When all risk-adjusted performance measures are the same: In praise of the Sharpe ratio

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Abstract

The classical mean-variance investment model is simple, elegant, and popular. As such, it is also subject to criticisms. One unsatisfactory feature of the model is that variance treats the upside and downside equally as risks. In this regard, the Lower Partial Moments (LPM), Value-at-Risk, and Conditional Value-at-Risk, are more attractive as alternative risk measures, since they only measure the tail loss or the downside. In the meanwhile, considerable amount of recent research efforts have been paid to the so-called Q-radial distributions, which are capable of better modeling the investment returns. In this paper we show that if the investment return rates follow a Q-radial distribution, then the LPM, VaR, or CVaR related Risk Adjusted Performance Measures (RAPM), such as the Sortino ratio, the Omega Statistic, the upside potential ratio, the normalized LPM, the VaR ratio, the CVaR ratio, and the Rachev ratio, are all equivalent to the ordinary Sharpe ratio, which is easy to compute and optimize. Conversely, if all normalized LPM's are equivalent to the Sharpe ratio, then the underlying distribution must be Q-radial. Therefore, this property characterizes the class of distributions in which the Sharpe ratio is essentially the only risk adjusted performance measure.

1 Introduction

Quantitative risk management plays a key role in quantitative finance. Financial institutions are typically equipped with a quantitative risk control division to evaluate the risks before trading the

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financial instruments, and to monitor the risks after the trades. At the time of financial turmoils, as is currently the case, the role of quantitative risk control in financial engineering is particularly prominent. As is common in any engineering discipline, a first step in financial engineering is to measure risks and rewards of an investment portfolio in some computable terms, and such quantities will in turn enable us to optimize, control, and verify the performance of the solutions. The celebrated mean-variance model of Markowitz ([15], 1952) was a pioneer in this practice, where the risk was modeled by the variance of the return, and the reward by the mean of the return. The mean-variance portfolio selection then sets out to minimize the variance of the portfolio while maintaining the mean of the portfolio to be no less than a given threshold. The performance of a portfolio, which measures its efficiency, should take into account the tradeoff between the reward and the risk. Such measurement is made explicit by Sharpe ([26]) as a ratio, known as the *Sharpe ratio*. In particular, according to a refined notion formulated by Sharpe [27] in 1994, the Sharpe ratio of a portfolio is:

$$S(R) := \frac{\mathsf{E}[X - R]}{\sqrt{\mathrm{Var}[X - R]}}$$

where X is return of the portfolio, R is a benchmark index, e.g. the risk-free rate of return, $\mathsf{E}[X - R]$ is the expected excess return over the benchmark (Sharpe [27]). Hence, the Sharpe ratio is a measure of efficiency, in terms of the reward, given the level of risks being taken. In this sense, an asset with higher Sharpe ratio is naturally more attractive since it is perceived as making more efficient use of the risks being taken. The derivation of the Sharpe ratio, however, depends entirely on the mean and variance as the measurements for the reward and risks.

Now, in the mean-variance realm, *risk* is treated identical as *uncertainty*, which is debatable and is somewhat unsatisfactory. For instance, any uncertain gain above the expectation is usually not considered a risk in the ordinary sense. Markowitz's contemporary, Roy ([23]), made a note of this subtle difference. Roy argued that an investor would prefer safety of principal first and will set some minimum acceptable return that will conserve the principal. Markowitz ([16], 1959) recognized the importance of this idea and proposed a downside risk measure known as the *semivariance* to replace the ordinary variance, since the semivariance is only concerned with the downside, which was the first time that the downside risk has been included in a portfolio selection model. The semivariance measure is more consistent with the perception of the investment risk of a typical investor. However, the attitude towards risks can be vastly different. Since the semivariance is based on the *second* moment of the downside, it is natural to consider general nth moments of downside to suit different investors. This is why Bawa [1] and Fishburn [9] introduced in the 1970's the so-called *lower partial moment* (LPM) of the downside:

$$\mathrm{LPM}_n(R) = \mathsf{E}[(R - X)_+{}^n],$$

where n can be any non-negative number, and R is a benchmark index. Specifically, if $n = 0$ then the risk measure LPM_0 becomes the probability of the asset return falling below the benchmark return; if $n = 1$ then LPM_1 becomes the expected shortfall of the return below a benchmark index; if $n = 2$ then LPM_2 is analogous to semivariance, where the deviations are determined with respect to the benchmark rather than the deviation around the mean. Since its introduction, LPM_n has been widely used in a series of risk adjusted performance measures (RAPM). Some of these measures are listed below:

The Sortino ratio:	$SO(R) := \frac{\mathsf{E}[X-R]}{\sqrt{\mathrm{LPM}_2(R)}}$;
The Omega:	$\Omega(R) := \frac{\int_R^{\infty}[1-F(X)]dX}{\int_{-\infty}^{R} F(X)dX}$, where $F(\cdot)$ is the cdf of X;
The Kappa ratio:	$K_n(R) := \frac{\mathsf{E}[X-R]}{\sqrt[n]{\mathrm{LPM}_n(R)}}$;
The Upside Potential ratio:	$UP(R) := \frac{\mathsf{E}[(X-R)_+]}{\sqrt{\mathrm{LPM}_2(R)}}$.

Let us briefly review here the historical developments of these notions. Actually, the so-called Sortino ratio was initially proposed by Brian Rom in 1986 as an element of his company's (called "Investment Technologies") post-modern portfolio theory and portfolio optimization software (see also [20, 21, 22]). Later, Sortino and Van der Meer [28] used this notion as a tool to capture the essence of the downside risk. Technically, the Sortino ratio is a modification of the Sharpe ratio, but it only penalizes the events which fall below the benchmark index, while the original Sharpe ratio appears to penalize both up and downside deviations. It is therefore a more appealing measure of risk-adjusted performance than the original Sharpe ratio. For more discussions on the Sortino ratio we refer to [28, 30, 29, 20, 21, 22]. The Omega Statistic, as proposed by Shadwick and Keating [25], is another alternative to the Sharpe ratio, which also measures the risk-adjusted performance. It provides a risk-reward evaluation of a return distribution, relative to any user-specified benchmark. This evaluation weighs the benefit of the gains over the detrimental impact of the losses. The Kappa ratio of Kaplan and Knowles [13] was presented as a general downside risk-adjusted performance measurement, by allowing to vary the parameter n. It is clear that $\Omega(R) = 1 + \frac{\mathsf{E}(X-R)}{\mathrm{LPM}_1(R)}$, and so the Omega is equal to $K_1(R) + 1$ and the Sortino ratio is equal to $K_2(R)$. The upside potential ratio was first introduced by Sortino *et al.*, in [31]. As alluded to in the famous Nobel prize-winning work of Kahneman and Tversky [12], viz. the prospect theory, an individual behaves risk-aversely if the return exceeds his/her reference point, but could act risk-seekingly otherwise. The upside potential ratio offers a new perspective on the tradeoff between the risk and reward. That is well suited for investors seeking the best performance above the reference point, subject to the risk of falling below. Value-at-Risk (VaR) is the current standard benchmark for firm-wide measures of risk, but it lacks subadditivity and convexity. As an alternative risk measure, Rockafellar and Uryasev [19] introduced conditional value-at-risk (CVaR), which is coherent and tractable. It is natural to involve these two new risk measures into RAPM. Therefore, some new performance measures have recently been proposed, such as the VaR ratio [8, 17], the CVaR ratio [17], and the Rachev ratio [2]. Most of the financial risk models where LPM_n or VaR is involved have the multivariate normal distribution in mind to model the investment returns, due to its tractability. However, there are strong evidences suggesting that the normal distribution may not be ideal in modeling the investment returns. One often quoted problem is the light tail of the normal distribution, which is not supported by the historical financial data. Another issue is the 'independence' of the extremal events for the normal distribution, although the extreme events can be modeled by some tail-dependence measures. To tackle these issues of the normal distribution, a natural remedy would be to consider a broader class of distributions, allowing some freedom in modeling the tail distributions for instance. A good candidate in this category is the so-called *class of elliptical distributions*; see e.g. [7, 14, 18]. The class of elliptical distributions is also known as the *radial distributions* in Operations Research and the statistical mechanics literatures [3, 4]. For clarity, hereafter we shall adhere to the terminology radial distribution. By an affine transformation, a radial distribution induces a Q-radial distribution, and Q-radial distributions are a generalization of the multivariate normal distribution. As a consequence, the class of Q-radial distributions retain many of the nice properties inherited from the normal distribution; we refer to Fang, Kotz, Ng [7] for a discussion. Moreover, the class of Q-radial distributions contains many other interesting and useful distributions, e.g. the uniform distribution on an ellipsoidal support, the multivariate student-t, the truncated multivariate normal distribution, the Laplace distribution etc. Within the class of Q-radial distributions, it is allowed to have a heavier tail than that of the normal distribution. Furthermore, some Q-radial distribution can very well model the so-called tail dependence phenomenon in insurance and financial data analysis; see Embrechts *et al.* [6] and Schmidt [24]. Embrechts *et al.* [6] also showed that the elliptical class preserves the property that the Markowitz variance-minimizing portfolio minimizes any coherent risk measures and LPM_n.

The main contribution of this paper is to pin down a linkage between the above LPM_n, VaR, and CVaR-related risk adjusted performance measures and the Q-radial distributions. We show that if the rate of return follows a Q-radial distribution, then all the above risk adjusted performance measures, including the Sortino ratio, the Omega Statistic, the Kappa ratio, the upside potential ratio, the normalized LPM_n, the VaR ratio, the CVaR ratio, and the Rachev ratio, can all be viewed as monotonic transformations of the Sharpe ratio. As a byproduct, it follows that if the risk is modeled by any of the above risk measures then the efficient frontier (the 'tradeoff curve' between the risk and the reward) coincides with that of the mean-variance model. Conversely, we show that if the normalized LPM_n performance measure is equivalent to the Sharpe ratio for all n, then the return rates must necessarily follow a certain Q-radial distribution. Essentially, that is to say, the Sharpe ratio is *the* risk-adjusted performance measure (RAPM) if the asset return follows some Q-radial distribution, and vice versa. The rest of the paper is organized as follows. In Section 2, we introduce the multivariate Q-radial distributions. In Section 3, we show the equivalence between the LPM_n, VaR, and CVaR-related RAPM's and the Sharpe ratio. Moreover, we prove that this LPM_n-related equivalence actually is a characteristic for the Q-radial distributions. Since the Sharpe ratio is such a key quantity to be optimized within the class of Q-radial distributions, in Section 4 we show how to find a portfolio that maximizes the Sharpe ratio, by means of convex conic optimization.

2 The Radial Distributions

Let us formally introduce the definition and discuss properties of the multivariate radial distributions.

Definition 2.1 *A random vector $\xi \in \Re^n$ has a* radial distribution *if the probability density function of ξ, to be denoted $f_\xi(x)$ here, only depends on the Euclidean norm, i.e.,*

$$f_\xi(x) = g_n(\|x\|).$$

The function $g_n(\cdot)$ is called the defining function of ξ.

To get an impression, a typical radial distribution by means of a contour diagram is shown in Figure 1.

By its symmetry, the mean of ξ is the zero vector, and its covariance matrix is

$$\Sigma_\xi = \left(V_n \int_0^\infty r^{n+1} g_n(r) \, \mathrm{d}r \right) I_n,$$

where V_n is the volume of the Euclidean ball of unit radius in $\Re^n$.

Choosing the defining function appropriately we find several familiar distributions. For example, the multivariate normal distribution is generated by $g_n(x) = (2\pi)^{-n/2} \cdot e^{-x^2/2}$; the multivariate logistic distribution is generated by $g_n(x) = c \cdot \frac{\exp(-x^2/2)}{[1+\exp(-x^2/2)]^2}$; the double exponential distribution is generated by $g_n(x) = c \cdot \exp(-|x|)$; the multivariate student-t distribution with m degrees of freedom is generated by $g_n(x) = c \cdot (1 + \frac{x^2}{m})^{-(n+m)/2}$, where c is a generic normalizing constant.

The multivariate t distribution with m degree of freedom yields an example of radial distribution used in quantitative finance. Its density function has the following form:

$$f(x_1, \cdots, x_n) = c \left[1 + \frac{1}{m}(x_1^2 + \cdots + x_n^2) \right]^{-\frac{m+n}{2}}.$$

Empirical evidences have suggested that the distribution of financial returns tends to have a fatter tail than a normal distribution. It is well known that the t distribution indeed has a fat tail and is therefore a candidate for simulating more realistic tail events of the investment returns: events that occur with non-negligible chances and will cause some major gains or losses.

The multivariate normal distribution is invariant under orthonormal linear transformation, which is a common heritage of all radial distributions. Below we will present some more properties of the radial distributions, the proof of which can be found in the appendix.

Proposition 2.1 *(1). A random vector $\xi \in \Re^n$ follows a radial distribution if and only if for every orthonormal matrix $U \in \Re^{n \times n}$, $U\xi$ follows the same distribution.*

(2). The probability density function of $\eta := a^{\mathrm{T}}\xi$ is

$$f_\eta(x) = \frac{S_{n-1}}{\|a\|} \int_0^\infty \rho^{n-2} g_n \left(\sqrt{\rho^2 + \left(\frac{x}{\|a\|} \right)^2} \right) \mathrm{d}\rho, \tag{1}$$

and its cumulative distribution function is

$$\mathrm{Prob}\left\{\eta \le x\right\} = S_{n-1} \int_0^\infty \int_{-\infty}^{\frac{x}{\|a\|}} \rho^{n-2} g_n \left(\sqrt{\rho^2 + \gamma^2} \right) \, \mathrm{d}\gamma \mathrm{d}\rho, \tag{2}$$

where $S_{n-1} = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}$ denotes the spherical area of the Euclidean ball of unit radius in $\Re^{n-1}$.

Any normal distribution can be considered as an affine transformation of the standard normal distribution. In the same spirit, we shall also work with the standard cdf and pdf for each radial distribution, which turns out to greatly simplify our analysis later. Observe that the statistical properties of η are *independent* of the direction of a, but are only dependent on its norm $\|a\|$. This motivates us to introduce the notion of a *standard* pdf of η to be

$$\psi_n(x) = f_\eta(x) = S_{n-1} \int_0^\infty \rho^{n-2} \, g_n \left(\sqrt{\rho^2 + x^2} \right) \mathrm{d}\rho, \tag{3}$$

which corresponds to the case where $\|a\| = 1$, and the standard cdf of η is

$$\Psi_n(x) = \mathrm{Prob}\left\{\eta \le x\right\} = \int_{-\infty}^x \psi_n(\gamma) \mathrm{d}\gamma. \tag{4}$$

Observe that in this context, $\psi_n(x)$ is an even function in x. Thus, $\Psi_n(-x) = 1 - \Psi_n(x)$.

An affine linear transformation may in general ruin the radial distribution. It is however also possible to turn a non-radial distribution into a radial distribution by some affine transformation.

Definition 2.2 *A random vector $\omega \in \Re^m$ has a Q-*radial distribution *with defining function $g_n(\cdot)$ if*

$$\omega - \mathsf{E}[\omega] = Q\xi,$$

where $Q \in \Re^{m \times n}$ is a full column rank matrix and $\xi \in \Re^n$ is a random vector having a radial distribution with the pdf

$$f_\xi(x) = g_n(\|x\|). \tag{5}$$

For the above distribution ω, we have

$$\mathrm{Var}(\omega) = \frac{1}{\nu^2} Q Q^{\mathrm{T}}, \tag{6}$$

with $\nu := \left(V_n \int_0^\infty r^{n+1} g_n(r)\mathrm{d}r\right)^{-1/2}$. In other words,

$$\mathrm{Var}(\omega^{\mathrm{T}}x) = \frac{1}{\nu^2}\|x^{\mathrm{T}}Q\|^2. \tag{7}$$

It is possible to compute the pdf, cdf, value at risk (VaR), conditional value at risk (CVaR), and other related statistical quantities for the Q-radial distribution explicitly, as the next proposition shows.

Proposition 2.2 *(1). Suppose that* $\zeta = b^{\mathrm{T}}\omega$, $\mathsf{E}[\omega] = \mu$. *The probability density function of* ζ *is*

$$f_\zeta(x) = \frac{1}{\|b^{\mathrm{T}}Q\|} \cdot \psi_n\left(\frac{x - b^{\mathrm{T}}\mu}{\|b^{\mathrm{T}}Q\|}\right),$$

and its cumulative distribution function is

$$\mathrm{Prob}\left\{\zeta \le x\right\} = \Psi_n\left(\frac{x - b^{\mathrm{T}}\mu}{\|b^{\mathrm{T}}Q\|}\right).$$

(2). Suppose $\mathsf{E}[\omega] = \mu$. *For any* $\beta \in (0, 0.5]$, $R \in \Re$, $x \in \Re^m$, $n \le m$, *the chance constraint*

$$\mathrm{Prob}\left\{\omega^{\mathrm{T}}x \le R\right\} \le \beta$$

is equivalent to

$$x^{\mathrm{T}}\mu - R \ge -\tau \cdot \|x^{\mathrm{T}}Q\|,$$

where $\tau = \sup\{t \mid \Psi_n(t) = \beta\} \le 0$, *whenever* $\beta \in (0, 0.5]$. *The last constraint is in the form of* Second Order Cone (SOC) *constraint.*

(3). Suppose that $f(x, \omega, \omega_0) = x^{\mathrm{T}}\omega - \omega_0$, *and* $\mathsf{E}[\omega] = \mu$, $\mathsf{E}[\omega_0] = \mu_0$, $\mathrm{Cov}(\omega, \omega_0) = \begin{pmatrix} H & h \\ h^{\mathrm{T}} & h_0 \end{pmatrix}$. *Then*

$$\mathrm{VaR}_\beta(x) = \mu(x) + \kappa_\beta \cdot \sigma(x), \tag{8}$$

$$\mathrm{CVaR}_\beta(x) = \mu(x) + \gamma_\beta \cdot \sigma(x), \tag{9}$$

where

$$\mu(x) = x^{\mathrm{T}}\mu - \mu_0, \quad \sigma(x) = \nu\sqrt{x^{\mathrm{T}}Hx - 2x^{\mathrm{T}}h + h_0};$$
$$\kappa_\beta = \inf\{t \mid \Psi_n(t) = \beta\}, \quad \gamma_\beta = \frac{1}{1-\beta}\int_{\kappa_\beta}^{\infty} y\psi_n(y)\ \mathrm{d}y.$$

The chance constraints, VaR, and CVaR can all be conveniently worked out for normal distributions. Proposition 2.2 shows that this is actually also the case for any radial distribution. As is well known, chance constraints usually lead to nonconvexity. In the case of radial distribution, however, the chance constraint can be represented using the Second Order Cones, which is convex and numerically tractable. It is also well known that in general the chance constraint is related to VaR; due to its intractability, Rockafellar and Uryasev [19] introduced the notion of Conditional Value-at-Risk (CVaR), which turns out to be a coherent risk measure. Interestingly, in the case of radial distribution, VaR is itself a coherent risk measure, and the formulas of VaR and CVaR will all have simple expressions; see e.g. Embrechts *et al.* [6]. For the benefit of the reader, we shall present a self-contained proof below for convenience.

Proof of Proposition 2.2.

(1). The assertion follows from the observation that

$$\text{Prob}\,\{\zeta \le x\} = \text{Prob}\,\{b^{\mathrm{T}}Q\xi \le x - b^{\mathrm{T}}\mu\}.$$

(2). Since

$$\text{Prob}\,\{\omega^{\mathrm{T}}x \le R\} = \Psi_n\left(\frac{R - x^{\mathrm{T}}\mu}{\|x^{\mathrm{T}}Q\|}\right) \le \beta$$

and $\psi(x)$ is symmetric around the origin, τ is non-positive if $\beta \le 0.5$.

(3). It follows from Rockafellar and Uryasev [19] that

$$\text{CVaR}_\beta(x) = \min_{\alpha\in\Re} F_\beta(x,\alpha)$$

where

$$F_\beta(x,\alpha) \triangleq \alpha + \frac{1}{1-\beta}\int_{\xi\in\Re^m} [f(x,\xi)-\alpha]_+ \,\mathrm{d}\pi(\xi).$$

Under our assumptions,

$$F_\beta(x,\alpha) = \alpha + \frac{1}{1-\beta}\left[(\mu(x)-\alpha)\cdot\Psi_n\left(\frac{\mu(x)-\alpha}{\sigma(x)}\right) + \sigma(x)\cdot\int_{\frac{\mu(x)-\alpha}{\sigma(x)}}^{\infty} y\psi_n(y)\,\mathrm{d}y\right].$$

By the first order optimality condition,

$$0 = \frac{\partial\,(1-\beta)F_\beta(x,\alpha)}{\partial\alpha} = (1-\beta) - \Psi_n\left(\frac{\mu(x)-\alpha^*}{\sigma(x)}\right),$$

it follows that

$$\beta = \Psi_n\left(\frac{\alpha^* - \mu(x)}{\sigma(x)}\right),$$

and

$$\text{VaR}_\beta(x) = \inf\arg\min_{\alpha\in\Re} F_\beta(x,\alpha) = \mu(x) + \kappa_\beta\cdot\sigma(x).$$

Q.E.D.

3 Measuring the Downside Risk

In this section we study the risk adjusted performance measures when the underlying distribution is actually Q-radial. Let us start by noting a simple observation, which, however, plays a key role in our analysis.

Lemma 3.1 *Suppose that $\psi_n(y)$ is a density function of a probability distribution, and the following function is finite*

$$h_k(z) := \int_{-z}^{\infty} (y+z)^k \psi_n(y) \mathrm{d}y, \tag{10}$$

for $k \geq 0$. Then,

$$(h_k)'(z) = k h_{k-1}(z),$$

and

$$h_k(z) \geq 0, \quad \forall z \in \Re.$$

In other words, $h_k(\cdot)$ is an increasing function, for any $k \geq 0$.

Proof. Obviously, for any $k \in \Re$, $h_k(\cdot)$ is always nonnegative. If $k=0$, then $(h_0)'(z) = \psi_n(-z) \geq 0$. If $k > 0$, then

$$(h_k)'(z) = \int_{-z}^{\infty} k(y+z)^{k-1}\psi_n(y)\mathrm{d}y + (-z+z)^k \psi_n(-z) = k h_{k-1}(z) \geq 0,$$

and so $h_k(z)$ is increasing for any $k \geq 0$. □

Consider n financial assets S_j, $j = 1, ..., n$, and $x = (x_1, \cdots, x_n)$ is a portfolio of investments on these n assets. Let us denote the rate of return of the jth asset be ω_j, $j = 1, ..., n$. The benchmark rate of return is denoted to be R. Suppose that the mean of ω is μ and the covariance of ω is Γ. Then the Sharpe ratio of the return on the portfolio is

$$S(R) = \frac{x^{\mathrm{T}}\mu - R}{\sqrt{x^{\mathrm{T}}\Gamma x}}.$$

Furthermore, if ω is a Q-radial distribution, then

$$S(R) = \nu \cdot \frac{x^{\mathrm{T}}\mu - R}{\|x^{\mathrm{T}}Q\|}$$

where $\nu = \left(V_n \int_0^{\infty} r^{n+1} g_n(r) \mathrm{d}r\right)^{-1/2}$; see (6).

In this section, the loss function shall be denoted as $R - x^{\mathrm{T}}\omega$. In this case, VaR and CVaR can be explicitly expressed, using part (3) of Proposition 2.2:

$$\begin{aligned} \mathrm{VaR}_\beta(x) &:= \min\{\alpha \in \Re \mid \mathrm{Prob}\{R - x^{\mathrm{T}}\omega \le \alpha\} \ge \beta\} = R - x^{\mathrm{T}}\mu + \kappa_\beta \|x^{\mathrm{T}}Q\|, && (11) \\ \mathrm{CVaR}_\beta(x) &:= \mathsf{E}[R - x^{\mathrm{T}}\omega \mid R - x^T\omega \ge \mathrm{VaR}_\beta(x)] = R - x^{\mathrm{T}}\mu + \gamma_\beta \|x^{\mathrm{T}}Q\|, && (12) \end{aligned}$$

where

$$\begin{aligned} \kappa_\beta &:= \inf\{t \mid \Psi_n(t) = \beta\}, && (13) \\ \gamma_\beta &:= \frac{1}{1-\beta}\int_{\kappa_\beta}^{\infty} y\psi_n(y)\ \mathrm{d}y. && (14) \end{aligned}$$

Our main result is the following theorem.

Theorem 3.1 *Suppose the benchmark return index is R, and that ω is multivariate Q-radially distributed with mean $\mu \in \Re^n$. Then the following eleven utility measures are all monotonic transformations of the Sharpe ratio:*

(a). The normalized LPM_k and UPM_k:

$$\frac{\mathsf{E}[(R - x^{\mathrm{T}}\omega)_+{}^k]}{\left[\sqrt{\mathrm{Var}(R - x^{\mathrm{T}}\omega)}\right]^k} = \nu^k h_k\left(\frac{R - x^{\mathrm{T}}\mu}{\|x^{\mathrm{T}}Q\|}\right), \quad \forall\, k \ge 0,$$

and

$$\frac{\mathsf{E}[(x^{\mathrm{T}}\omega - R)_+{}^k]}{\left[\sqrt{\mathrm{Var}(x^{\mathrm{T}}\omega - R)}\right]^k} = \nu^k h_k\left(\frac{x^{\mathrm{T}}\mu - R}{\|x^{\mathrm{T}}Q\|}\right), \quad \forall\, k \ge 0.$$

(b). The upside potential ratio:

$$\frac{\mathsf{E}[(x^{\mathrm{T}}\omega - R)_+]}{\sqrt{\mathsf{E}[(R - x^{\mathrm{T}}\omega)_+{}^2]}} = \frac{h_1\left(\frac{x^{\mathrm{T}}\mu - R}{\|x^{\mathrm{T}}Q\|}\right)}{\sqrt{h_2\left(\frac{R - x^{\mathrm{T}}\mu}{\|x^{\mathrm{T}}Q\|}\right)}}.$$

(c). The generalized upside potential ratio :

$$\frac{\sqrt[k]{\mathsf{E}[(x^{\mathrm{T}}\omega - R)_+{}^k]}}{\sqrt[l]{\mathsf{E}[(R - x^{\mathrm{T}}\omega)_+{}^l]}} = \frac{\sqrt[k]{h_k\left(\frac{x^{\mathrm{T}}\mu - R}{\|x^{\mathrm{T}}Q\|}\right)}}{\sqrt[l]{h_l\left(\frac{R - x^{\mathrm{T}}\mu}{\|x^{\mathrm{T}}Q\|}\right)}}, \quad \forall\, k, l \ge 0.$$

(d). The Omega statistic:

$$\Omega(R) := \frac{\int_R^{\infty} \mathrm{Prob}\{x^{\mathrm{T}}\omega \ge t\}\ \mathrm{d}t}{\int_{-\infty}^{R} \mathrm{Prob}\{x^{\mathrm{T}}\omega \le t\}\ \mathrm{d}t} = 1 + \frac{\frac{x^{\mathrm{T}}\mu - R}{\|x^{\mathrm{T}}Q\|}}{h_1\left(\frac{R - x^{\mathrm{T}}\mu}{\|x^{\mathrm{T}}Q\|}\right)}.$$

(e). The Sortino ratio:

$$\frac{\mathsf{E}[x^{\mathrm{T}}\omega - R]}{\sqrt{\mathsf{E}[(R - x^{\mathrm{T}}\omega)_+{}^2]}} = \frac{\frac{x^{\mathrm{T}}\mu - R}{\|x^{\mathrm{T}}Q\|}}{\sqrt{h_2\left(\frac{R - x^{\mathrm{T}}\mu}{\|x^{\mathrm{T}}Q\|}\right)}}. \tag{15}$$

(f). The Kappa ratio:

$$\frac{\mathsf{E}[x^{\mathrm{T}}\omega - R]}{\sqrt[k]{\mathsf{E}[(R - x^{\mathrm{T}}\omega)_+{}^k]}} = \frac{\frac{x^{\mathrm{T}}\mu - R}{\|x^{\mathrm{T}}Q\|}}{\sqrt[k]{h_k\left(\frac{R - x^{\mathrm{T}}\mu}{\|x^{\mathrm{T}}Q\|}\right)}}, \quad \forall\, k \geq 0. \tag{16}$$

(g). The VaR ratio (the Excess Return on VaR):

$$\frac{\mathsf{E}[x^{\mathrm{T}}\omega - R]}{\mathrm{VaR}_\beta(x)} = \frac{\frac{x^{\mathrm{T}}\mu - R}{\|x^{\mathrm{T}}Q\|}}{\frac{R - x^{\mathrm{T}}\mu}{\|x^{\mathrm{T}}Q\|} + \kappa_\beta}. \tag{17}$$

(h). The CVaR ratio (the STARR ratio, or the Conditional Sharpe ratio):

$$\frac{\mathsf{E}[x^{\mathrm{T}}\omega - R]}{\mathrm{CVaR}_\beta(x)} = \frac{\frac{x^{\mathrm{T}}\mu - R}{\|x^{\mathrm{T}}Q\|}}{\frac{R - x^{\mathrm{T}}\mu}{\|x^{\mathrm{T}}Q\|} + \gamma_\beta}. \tag{18}$$

(i). The Rachev ratio:

$$\frac{\mathsf{E}[x^{\mathrm{T}}\omega - R \mid x^{\mathrm{T}}\omega - R \geq \mathrm{VaR}_\alpha(x)]}{\mathrm{CVaR}_\beta(x)} = \frac{\frac{x^{\mathrm{T}}\mu - R}{\|x^{\mathrm{T}}Q\|} + \gamma_\alpha}{\frac{R - x^{\mathrm{T}}\mu}{\|x^{\mathrm{T}}Q\|} + \gamma_\beta}. \tag{19}$$

(j). The VaR-based upside potential ratio:

$$\frac{\sqrt[k]{\mathsf{E}[(x^{\mathrm{T}}\omega - R)_+{}^k]}}{\mathrm{VaR}_\beta(x)} = \frac{\sqrt[k]{h_k\left(\frac{x^{\mathrm{T}}\mu - R}{\|x^{\mathrm{T}}Q\|}\right)}}{\frac{R - x^{\mathrm{T}}\mu}{\|x^{\mathrm{T}}Q\|} + \kappa_\beta}, \quad \forall\, k \geq 0. \tag{20}$$

(k). The CVaR-based upside potential ratio:

$$\frac{\sqrt[k]{\mathsf{E}[(x^{\mathrm{T}}\omega - R)_+{}^k]}}{\mathrm{CVaR}_\beta(x)} = \frac{\sqrt[k]{h_k\left(\frac{x^{\mathrm{T}}\mu - R}{\|x^{\mathrm{T}}Q\|}\right)}}{\frac{R - x^{\mathrm{T}}\mu}{\|x^{\mathrm{T}}Q\|} + \gamma_\beta}, \quad \forall\, k \geq 0. \tag{21}$$

Proof. Clearly, (b) and (d), (e) are all special cases of (c) and (f) respectively, so we will skip the proofs for these three assertions.

(a). We first observe that

$$\begin{aligned}
\mathsf{E}[(\omega^{\mathrm{T}}x - R)_+{}^k] &= \int_{-\frac{x^{\mathrm{T}}\mu - R}{\|x^{\mathrm{T}}Q\|}}^{\infty} (y\|x^{\mathrm{T}}Q\| + x^{\mathrm{T}}\mu - R)^k \psi_n(y)\,\mathrm{d}y \\
&= \|x^{\mathrm{T}}Q\|^k \int_{-\frac{x^{\mathrm{T}}\mu - R}{\|x^{\mathrm{T}}Q\|}}^{\infty} \left(y + \frac{x^{\mathrm{T}}\mu - R}{\|x^{\mathrm{T}}Q\|}\right)^k \psi_n(y)\,\mathrm{d}y \\
&= \|x^{\mathrm{T}}Q\|^k h_k\left(\frac{x^{\mathrm{T}}\mu - R}{\|x^{\mathrm{T}}Q\|}\right), \qquad (22)
\end{aligned}$$

and so it further follows from (10) and (7) that

$$\frac{[\mathsf{E}(x^{\mathrm{T}}\omega - R)_+{}^k]}{\left[\sqrt{\mathrm{Var}(x^{\mathrm{T}}\omega - R)}\right]^k} = \nu^k h_k\left(\frac{x^{\mathrm{T}}\mu - R}{\|x^{\mathrm{T}}Q\|}\right). \qquad (23)$$

This proves the case for UPM_k in (a), and the case for LPM_k is similar.

(c). Using (22) we have

$$\frac{\sqrt[k]{\mathsf{E}[(x^{\mathrm{T}}\omega - R)_+{}^k]}}{\sqrt[l]{\mathsf{E}[(R - x^{\mathrm{T}}\omega)_+{}^l]}} = \frac{\sqrt[k]{h_k\left(\frac{x^{\mathrm{T}}\mu - R}{\|x^{\mathrm{T}}Q\|}\right)}}{\sqrt[l]{h_l\left(\frac{R - x^{\mathrm{T}}\mu}{\|x^{\mathrm{T}}Q\|}\right)}}.$$

(f). The claimed relation follows from the derivation:

$$\begin{aligned}
\frac{\mathsf{E}[x^{\mathrm{T}}\omega - R]}{\sqrt[k]{\mathsf{E}[(R - x^{\mathrm{T}}\omega)_+{}^k]}} &= \frac{x^{\mathrm{T}}\mu - R}{\|x^{\mathrm{T}}Q\| \cdot \sqrt[k]{h_k\left(\frac{R - x^{\mathrm{T}}\mu}{\|x^{\mathrm{T}}Q\|}\right)}} \\
&= \frac{\frac{x^{\mathrm{T}}\mu - R}{\|x^{\mathrm{T}}Q\|}}{\sqrt[k]{h_k\left(\frac{R - x^{\mathrm{T}}\mu}{\|x^{\mathrm{T}}Q\|}\right)}}.
\end{aligned}$$

(g) and (h). The equivalence relations follow from the expressions for VaR and CVaR as given in (11) and (12):

$$\begin{aligned}
\frac{\mathsf{E}[x^{\mathrm{T}}\omega - R]}{\mathrm{VaR}_\beta(x)} &= \frac{x^{\mathrm{T}}\mu - R}{R - x^{\mathrm{T}}\mu + \kappa_\beta\|x^{\mathrm{T}}Q\|} &= \frac{\frac{x^{\mathrm{T}}\mu - R}{\|x^{\mathrm{T}}Q\|}}{\frac{R - x^{\mathrm{T}}\mu}{\|x^{\mathrm{T}}Q\|} + \kappa_\beta}, \\
\frac{\mathsf{E}[x^{\mathrm{T}}\omega - R]}{\mathrm{CVaR}_\beta(x)} &= \frac{x^{\mathrm{T}}\mu - R}{R - x^{\mathrm{T}}\mu + \gamma_\beta\|x^{\mathrm{T}}Q\|} &= \frac{\frac{x^{\mathrm{T}}\mu - R}{\|x^{\mathrm{T}}Q\|}}{\frac{R - x^{\mathrm{T}}\mu}{\|x^{\mathrm{T}}Q\|} + \gamma_\beta}.
\end{aligned}$$

(i). In this case, the numerator is CVaR_α, with the loss function being $x^{\mathrm{T}}\omega - R$, which can be expressed as $x^{\mathrm{T}}\mu - R + \gamma_\alpha \|x^{\mathrm{T}}Q\|$, as indicated by (9). Thus, the ratio can be written as

$$\frac{\mathsf{E}[x^{\mathrm{T}}\omega - R \mid x^{\mathrm{T}}\omega - R \geq \mathrm{VaR}_\alpha(x)]}{\mathrm{CVaR}_\beta(x)} = \frac{x^{\mathrm{T}}\mu - R + \gamma_\alpha\|x^{\mathrm{T}}Q\|}{R - x^{\mathrm{T}}\mu + \gamma_\beta\|x^{\mathrm{T}}Q\|} = \frac{\frac{x^{\mathrm{T}}\mu - R}{\|x^{\mathrm{T}}Q\|} + \gamma_\alpha}{\frac{R - x^{\mathrm{T}}\mu}{\|x^{\mathrm{T}}Q\|} + \gamma_\beta}.$$

(j) and (k). Applying (11), (12) and (22), we have

$$\begin{aligned}
\frac{\sqrt[k]{\mathsf{E}[(x^{\mathrm{T}}\omega - R)_+{}^k]}}{\mathrm{VaR}_\beta(x)} &= \frac{\|x^{\mathrm{T}}Q\| \cdot \sqrt[k]{h_k\left(\frac{R - x^{\mathrm{T}}\mu}{\|x^{\mathrm{T}}Q\|}\right)}}{R - x^{\mathrm{T}}\mu + \kappa_\beta\|x^{\mathrm{T}}Q\|} &= \frac{\sqrt[k]{h_k\left(\frac{x^{\mathrm{T}}\mu - R}{\|x^{\mathrm{T}}Q\|}\right)}}{\frac{R - x^{\mathrm{T}}\mu}{\|x^{\mathrm{T}}Q\|} + \kappa_\beta}, \quad \forall\, k \geq 0, \\
\frac{\sqrt[k]{\mathsf{E}[(x^{\mathrm{T}}\omega - R)_+{}^k]}}{\mathrm{CVaR}_\beta(x)} &= \frac{\|x^{\mathrm{T}}Q\| \cdot \sqrt[k]{h_k\left(\frac{R - x^{\mathrm{T}}\mu}{\|x^{\mathrm{T}}Q\|}\right)}}{R - x^{\mathrm{T}}\mu + \gamma_\beta\|x^{\mathrm{T}}Q\|} &= \frac{\sqrt[k]{h_k\left(\frac{x^{\mathrm{T}}\mu - R}{\|x^{\mathrm{T}}Q\|}\right)}}{\frac{R - x^{\mathrm{T}}\mu}{\|x^{\mathrm{T}}Q\|} + \gamma_\beta}, \quad \forall\, k \geq 0.
\end{aligned}$$

Q.E.D.

It is interesting to note that the above mentioned equivalence actually characterizes the class of Q-radial distributions. In particular, we shall show below that if the normalized LPM_n is a monotone transformation of the Sharpe ratio for any nonnegative integer n, then the underlying distribution is necessarily Q-radial.

Theorem 3.2 *Consider a given distribution ω with finite moments and nonsingular variance. Suppose that $\frac{\sqrt[k]{\mathsf{E}[(x^{\mathrm{T}}\omega - R)_+{}^k]}}{\sqrt{\mathrm{Var}(R - x^{\mathrm{T}}\omega)}}$ is a function of the Sharpe ratio $\frac{x^{\mathrm{T}}\mathsf{E}[\omega] - R}{\|x^{\mathrm{T}}\mathrm{Var}(\omega)^{1/2}\|}$ for any R, $x \in \Re^n$ and nonnegative k, then ω is Q-radially distributed.*

Proof. Denote $\mathsf{E}[\omega] = \mu$ and $\mathrm{Var}(\omega) = \Gamma$. According to the hypothesis of the theorem, for any given integer k, there exists some function $q_k(\cdot)$, such that

$$\mathsf{E}[(x^{\mathrm{T}}\omega - R)_+{}^k] = q_k\left(\frac{x^{\mathrm{T}}\mu - R}{\|x^{\mathrm{T}}\Gamma^{\frac{1}{2}}\|}\right) \cdot \|x^{\mathrm{T}}\Gamma^{\frac{1}{2}}\|^k,$$

and so

$$\mathsf{E}[(R - x^{\mathrm{T}}\omega)_+{}^k] = q_k\left(\frac{R - x^{\mathrm{T}}\mu}{\|x^{T}\Gamma^{\frac{1}{2}}\|}\right) \cdot \|x^{\mathrm{T}}\Gamma^{\frac{1}{2}}\|^k.$$

Thus for any $k \geq 0$, we have

$$\begin{aligned}
\mathsf{E}[(x^{\mathrm{T}}\omega - R)^k] &= \mathsf{E}[(x^{\mathrm{T}}\omega - R)_+{}^k + (-1)^k \cdot (R - x^{\mathrm{T}}\omega)_+{}^k] \\
&= \left[q_k\left(\frac{x^{\mathrm{T}}\mu - R}{\|x^{\mathrm{T}}\Gamma^{\frac{1}{2}}\|}\right) + (-1)^k \cdot q_k\left(\frac{R - x^{\mathrm{T}}\mu}{\|x^{\mathrm{T}}\Gamma^{\frac{1}{2}}\|}\right)\right] \cdot \|x^{\mathrm{T}}\Gamma^{\frac{1}{2}}\|^k.
\end{aligned}$$

Since ω has finite moments, we may expand the characteristic function of the random vector $\Gamma^{-\frac{1}{2}}(\omega - \mu)$ to yield

$$
\begin{aligned}
& \mathsf{E}\left[\exp\left(\boldsymbol{i}x^{\mathrm{T}}\Gamma^{-\frac{1}{2}}(\omega-\mu)\right)\right] \\
= & \sum_{m=0}^{\infty}\frac{\boldsymbol{i}^m}{m!}\cdot\mathsf{E}\left[x^{\mathrm{T}}\Gamma^{-\frac{1}{2}}(\omega-\mu)\right]^m \\
= & \sum_{m=0}^{\infty}\frac{\boldsymbol{i}^m}{m!}\cdot[q_m(0)+(-1)^m\cdot q_m(0)]\cdot\|x^{\mathrm{T}}\Gamma^{-\frac{1}{2}}\Gamma^{\frac{1}{2}}\|^m \\
= & \sum_{m=0}^{\infty}\frac{\boldsymbol{i}^m}{m!}\cdot[q_m(0)+(-1)^m\cdot q_m(0)]\cdot\|x\|^m.
\end{aligned}
$$

Let U be an arbitrary $n\times n$ orthonormal matrix. We have

$$
\begin{aligned}
& \mathsf{E}\left[\exp\left(\boldsymbol{i}x^{\mathrm{T}}U\Gamma^{-\frac{1}{2}}(\omega-\mu)\right)\right] \\
= & \sum_{m=0}^{\infty}\frac{\boldsymbol{i}^m}{m!}\cdot\mathsf{E}\left[x^{\mathrm{T}}U\Gamma^{-\frac{1}{2}}(\omega-\mu)\right]^m \\
= & \sum_{m=0}^{\infty}\frac{\boldsymbol{i}^m}{m!}\cdot[q_m(0)+(-1)^m\cdot q_m(0)]\cdot\|x^{T}U\Gamma^{-\frac{1}{2}}\Gamma^{\frac{1}{2}}\|^m \\
= & \sum_{m=0}^{\infty}\frac{\boldsymbol{i}^m}{m!}\cdot[q_m(0)+(-1)^m\cdot q_m(0)]\cdot\|x^{\mathrm{T}}U\|^m \\
= & \sum_{m=0}^{\infty}\frac{\boldsymbol{i}^m}{m!}\cdot[q_m(0)+(-1)^m\cdot q_m(0)]\cdot\|x\|^m \\
= & \mathsf{E}\left[\exp\left(\boldsymbol{i}x^{\mathrm{T}}\Gamma^{-\frac{1}{2}}(\omega-\mu)\right)\right].
\end{aligned}
$$

Because of the one-to-one correspondence between the characteristic function and the distribution, this implies that $U\Gamma^{-\frac{1}{2}}(\omega-\mu)$ has the same distribution as $\Gamma^{-\frac{1}{2}}(\omega-\mu)$. Hence ω is Q-radially distributed, with $Q=\nu\Gamma^{\frac{1}{2}}$.

Q.E.D.

4 Optimizing the Sharpe Ratio via Conic Programming

Since the Sharpe ratio plays such an instrumental role in the risk adjusted performance measures if the return rates follow the Q-radial distribution, it is important to solve the optimization model where the objective is simply the Sharpe ratio. In the case of Q-radial distribution, this essentially solves all the optimization models where the risk adjusted performance measure is any of which in the list of Theorem 3.1.

Suppose that the return rates ω satisfy $\mathsf{E}[\omega] = \mu$ and $\mathrm{Var}(\omega) = \Gamma$. Let $\mathcal{F}$ be the feasible set for the portfolio selection. Consider the following optimization model which aims at maximizing the Sharpe ratio:

$$(SRO) \qquad \max_{x \in \mathcal{F}} \frac{x^{\mathrm{T}}\mu - R}{\sqrt{x^{\mathrm{T}}\Gamma x}}. \tag{24}$$

Note that the objective function above is non-concave. Denote $\hat{R} := \max_{x \in \mathcal{F}} x^{\mathrm{T}}\mu$. Suppose that $\Gamma \succ 0$, $0 \leq R \leq \hat{R}$ and $0 \notin \mathcal{F}$. If $\mathcal{F}$ is an LMI (*Linear Matrix Inequality*) representable set, i.e., there exist symmetric matrices A_i's and B, such that

$$\mathcal{F} = \left\{ x \in \Re^n \;\middle|\; \sum_{i=1}^{n} A_i x_i \succeq B \right\},$$

then optimization problem (SRO) can be equivalently cast as a conic convex program

$$\begin{array}{lll}
(SDP) & \max\limits_{y,t} & y^{\mathrm{T}}\mu - Rt \\
& \text{s.t.} & \begin{pmatrix} 1 & y^{\mathrm{T}}\Gamma^{1/2} \\ \Gamma^{1/2}y & I \end{pmatrix} \succeq 0 \\
& & t \geq 0 \\
& & \sum\limits_{i=1}^{n} A_i y_i \succeq tB,
\end{array}$$

which is essentially an SDP problem (*Semidefinite Program*). Note that the first constraint is nothing but $1 \geq y^{\mathrm{T}}\Gamma y$ by the Schur complement relation.

The underlying methodology leading to this observation is the so-called Charnes-Cooper transformation [5]. To show that the transformation is correct, we take x to be any feasible solution for (SRO). Then

$$(y^{\mathrm{T}}, t) := \left(\frac{x^{\mathrm{T}}}{\sqrt{x^{\mathrm{T}}\Gamma x}}, \frac{1}{\sqrt{x^{\mathrm{T}}\Gamma x}} \right)$$

is a feasible solution for (SDP). Thus, $v(SRO) \leq v(SDP)$.

On the other hand, take (y^{T}, t) to be an optimal solution for (SDP). If $t > 0$, then $x = y/t$ is a feasible solution for (SRO), thus $v(SRO) \geq v(SDP)$. If $t = 0$, then for any feasible solution x of (SRO) it follows that $x(s) = x + sy$ remains feasible for (SRO) for all $s \geq 0$. Thus

$$\frac{x(s)^{\mathrm{T}}\mu - R}{\sqrt{x(s)^{\mathrm{T}}\Gamma x(s)}} \leq v(SRO).$$

Observing that

$$\lim_{s \to \infty} \frac{x(s)^{\mathrm{T}}\mu - R}{\sqrt{x(s)^{\mathrm{T}}\Gamma x(s)}} = \frac{y^{\mathrm{T}}\mu}{\sqrt{y^{\mathrm{T}}\Gamma y}} \geq y^{\mathrm{T}}\mu = v(SDP),$$

we have $v(SRO) \geq v(SDP)$. Hence, $v(SRO) = v(SDP)$, and the optimal solutions can be deduced from one to the other problem accordingly.

If $\mathcal{F}$ is SOCP representable, e.g,

$$\mathcal{F} = \{x \in \Re^n \mid a_i^{\mathrm{T}} x + b_i \geq \|C_i x + d_i\|\},$$

where $a_i \in \Re^n$, $b_i \in \Re$, $C_i \in \Re^{m_i \times n}$, and $d_i \in \Re^{m_i}$, $i = 1, ..., k$, then (SRO) can be equivalently cast as

$$\begin{array}{llll} (SOCP) & \max\limits_{y,t} & y^{\mathrm{T}}\mu - Rt \\ & \text{s.t.} & \|\Gamma^{1/2} y\| \leq 1 \\ & & t \geq 0 \\ & & a_i^{\mathrm{T}} y + b_i t \geq \|C_i y + d_i t\|, \ i = 1, ..., k. \end{array}$$

The analysis leading to this formulation is similar to the SDP case. In either cases (SRO) can be solved by, e.g., SeDuMi of Jos Sturm [32]; or, for a friendly modeling interface, one may resort to CVX of Grant *et al.* [10] and [11].

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A Proof of Proposition 2.1

(1). "$\Longrightarrow$"

The characteristic function of $U\xi$ is

$$\begin{aligned}
\mathsf{E}\left[\exp(\boldsymbol{i}t^{\mathrm{T}}U\xi)\right] &= \int_{x\in\Re^n} \exp\left(\boldsymbol{i}t^{\mathrm{T}}Ux\right) g_n(\|x\|)\mathrm{d}x \\
&= \int_{x\in\Re^n} \exp\left(\boldsymbol{i}t^{\mathrm{T}}Ux\right) g_n(\|Ux\|)\mathrm{d}(Ux) \\
&= \int_{y\in\Re^n} \exp\left(\boldsymbol{i}t^{\mathrm{T}}y\right) g_n(\|y\|)\mathrm{d}y \\
&= \mathsf{E}[\exp(\boldsymbol{i}t^{\mathrm{T}}\xi)].
\end{aligned}$$

"$\Longleftarrow$"

Since the density function is invariant under any unitary transformation, it is only related to the radius. Thus, it has a radial distribution.

(2). According to Proposition 2.1, we can always rotate the coordinates so as to ensure that the first component of the vector is the coordinate of the first axis, while the distribution is kept invariant. Therefore,

$$\begin{aligned} F(x) &:= \text{Prob}\ \{\eta \le x\} \\ &= \text{Prob}\ \left\{a^{\mathrm{T}}\xi \le x\right\} \\ &= \text{Prob}\ \left\{\|a\|\tilde{\xi}_1 \le x\right\} \\ &= \int_{\|a\|\tilde{\xi}_1 \le x} g_n(\|\tilde{\xi}\|)\mathrm{d}\tilde{\xi}. \end{aligned}$$

Consider the spherical coordinates and let

$$E_n = \{(r, \theta_1, \cdots, \theta_{n-1}) \mid r \in \Re_+,\ 0 \le \theta_1 \cdots, \theta_{n-2} \le \pi,\ 0 \le \theta_{n-1} \le 2\pi\}$$

and so

$$\begin{aligned} F(x) &= \int_{\|a\| r\cos\theta_1 \le x, (r,\theta_1,\cdots,\theta_{n-1})\in E_n} r^{n-1} g_n(r)(\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}\cdots\sin\theta_{n-2}\ \mathrm{d}r\mathrm{d}\theta_1\cdots\mathrm{d}\theta_{n-1} \\ &= S_{n-1}\int_{\|a\| r\cos\theta_1 \le x, \theta_1\in[0,\pi],\ r\in\Re_+} r^{n-1} g_n(r)(\sin\theta_1)^{n-2}\mathrm{d}r\mathrm{d}\theta_1 \\ &= S_{n-1}\int_0^{\infty}\int_{-\infty}^{\frac{x}{\|a\|}} \rho^{n-2} g_n\left(\sqrt{\rho^2+\gamma^2}\right)\mathrm{d}\gamma\mathrm{d}\rho. \end{aligned}$$

Its probability density function is given by

$$\begin{aligned} f_\eta(x) &= F'(x) \\ &= \frac{S_{n-1}}{\|a\|}\int_0^{\infty} \rho^{n-2} g_n\left(\sqrt{\rho^2+\left(\frac{x}{\|a\|}\right)^2}\right)\mathrm{d}\rho. \end{aligned}$$

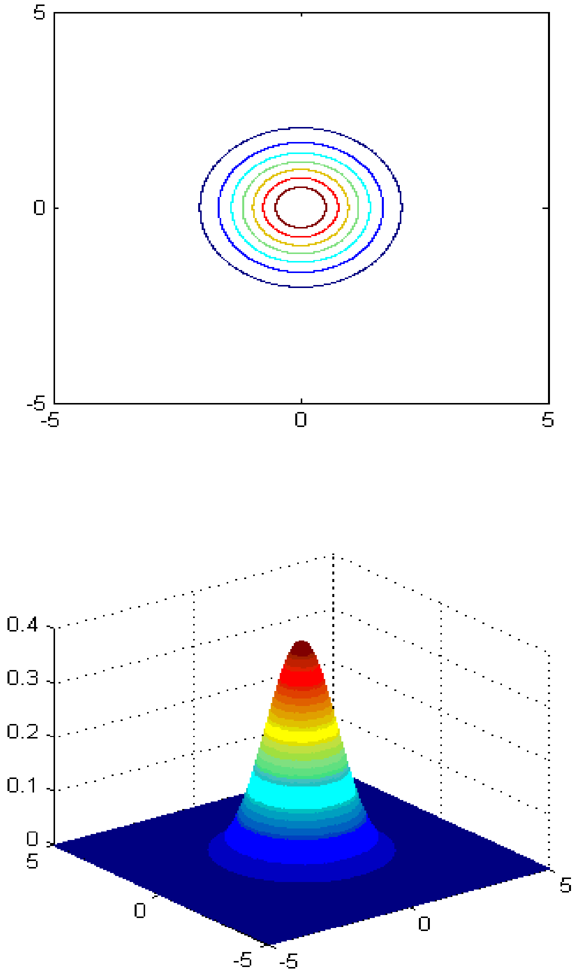

Figure 1: Contour map and the probability density function of a radial distribution