j\ell} + X_{i\ell} X_{jk} \right) \\ & \text{subject to} & \operatorname{tr}(X) = 1, \quad \|X\|_F^2 \le \lambda. \end{array} \tag{5.9}$$

Then the restriction (5.5) corresponds to $\lambda = 1/(n-1)$ and the relaxation (5.6) corresponds to $\lambda = 1$. Moreover, since the trust-region subproblem (5.9) has (essentially) one quadratic constraint, it follows from [18] that the SDP relaxation of (5.9) is tight. Therefore, $v(\lambda)$ is equal to the maximum value of a SDP whose linear constraint is parameterized by λ. It follows that $v(\lambda)$ is concave in λ. Moreover, it can be checked that for $\lambda = 1/n$, there is exactly one feasible solution for (5.9), namely, $X = \frac{1}{n} I$. Let $\hat{X}$ be an optimal solution of the restriction (5.5). Then, it follows from the concavity of $v(\lambda)$ that

$$\begin{array}{rcll} g(\hat{X}) & = & v(1/(n-1)) & \\ & \ge & \dfrac{n^2 - 2n}{(n-1)^2}\, v(1/n) + \dfrac{1}{(n-1)^2}\, v(1) & \\ & = & \dfrac{n^2 - 2n}{n^2 (n-1)^2}\, g(I) + \dfrac{1}{(n-1)^2}\, v(1) & (\text{since } v(1/n) = g(I/n) = g(I)/n^2) \\ & \ge & \dfrac{1}{(n-1)^2}\, v(1) & (\text{since } g(I) \ge 0) \\ & \ge & \dfrac{1}{(n-1)^2}\, g_{\max}, & \end{array}$$

where $g_{\max}$ is the optimal value of (5.2), and the last inequality is due to the fact that (5.6) is a relaxation of (5.2). This implies that the optimal solution of (5.5) $\hat{X}$ is an $\frac{1}{(n-1)^2}$-factor approximation solution for (5.2). Combining this bound with Theorem 4.1 shows that if $m = 1$

then we can find an $O(1/n^2)$-factor approximate solution for (1.1), provided that $g(I) \geq 0$ and $g_{\max} > 0$. This completes the proof of Theorem 5.1. **Q.E.D.**

In case that the conditions $g(I) \geq 0$ and $g_{\max} > 0$ are not satisfied, then there cannot be any polynomial-time constant factor approximation ratio in general, unless P=NP (see [2, Section 3.4]). However it is still possible to use Theorem 5.1 to derive $(1-O(1/n^2))$-*relative approximation* results when $A_1 \succ 0$ (see Theorem 5.2 below).

Theorem 5.2 *Let $A_1 \succ 0$. Then we can find a feasible approximate solution $\hat{x}$ for the quartic maximization problem* (5.1) *in polynomial time satisfying*

$$\frac{f_{\max} - f(\hat{x})}{f_{\max} - \underline{f}} \leq \left(1 - O(n^{-2})\right) \tag{5.10}$$

where the constant in $O(\cdot)$ notation is universal (i.e., independent of problem data), and

$$\begin{array}{rll} \underline{f} = & \textit{minimize} & \displaystyle\sum_{1 \leq i,j,k,\ell \leq n} a_{ijk\ell} x_i x_j x_k x_\ell \\ & \textit{subject to} & x^{\mathrm{T}} A_1 x = 1. \end{array} \tag{5.11}$$

Proof. Without loss of generality, we assume $A_1 = I$. Notice that $f_{\max} \geq 0$ since $x = 0$ is feasible. We consider two cases. The first case is when $f_{\max} = 0$. In this case, we set $\hat{x} = 0$ which is the global optimal solution of (5.1).

The second case is when $f_{\max} > 0$, in which case we know the optimum of (5.1) is attained at the boundary of the unit ball. Define the quartic function $h(x) = \|x\|^4$ whose quadratic matrix relaxation (under symmetric matrix lifting (3.3)) is

$$H(X) = \sum_{i,j} \frac{X_{ii}X_{jj} + X_{ij}^2 + X_{ji}^2}{3}.$$

Consider

$$\begin{array}{ll} \text{maximize} & \bar{f}(x) \equiv \displaystyle\sum_{1 \leq i,j,k,\ell \leq n} a_{ijk\ell} x_i x_j x_k x_\ell - \mu h(x) \\ \text{subject to} & \|x\|^2 = 1, \end{array} \tag{5.12}$$

where $\mu = g(I/n)/H(I/n)$ is a fixed parameter. Since the optimum of (5.1) is attained at the boundary of the unit ball, it follows that $\bar{f}_{\max} = f_{\max} - \mu$, where $\bar{f}_{\max}$ is the optimum value of (5.12). Moreover, the SDP relaxation of (5.12) (under symmetric matrix lifting (3.3)) is

$$\begin{array}{ll} \text{maximize} & \bar{g}(X) \equiv g(X) - \mu H(X) \\ \text{subject to} & \mathrm{tr}(X) = 1,\ X \succeq 0. \end{array} \tag{5.13}$$

Clearly, $\bar{g}(I/n) = 0$, so the argument for Theorem 5.2 is applicable to (5.13). In particular, we can use the relation (5.4) to approximate the feasible set of (5.13) by a second order cone constraint, leading to a second order cone program which is a restriction of (5.13). Moreover, by the concavity of $v(\lambda)$ (cf. (5.9)), we can show that the optimal solution of the resulting second order cone program serves as an $1/(n-1)^2$-factor approximation for (5.13). Using this approximate solution in Gaussian randomized rounding and invoking Theorem 4.1, we conclude that it is possible to find an approximate solution $\bar{x}$ for (5.12) in polynomial time with $\|\bar{x}\| = 1$ such that

$$\bar{f}(\bar{x}) \geq \alpha \bar{f}_{\max}, \quad \text{where } \alpha = O(1/n^2).$$

This implies that

$$f(\bar{x}) - \mu \geq \alpha \left(f_{\max} - \mu \right),$$

or equivalently

$$f(\bar{x}) - \underline{f} \geq \alpha(f_{\max} - \underline{f}) + (1-\alpha)\left(\mu - \underline{f}\right). \tag{5.14}$$

Now we show the last term is nonnegative. Notice that for any vector ξ we have

$$\underline{f} \cdot h(\xi) \leq f(\xi) \leq f_{\max} \cdot h(\xi).$$

For any $X \succeq 0$, let ξ be drawn from the Gaussian distribution $N(0, X)$. Using (3.2) and taking expectation, we have

$$\underline{f} \cdot H(X) \leq g(X) \leq f_{\max} \cdot H(X).$$

By the definition of μ, this further shows that

$$\underline{f} \leq \min_{\mathrm{tr}(X)=1,\, X \succeq 0} \frac{g(X)}{H(X)} \leq \mu \leq \max_{\mathrm{tr}(X)=1,\, X \succeq 0} \frac{g(X)}{H(X)} \leq f_{\max}.$$

Combining this with (5.14) yields

$$f(\bar{x}) - \underline{f} \geq \alpha(f_{\max} - \underline{f}).$$

Finally, we combine the two cases by letting $\hat{x} = \mathrm{argmax}_{x=0,\bar{x}} f(x)$. Therefore, we have $f(\hat{x}) = \max\{0, f(\bar{x})\}$. It can be checked that in either case ($f_{\max} = 0$ or $f_{\max} > 0$), we have

$$f_{\max} - f(\hat{x}) \leq (1-\alpha)(f_{\max} - \underline{f}), \quad \alpha = O(1/n^2).$$

In other words, we can find an approximate solution $\hat{x}$ for (5.1) in polynomial time with a guaranteed *relative* approximation ratio of $(1 - O(1/n^2))$, regardless the sign of $g(I)$. **Q.E.D.**

The case of a ball-constrained quartic *minimization* problem can be treated similarly.

Theorem 5.3 *Let $A_1 \succ 0$ and consider the quartic minimization problem*

$$\begin{array}{ll} \textit{minimize} & f(x) = \sum\limits_{1 \le i,j,k,\ell \le n} a_{ijk\ell} x_i x_j x_k x_\ell \\ \textit{subject to} & x^{\mathrm{T}} A_1 x \ge 1. \end{array} \tag{5.15}$$

Assume the minimum value of (5.15) *is finite, i.e.,* $f_{\min} > -\infty$*. Then, we can find a feasible approximate solution* $\hat{x}$ *for* (5.15) *in polynomial time satisfying*

$$\frac{f(\hat{x}) - f_{\min}}{\bar{f} - f_{\min}} \le \left(1 - O(n^{-2})\right)$$

where the constant in $O(\cdot)$ *notation is universal (i.e., independent of problem data), and*

$$\begin{array}{lll} \bar{f} = & \textit{maximize} & \sum\limits_{1 \le i,j,k,\ell \le n} a_{ijk\ell} x_i x_j x_k x_\ell \\ & \textit{subject to} & x^{\mathrm{T}} A_1 x = 1. \end{array} \tag{5.16}$$

Proof. Without loss of generality, we assume $A_1 = I$. Under the assumption $f_{\min} > -\infty$ (which is equivalent to $f_{\min} \ge 0$), the minimum of $f(x)$ must be attained on the boundary of the unit ball. In this case, the minimization problem (5.15) is equivalent to maximizing $-f$ over the unit sphere. The latter problem is exactly the second case considered in the proof of Theorem 5.2. It follows that we can find a unit-norm vector $\hat{x}$ in polynomial-time such that

$$\frac{-f_{\min} + f(\hat{x})}{-f_{\min} + \bar{f}} \le \left(1 - O(n^{-2})\right)$$

where we have replaced $f(\hat{x})$, $f_{\max}$ and $\underline{f}$ in (5.10) by $-f(\hat{x})$, $-f_{\min}$ and $-\bar{f}$ respectively. This completes the proof of Theorem 5.3. **Q.E.D.**

When there are two or more constraints ($m \ge 2$), it is not clear whether there are efficient approximation algorithms for solving (3.4)–(3.5). This is an interesting topic of further research.

6 Extensions and Discussions

In principle, our symmetric mapping (3.3) and analysis can be extended to higher order polynomial optimization problems. Specifically, when x is generated according to a vector Gaussian distribution, it is possible to represent the expected value of a polynomial function of x in terms of the covariance matrix of x. For instance, for the 6-th order polynomial, it is known that if $x \in N(0, X)$, then

$$
\begin{aligned}
& \mathrm{E}[x_1x_2x_3x_4x_5x_6] \\
= \; & X_{12}X_{34}X_{56} + X_{12}X_{35}X_{46} + X_{12}X_{36}X_{45} + X_{13}X_{24}X_{56} + X_{13}X_{25}X_{46} + X_{13}X_{26}X_{45} \\
& + X_{14}X_{23}X_{56} + X_{14}X_{25}X_{36} + X_{14}X_{26}X_{35} + X_{15}X_{23}X_{46} + X_{15}X_{24}X_{36} + X_{15}X_{26}X_{34} \\
& + X_{16}X_{23}X_{45} + X_{16}X_{24}X_{35} + X_{16}X_{25}X_{34}.
\end{aligned}
$$

Therefore, if one wishes to solve the following $2d$-th order polynomial maximization problem

$$
\begin{array}{ll}
\text{maximize} & f_{2d}(x) = \displaystyle\sum_{1 \le i_1, \cdots, i_{2d} \le n} a_{i_1 \cdots i_{2d}} x_{i_1} \cdots x_{i_{2d}} \\
\text{subject to} & x^{\mathrm{T}} A_i x \le 1, \; i = 1, ..., m,
\end{array}
\tag{6.1}
$$

then the corresponding (non-convex) SDP relaxation problem is

$$
\begin{array}{ll}
\text{maximize} & p_d(X) \\
\text{subject to} & A_i \bullet X \le 1, \; i = 1, ..., m \\
& X \succeq 0,
\end{array}
\tag{6.2}
$$

where $p_d(X)$ is a d-th order polynomial in X.

Theorem 6.1 *Suppose that* (6.2) *has an α-approximation solution, then* (6.1) *admits an overall* $O\left(\frac{\alpha}{(\ln(mn))^d}\right)$ *approximation solution.*

The new technical tool required for this general result is the so called hyper-contractive property of Gaussian distributions (Proposition 6.5.1 of [8]).[3]

Proposition 6.1 *Suppose that f is a multivariate polynomial with degree r. Let $x \in N(0, I)$. Suppose that $p > q > 0$. Then*

$$
(\mathrm{E}|f(x)|^p)^{1/p} \le \kappa_r c_{pq}^r (\mathrm{E}|f(x)|^q)^{1/q}
$$

where κ_r is a constant depending only on r, and $c_{pq} = \sqrt{(p-1)(q-1)}$.

Proposition 6.1 is more general than Proposition 4.2 which deals with the special case of $r = 4$. However, the proof of Proposition 6.1 [8, page 162] was based on the Paley-Zygmund inequality and was non-constructive in the sense that it did not provide an explicit estimate of the constant κ_r for the general r case. In contrast, our constructive proof of Proposition 4.2 does contain an

[3]Professor Anthony So pointed this reference to us after we completed the analysis of $r = 4$ case.

explicit estimate of κ_r for the $r = 4$ case. Repeating the same 4th order moment calculation for a general r-degree polynomial as in the proof of Proposition 4.2 is extremely messy if not impossible.

To prove Theorem 6.1 we need to establish a probability estimation as in the proof of Theorem 4.1. To this end, we may apply Lemma 4.2, in combination of Proposition 6.1 to obtain a lower bound, although we will no longer have explicit estimates of the constants to completely determine the final approximation ratio. More specifically, let $f(x) := f_{2d}(x) - p_d(X)$. Since $p_d(X) = \mathrm{E}[f_{2d}(x)]$, according to Lemma 4.2,

$$\mathrm{Prob}\{f_{2d}(x) \geq p_d(X)\} \geq \frac{(\mathrm{E}|f(x)|^2)^2}{4\mathrm{E}|f(x)|^4}. \tag{6.3}$$

By Proposition 6.1, letting $p = 4$, $q = 2$ and $r = 2d$, we have

$$\left(\mathrm{E}|f(x)|^4\right)^{1/4} \leq \kappa_{2d} 3^d \left(\mathrm{E}|f(x)|^2\right)^{1/2},$$

and so it follows from (6.3) that

$$\mathrm{Prob}\{f_{2d}(x) \geq p_d(X)\} \geq \frac{1}{81^d \kappa_{2d}^4}. \tag{6.4}$$

By using (4.2) and (6.4) we can prove Theorem 6.1 in a manner identical to the proof of Theorem 4.1. Finally, we note that (6.2) is itself a hard problem in general. The effectiveness of the SDP relaxation method depends on our ability to (approximately) solve the relaxed problem (6.2). This is an interesting topic of further research.

The odd-degree polynomial optimization problem can be reduced to the even-degree one by homogenization. Consider

$$\begin{array}{ll} \text{maximize} & f_{2d+1}(x) = \displaystyle\sum_{1 \leq i_1, \cdots, i_{2d+1} \leq n} a_{i_1 \cdots i_{2d+1}} x_{i_1} \cdots x_{i_{2d+1}} \\ \text{subject to} & x^{\mathrm{T}} A_i x \leq 1,\ i = 1, ..., m, \end{array} \tag{6.5}$$

which can be turned into

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{1 \leq i_1, \cdots, i_{2d+1} \leq n} a_{i_1 \cdots i_{2d+1}} x_{i_1} \cdots x_{i_{2d+1}} x_0 \\ \text{subject to} & x^{\mathrm{T}} A_i x \leq 1,\ i = 1, ..., m, \\ & x_0^2 \leq 1. \end{array}$$

An interesting special case of (6.5) is optimization of the 3rd order polynomial over a unit sphere:

$$\begin{array}{ll} \text{maximize} & \sum_{1 \leq i,j,k \leq n} a_{ijk} x_i x_j x_k \\ \text{subject to} & x^{\mathrm{T}} x = 1. \end{array} \tag{6.6}$$

This problem is related to the computation of the eigenvalues of a tensor form (see Qi [17]). Nesterov [15] proved that the optimization model (6.6) is NP-hard.

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