Design of Radar Receive Filters Optimized According to $L_p$-norm Based Criteria

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Abstract

This paper deals with the design of radar receive filters which optimize either the $L_p$-norm of the vector containing the filter sidelobe energies or the Inverse Signal to Noise Ratio (ISNR) under an upper-bound constraint on the previously mentioned $L_p$-norm. In both the cases, we prove that the filter design can be formulated as a convex optimization Second Order Cone Programming (SOCP) problem which can be efficiently solved with a polynomial time computational complexity resorting to interior point methods. At the analysis stage, we assess the performance of the receive filters in correspondence of different values of the parameter $p$ highlighting the performance compromises between the Integrated Sidelobe Level (ISL) and the Peak Sidelobe Level (PSL).

Index Terms

I. INTRODUCTION

The design of optimized receive filters for pulse compression radar systems is an hot research topic among the radar signal processing community since the end of sixties. To the authors’ best knowledge, the early studies can be dated back to 1967-1968 [1], [2], with reference to the IEEE journals, while to 1970 [3], [4], in the context of Russian literature. In [5], a literary survey on this interesting problem is provided together with some new contributions concerning issues related to the filter length and the choice of the design criterion. According to [5], the receiving filters proposed over the years can be classified into two main categories. The former, data independent class, does not require any prior knowledge about the surrounding environment, whereas the latter, data dependent class, depends on the assumed (possibly estimated) parameters of the environment. With reference to the former class, we quote [4], [6], and [7] where the minimum Integrated Sidelobe Level (ISL) filter [4] and the minimum Peak Sidelobe Level (PSL) filter [6], [7] are respectively designed. While the minimum ISL system shares a closed form solution, the computation of the minimum PSL filter requires the solution of a Linear Programming (LP) problem [6], [7], with reference to real optimization variables and transmitted code sequence, or the solution of a convex optimization Second Order Cone Programming (SOCP) problem [5] in the case of complex variables. Indeed, SOCP [8] problems represent a family of convex optimization programs of great interest for many signal processing applications such as beamforming [9] and target localization [10].

In [11], a filter design criterion based on the minimization of the $L_p$-norm of the vector containing the filter sidelobe energies is proposed. Interestingly, it includes as special cases either the minimum ISL or the minimum PSL criterion. However, the solution method, given in [11] for integers $p > 1$, is based on iterative numerical methods without any theoretical convergence guarantee. Starting from this last observation, in this paper, we first show that the design problem of [11] can be formulated as a convex optimization problem which belongs to the class of SOCP problems. Then, we consider a new criterion for the design of radar receive filters based on the minimization of the Inverse Signal to Noise Ratio (ISNR) under a constraint on the $L_p$-norm of the vector containing the energies of the filter sidelobes.

The paper is organized as follows. In Section II, we deal with problem formulation and show the SOCP-representation of the filter optimization problems. In Section III, we present the performance analysis of the two methods. Finally, conclusions are given in Section IV.

A. Notation

We adopt the notation of using boldface for vectors $\mathbf{a}$ and matrices $\mathbf{A}$. The $i$-th element of $\mathbf{a}$ and the $(l, m)$-th entry of $\mathbf{A}$ are respectively denoted by $a(i)$ and $A(l, m)$. The transpose operator and the conjugate transpose operator are denoted by the symbols $(\cdot)^T$ and $(\cdot)^H$ respectively. The letter $j$ represents December 1, 2010
the imaginary unit (i.e. $j = \sqrt{-1}$). $\mathbb{C}$ is the set of real and complex numbers. For any complex number $x$, we use $\Re(x)$ to denote the real part of $x$, $|x|$ and $\arg(x)$ represent the modulus and the argument of $x$. The Euclidean norm of the vector $x$ is denoted by $\|x\|$. The $L_p$-norm of a $N$-dimensional vector $x$ ($p \geq 1$) is defined as $\left[|x(1)|^p + \ldots + |x(N)|^p\right]^{\frac{1}{p}}$. Finally, $\mathbf{0}$ denotes a zero vector or matrix as long as the size of it is clear in the context.

II. PROBLEM FORMULATION AND MISMATCHED FILTER DESIGN

Assume that the transmitted signal is a coded pulse; denote by $M$ the number of subpulses and by $[s(1), \ldots, s(M)]^T$ the radar code. The waveform at the receiver end is down-converted to baseband, undergoes a subpulse matched filtering operation, and then is sampled. The vector $\mathbf{r} = [r(1), \ldots, r(P)]^T$ ($P = 2L + M$, with $L$ being a design parameter) of the samples from the range cell under test can be written as [5], [12]\footnote{See these references for more details on the system model.}

$$\mathbf{r} = \alpha_0 \mathbf{s} + \sum_{n=-N+1, n \neq 0}^{N-1} \alpha_n \mathbf{J}_n \mathbf{s} + \mathbf{n},$$

where $N = P - L (= L + M)$, $\mathbf{s} = [0, s(1), \ldots, s(M), 0]^T \in \mathbb{C}^P$ ($\mathbf{0}$ is the zero row vector of dimension $L$), $\alpha_n$’s are complex scalars accounting for the Radar Cross Sections (RCS’s) of the range cells illuminated by the radar and for the channel propagation effects (in particular $\alpha_0$ refers to the RCS of the cell under test), $\mathbf{n}$ is the vector (assumed white) containing the filtered noise samples, and $\forall n \in \{1, \ldots, N - 1\}$

$$\mathbf{J}_n(l, m) = \begin{cases} 1 & \text{if } m - l = n \\ 0 & \text{if } m - l \neq n \end{cases} \quad (l, m) \in \{1, \ldots, P\}^2$$

denotes the shift matrix. Finally $\mathbf{J}_{-n} = \mathbf{J}_n^T$.

In order to estimate $\alpha_0$, as in [5], we focus on estimators whose analytic form is

$$\hat{\alpha}_0 = \frac{\mathbf{x}^H \mathbf{r}}{\mathbf{x}^H \mathbf{s}},$$

where $\mathbf{x}$ is a suitable $P$-dimensional complex vector (receive filter) which can be designed according to several criteria. In particular, if $\mathbf{x} = \mathbf{s}$, it is the classic matched filter to the signal $\mathbf{s}$. Otherwise, it is usually referred to, in open literature, as mismatched filter or instrumental variable filter. Relevant performance metrics to optimize in the design of a receive filter are related to the energies in the sidelobes of the filter, i.e. $\frac{|\mathbf{x}^H \mathbf{J}_n \mathbf{s}|^2}{|\mathbf{x}^H \mathbf{s}|^2}$, $n = \pm 1, \ldots, \pm (N - 1)$. Specifically, the minimum ISL filter can be obtained as an optimal solution to the optimization problem

$$\min_{\mathbf{x} \in \mathbb{C}^P} \sum_{n=-N+1, n \neq 0}^{N-1} \frac{|\mathbf{x}^H \mathbf{J}_n \mathbf{s}|^2}{|\mathbf{x}^H \mathbf{s}|^2},$$
whereas the minimum PSL filter coincides with an optimal solution to the optimization problem
\[
\min_{x \in \mathbb{C}^p} \max_{n=\pm 1, \ldots, \pm(N-1)} \frac{|x^H J_n s|^2}{|x^H s|^2}.
\]
Both the ISL and PSL approaches are included in the more general problem of minimizing the \(L_p\)-norm of the vector containing the energies of the sidelobes. This mismatched filter design criterion is proposed in [11], where an iterative algorithm attempting to obtain an optimal solution to the problem is introduced. However, the iterative technique of [11] has no known convergence properties even if simulation results show its effectiveness in some analyzed scenarios.

In the following, we prove that the design problem of [11] can be formulated as a convex optimization problem which belongs to the class of SOCP problems. As a consequence, it can be solved by a handy solver, e.g., sedumi [14] or cvx [15] through interior point methods with a polynomial-time computational complexity. Moreover, interior point methods have theoretical guaranteed convergence unlike the iterative method proposed in [11] whose convergence properties seem actually unknown.

Secondly, we propose a new criterion for the design of radar receive filters based on the minimization of the Inverse Signal to Noise Ratio (ISNR), i.e.
\[
\text{ISNR} = \frac{\|x\|^2}{|x^H s|^2},
\]
under a constraint on the \(L_p\)-norm of the vector containing the energies of the filter sidelobes. It is worth pointing out that if \(p = 1\), the new criterion reduces to that considered in [5, p. 100, problem (28)] where a solution method based on the Lagrange multipliers technique is also established. Remarkably, we prove that for a general integer value of \(p\) the proposed design criterion leads to a convex optimization SOCP problem.

A. Receive Filter Minimizing the \(L_p\)-norm of the Sidelobe Energies Vector

Let us denote by \(a_n = J_n s, n = \pm 1, \ldots, \pm(N-1), a_0 = s\) and
\[
A = [a_{-N+1}, \ldots, a_{-1}, a_1, \ldots, a_{N-1}]^H \in \mathbb{C}^{(2N-2) \times P}.
\]
The receive filter minimizing the \(L_p\)-norm of the vector whose elements are the sidelobe energies is an optimal solution to the optimization problem
\[
\min_{x \in \mathbb{C}^p} \sum_{n=1}^{N-1} \left[ \left( \frac{|x^H a_n|^2}{|a_0^H x|^2} \right)^p + \left( \frac{|x^H a_n|^2}{|a_0^H x|^2} \right)^p \right],
\]
where \(p \geq 1\) is an integer. Notice that, if \(p = 1\), we obtain the minimum ISL filter, while, if \(p = +\infty\), we come up with the minimum PSL vector. Values of \(p\) between these two extremes lead to different compromises between ISL and PSL.
In order to find an optimal solution to (3), it suffices to solve the following constrained $L_{2p}$-norm minimization problem

$$
\min_{x \in \mathbb{R}^p} \sum_{n=1}^{N-1} (|a_n^H x|^2 + |a_n^H x|^2) \\
\text{s.t.} \quad R(a_0^H x) = b.
$$

(4)

where $b > 0$ is any given positive real number. Indeed, suppose that $x^*$ is an optimal solution of (4), and it can be shown straightforwardly that $e^{-j \arg(a_n^H x^*)} x^*$ is an optimal solution of (3), and $v^*(3) = \frac{v^*(4)}{b^{2p}}$, where $v^*(\cdot)$ stands for the optimal value of problem ($\cdot$). As a consequence, to solve (3), it suffices to focus on (4) with $b = 1$.

In particular, when $p = 1$, problem (4) is equivalent to

$$
\min_{x \in \mathbb{C}^p} \quad x^H A^H Ax \\
\text{s.t.} \quad R(a_0^H x) = 1.
$$

(5)

It is well-known that the problem (5) has a closed-form solution $x^* = e^{i\theta} (A^H A)^{-1} a_0 / (a_0^H (A^H A)^{-1} a_0)$, for any given $\theta$, provided that $a_0$ is such that $A^H A$ is invertible or equivalently [13] $s(1) \neq 0$ and $s(N) \neq 0$. Therefore, we herein consider $p \geq 2$.

Now, we prove that (4) can be reformulated as a convex optimization problem which falls within the class of SOCP problems [8]. Toward the aforementioned goal, we first observe that problem (4) is equivalent to

$$
\min_{x, \{t_n\}, \{\xi_n\}} \quad \sum_{n=1}^{N-1} (t_n + t_{-n}) \\
\text{s.t.} \quad R(a_0^H x) = 1 \\
|a_n^H x|^2 \leq \xi_n, \quad n = \pm 1, \ldots, \pm (N-1) \\
\xi_n^p \leq t_n, \quad n = \pm 1, \ldots, \pm (N-1) \\
x \in \mathbb{C}^p, \xi_n \geq 0, t_n \geq 0, \quad n = \pm 1, \ldots, \pm (N-1),
$$

(6)

which, letting $a_n = a_{n1} + j a_{n2}$, $n = 0, \pm 1, \ldots, \pm (N-1)$, and $x = x_1 + j x_2$, can be further recast into

$$
\min_{x_1, x_2, \{t_n\}, \{\xi_n\}} \quad \sum_{n=1}^{N-1} (t_n + t_{-n}) \\
\text{s.t.} \quad a_{n1}^T x_1 + a_{n2}^T x_2 = 1 \\
\left| \begin{bmatrix} a_{n1}^T & a_{n2}^T \\ -a_{n2}^T & a_{n1}^T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right|^2 \leq \xi_n, \quad n = \pm 1, \ldots, \pm (N-1) \\
\xi_n^p \leq t_n, \quad n = \pm 1, \ldots, \pm (N-1) \\
x_1, x_2 \in \mathbb{R}^p, \xi_n \geq 0, t_n \geq 0, \quad n = \pm 1, \ldots, \pm (N-1).
$$

(7)

The second set of constraints can be represented as standard SOCP constraints. In fact, they can be reformulated as

$$
\begin{bmatrix}
    a_{n1}^T & a_{n2}^T & 0 \\
    -a_{n2}^T & a_{n1}^T & 0 \\
    0 & 0 & \frac{1}{2} \\
    0 & 0 & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\xi_n
\end{bmatrix}
= 
\begin{bmatrix}
    0 \\
    0 \\
    \frac{1}{2} \\
    -\frac{1}{2}
\end{bmatrix}
\in \mathcal{L}^4,
$$

(8)
\( n = \pm 1, \ldots, \pm (N-1) \), where \( \mathcal{L}^4 \) stands for the Second Order Cone (SOC)

\[
\{ z \in \mathbb{R}^K | \sqrt{z(1)^2 + \cdots + z(K-1)^2} \leq z(K) \} \tag{9}
\]

specifying \( K = 4 \). We now show that also the third set of constraints of (7) is SOC-representable, namely, it can be rewritten in terms of standard SOCP constraints. For instance, suppose that \( p = 2 \). Then, it is evident that

\[
\{(\xi, t) | 0 \leq \xi \leq \sqrt{t} \} = \left\{ (\xi, t) \mid \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \xi \\ \frac{t}{2} \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \in \mathcal{L}^3, \xi \geq 0 \right\}, \tag{10}
\]

where \( \mathcal{L}^3 \) denotes the 3-dimension SOC. Thus, in the case of \( p = 2 \), the corresponding \( L_4 \)-norm minimization problem (7) is equivalent to the following standard SOCP (cf. [8, Eqn. (2.1.2), page 50])

\[
\begin{align*}
\min_{x_1, x_2, \{t_n\}, \{\xi_n\}} & \sum_{n=1}^{N-1} (t_{n-1} + t_n) \\
\text{s.t.} & \begin{bmatrix} a_{01}^T & a_{02}^T \end{bmatrix} x_1 + a_{02}^T x_2 = 1 \\
& \begin{bmatrix} a_{n1}^T & a_{n2}^T & 0 \\ -a_{n2}^T & a_{n1}^T & 0 \\ 0 & 0 & 1/2 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \xi_n \\ \frac{t_n}{2} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \in \mathcal{L}^4, \ n = \pm 1, \ldots, \pm (N-1) \\
& \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \xi_n \\ \frac{t_n}{2} \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \in \mathcal{L}^3, \ n = \pm 1, \ldots, \pm (N-1) \\
& x_1, x_2 \in \mathbb{R}^P, \xi_n \geq 0, t_n \geq 0, n = \pm 1, \ldots, \pm (N-1).
\end{align*} \tag{11}
\]

In general, consider \( p \geq 3 \) and let \( l \) be such that \( 2^l \geq p > 2^{l-1} \) with \( p \geq 3 \) (thus \( l \geq 2 \)). We illustrate that \( \{(\xi, t) \mid \xi^p \leq t, \xi \geq 0 \} \) can be represented by the intersection of not more than \( 2(l-1) \) SOCP constraints with at most \( 2(l-2) + 1 \) auxiliary variables. To this end, observe that

\[
\{(\xi, t) \mid \xi^p \leq t, \xi \geq 0 \} = \{(\xi, t) \mid \xi \leq (\xi^{2^{l-p} t})^{1/2^l}, \xi \geq 0 \}, \tag{12}
\]

and the term \( (\xi^{2^{l-p} t})^{1/2^l} \) is the geometric mean of the \( 2^l \)-dimension vector:

\[
\begin{bmatrix} \xi & \cdots & \xi & t & 1 & \cdots & 1 \end{bmatrix}^{T}. \tag{13}
\]

Applying the equivalent reformulation of the hypograph of the geometric mean of \( 2^l \) variables (cf. [8, problem 10, page 69]) to (12), it leads to at most \( 2(l-1) \) SOCP constraints with at most \( 2(l-2) + 1 \) auxiliary variables\(^2\).

\(^2\)To be more concrete, in Appendix, we explain how to obtain an equivalent SOCP-reformulation for the set \( \{(\xi, t) \mid \xi \leq (\xi^{2^{l-p} t})^{1/2^l}, \xi \geq 0 \} \), for some specific values of \( p \).
In conclusion, we have shown that (3) is equivalent to the minimization of a linear function over the intersection of SOC’s and affine sets. The worst-case computational complexity of (3) (for instance, \(p = 2\)) is in the order \(O(N^{0.5}(2N + P)^3 \log(1/\zeta))\), where \(\zeta\) is the accuracy (see [8]).

B. Receive Filter Minimizing the ISNR under a Constraint on the \(L_p\)-norm of the Sidelobe Energies Vector

The design of the radar receive filter which minimizes the ISNR, under a constraint on the \(L_p\)-norm of the vector whose elements are the filter sidelobe energies, can be formulated as

\[
\begin{align*}
\min_{x \in \mathbb{C}^P} & \quad \frac{||x||^2}{|a_0^H x|^2} \\
\text{s.t.} & \quad \sum_{n=1}^{N-1} \left( \left( \frac{|a_n^H x|^2}{|a_0^H x|^2} \right)^p + \left( \frac{|a_{-n}^H x|^2}{|a_0^H x|^2} \right)^p \right) \leq \eta
\end{align*}
\]

(14)

where \(\eta > 0\) is the maximum allowed value for the \(L_p\)-norm of the sidelobe energies. A criterion to set \(\eta\) for \(p = 1\) is given in [5]. Evidently, if \(p > 1\), (14) is feasible if and only if

\[
\eta \geq \eta_{\text{min}} = \min_{x \in \mathbb{C}^P} \sum_{n=1}^{N-1} \left( \left( \frac{|a_n^H x|^2}{|a_0^H x|^2} \right)^p + \left( \frac{|a_{-n}^H x|^2}{|a_0^H x|^2} \right)^p \right).
\]

(15)

Additionally,

\[
\eta \geq \eta_{\text{max}} = \sum_{n=1}^{N-1} \left( \left( \frac{|a_n^H a_0|^2}{||a_0||^4} \right)^p + \left( \frac{|a_{-n}^H a_0|^2}{||a_0||^4} \right)^p \right),
\]

(16)

then an optimal solution \(x^*\) to (14) is the standard matched filter, namely \(x^* = a_0 = s\), and the optimum ISNR value is \(1/||s||^2\). Finally, for \(p = \infty\), (14) is tantamount to minimizing the ISNR under a PSL constraint.

In order to proceed further, we observe that problem (14) is equivalent to

\[
\begin{align*}
\min_{x, t} & \quad t \\
\text{s.t.} & \quad ||x||^2 \leq t \\
& \quad \sum_{n=1}^{N-1} (|a_n^H x|^{2p} + |a_{-n}^H x|^{2p}) \leq \eta \\
& \quad \Re(a_0^H x) = 1 \\
& \quad x \in \mathbb{C}^P, \; t \geq 0,
\end{align*}
\]

(17)

which, in terms of real variables and exploiting the SOC-representation of \(||x||^2 \leq t\), can be recast as
the following problem

\[
\min_{x_1, x_2, \{t_n\}, \{\xi_n\}, t} \quad t
\]
\[
\begin{bmatrix}
  I & 0 & 0 \\
  0 & I & 0 \\
  0 & 0 & \frac{1}{2} \\
  0 & 0 & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  t
\end{bmatrix}
- \begin{bmatrix}
  0 \\
  0 \\
  \frac{1}{2} \\
  -\frac{1}{2}
\end{bmatrix}
\in \mathcal{L}^{2P+2}
\]
\[
\begin{bmatrix}
  a_{n1}^T & a_{n2}^T & 0 \\
  -a_{n2}^T & a_{n1}^T & 0 \\
  0 & 0 & \frac{1}{2} \\
  0 & 0 & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \xi_n
\end{bmatrix}
- \begin{bmatrix}
  0 \\
  0 \\
  \frac{1}{2} \\
  -\frac{1}{2}
\end{bmatrix}
\in \mathcal{L}^4, \quad n = \pm 1, \ldots, \pm (N - 1)
\]
\[
\xi_n^p \leq t_n, \quad n = \pm 1, \ldots, \pm (N - 1)
\]
\[
\sum_{n=1}^{N-1} (t_{-n}^p + t_n^p) \leq \eta
\]
\[
a_{01}^T x_1 + a_{02}^T x_2 = 1
\]
\[
x_1, x_2 \in \mathbb{R}^P, t \geq 0, \xi_n \geq 0, t_n \geq 0, \quad n = \pm 1, \ldots, \pm (N - 1).
\]

Using the SOCP-representation of the constraint set \(\{(\xi, t) \mid \xi^p \leq t, \xi \geq 0\}\), discussed in the previous subsection, we can recast (18) as a standard SOCP problem, namely as the minimization of a linear function over the intersection of SOC’s and affine sets. The worst-case computational complexity for (18) (for instance, \(p = 2\)) is in the order \(O(N^{0.5}(2N + P)^3 \log(1/\zeta))\), where \(\zeta\) is the accuracy (see [8]).

In particular, when \(p\) is set to two, problem (18) becomes the following SOCP.

\[
\min_{x_1, x_2, \{t_n\}, \{\xi_n\}, t} \quad t
\]
\[
\begin{bmatrix}
  I & 0 & 0 \\
  0 & I & 0 \\
  0 & 0 & \frac{1}{2} \\
  0 & 0 & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  t
\end{bmatrix}
- \begin{bmatrix}
  0 \\
  0 \\
  \frac{1}{2} \\
  -\frac{1}{2}
\end{bmatrix}
\in \mathcal{L}^{2P+2}
\]
\[
\begin{bmatrix}
  a_{n1}^T & a_{n2}^T & 0 \\
  -a_{n2}^T & a_{n1}^T & 0 \\
  0 & 0 & \frac{1}{2} \\
  0 & 0 & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \xi_n
\end{bmatrix}
- \begin{bmatrix}
  0 \\
  0 \\
  \frac{1}{2} \\
  -\frac{1}{2}
\end{bmatrix}
\in \mathcal{L}^4, \quad n = \pm 1, \ldots, \pm (N - 1)
\]
\[
\sum_{n=1}^{N-1} (t_{-n}^p + t_n^p) \leq \eta
\]
\[
a_{01}^T x_1 + a_{02}^T x_2 = 1
\]
\[
x_1, x_2 \in \mathbb{R}^P, t \geq 0, \xi_n \geq 0, t_n \geq 0, \quad n = \pm 1, \ldots, \pm (N - 1).
\]
III. PERFORMANCE ANALYSIS

In this section, we assess the performance of the receive filters introduced in the previous section in terms of output modulus when the input is the transmitted sequence, ISL, PSL and ISNR. To this end, we resort to a four-phase, length $M = 34$, code with a very low PSL equal to $-19.49$ dB, designed according to the method described in Appendix-A of [12]. Moreover, we use cvx software [15] to solve SOCP problem.

In Figure 1, we show the output modulus of the receive filter in Section II-A for $P = 68$ and for some values of $p$. In the same figure, we also plot the outputs of the minimum ISL filter, the minimum PSL filter, and the matched filter. From the plots, we can notice that the parameter $p$ rules the tradeoff between ISL and PSL of the filter output. Additionally, increasing $p$, we obtain filter responses which are closer and closer to the minimum PSL filter output (accordingly, the filter response becomes flatter and flatter). This is of course expected and coherent with the observation that, increasing $p$, the $L_p$-norm tends to the $L_\infty$ one. To be more specific, in Table I, we report the tradeoff between ISL, PSL, ISNR which results varying $p$. Indeed, increasing $p$, we get lower and lower PSL values at the price of higher and higher ISL levels. Notice also that the value of $p$ does not significantly affect the actual ISNR whose value is approximately $-14$ dB.

In Figure 2, we analyze the performance of the receive filter in Section II-B. We consider the same transmitted code and the same filter length as in Figure 1. Moreover, we keep constant the value of $p = 3$ and let $\eta$ vary between $\eta_{\text{min}}$ and $\eta_{\text{max}}$, given respectively by (15) and (16). As expected from theoretical considerations, we get filter outputs which are between those of the minimum $L_p$-norm filter of Section II-A and the matched filter. On the other hand, the closer $\eta$ to $\eta_{\text{max}}$, the nearer the devised filter output to the matched filter response. Actually, when $\eta \geq \eta_{\text{max}}$ the two outputs coincide since the matched filter becomes a feasible solution for problem (14). Notice also that increasing $\eta$ we obtain improvements in the actual SNR value (hence a smaller ISNR) while a deterioration in the ISL/PSL. To quantify this effect, we have provided Table II where the values of ISL, PSL, and ISNR in correspondence of a given $\eta$ are given. In particular, for the chosen values of the parameters, we can obtain a maximum ISNR improvement of 1.4 dB with a maximum ISL (PSL) deterioration in the order of 4.8 dB (7.6 dB).

IV. CONCLUSIONS

In this paper, we have considered the design of radar receive filters according to some criteria involving the $L_p$-norm of the vector with entries given by the filter sidelobe energies. Firstly, we have addressed the design of the receive filter which minimizes the aforementioned $L_p$-norm. Secondly, we have devised the filter minimizing the ISNR with a constraint on the $L_p$-norm of the sidelobe energies vector. In both the cases, we have proved that the design can be formulated as a convex optimization SOCP problem. At
the analysis stage, we have analyzed the performance of the considered receive systems providing filter responses and highlighting the tradeoff between ISL, PSL, and ISNR. Possible future research tracks might include the extension of the framework to the design of data-dependent filter structures.

V. APPENDIX: CONVEX REFORMULATION OF THE CONSTRAINT (12) FOR SOME VALUES OF $p$

Suppose that $p = 3$. Then $l = 2$, $2^l - p = 1$, and $p - 1 = 2$. Suppose that $\xi \geq 0$. It follows that $\xi^p \leq t$ is equivalent to $\xi \leq (\xi t)^{1/4}$. This further is equivalent to $\exists y \geq 0$ such that $y \leq \sqrt{\xi t}$ and $\xi \leq \sqrt{y}$. In other words,

$$\{(\xi, t)|0 \leq \xi \leq (t)^{1/3}\} = \{(\xi, t)|0 \leq \xi, \exists y \geq 0, \text{s.t. } y \leq \sqrt{\xi t}, \xi \leq \sqrt{y}\}. \quad (20)$$

Moreover, for $y \geq 0$,

$$y \leq \sqrt{\xi t} \Leftrightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ t \\ y \end{bmatrix} \in \mathcal{L}^3, \quad (21)$$

and

$$\xi \leq \sqrt{y} \Leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} \xi \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ 1/2 \\ -1/2 \end{bmatrix} \in \mathcal{L}^3. \quad (22)$$

Assume $p = 4$. Then $l = 2$, $2^l - p = 0$, and $p - 1 = 3$. It is verified that

$$\{(\xi, t)|0 \leq \xi \leq (t)^{1/4}\} = \{(\xi, t)|0 \leq \xi, \exists y \geq 0, \text{s.t. } y \leq \sqrt{\xi t}, \xi \leq \sqrt{y}\}. \quad (23)$$

Assume $p = 5$. Then $l = 3$, $2^l - p = 3$, and $p - 1 = 4$. It is verified that for $\xi \geq 0$, the constraint $\xi^5 \leq t$ is equivalent to $\xi \leq (\xi^3 t)^{1/8}$, which in turn is equivalent to $\exists y \geq 0, z \geq 0, w \geq 0$, such that $y \leq \sqrt{\xi^3} = \xi$, $z \leq \sqrt{\xi^5}$, $w \leq \sqrt{y z}$, and $\xi \leq \sqrt{w}$.

Assume $p = 6$. Then $l = 3$, $2^l - p = 2$, and $p - 1 = 5$. It is verified that for $\xi \geq 0$, the constraint $\xi^6 \leq t$ is equivalent to $\xi \leq (\xi^2 t)^{1/8}$, which in turn is equivalent to $\exists y \geq 0, z \geq 0, w \geq 0$, such that $y \leq \sqrt{\xi^2} = \xi$, $z \leq \sqrt{\xi^6}$, $w \leq \sqrt{y z}$, and $\xi \leq \sqrt{w}$.

Assume $p = 7$. Then $l = 3$, $2^l - p = 1$, and $p - 1 = 6$. It is verified that for $\xi \geq 0$, the constraint $\xi^7 \leq t$ is equivalent to $\xi \leq (\xi t)^{1/8}$, which in turn is equivalent to $\exists y \geq 0, z \geq 0$, such that $y \leq \sqrt{\xi t}$, $z \leq \sqrt{y}$, and $\xi \leq \sqrt{w}$.

Assume $p = 8$. Then $l = 3$, $2^l - p = 0$, and $p - 1 = 7$. It is verified that for $\xi \geq 0$, the constraint $\xi^8 \leq t$ is equivalent to $\xi \leq t^{1/8}$, which in turn is equivalent to $\exists y \geq 0, z \geq 0$, such that $y \leq \sqrt{t}$, $z \leq \sqrt{y}$, and $\xi \leq \sqrt{w}$.

Assume $p = 15$. Then $l = 4$, $2^l - p = 1$, and $p - 1 = 14$. It is verified that for $\xi \geq 0$, the constraint $\xi^{15} \leq t$ is equivalent to $\xi \leq (\xi t)^{1/16}$, which in turn is equivalent to $\exists y \geq 0, z \geq 0, w \geq 0$, such that $y \leq \sqrt{\xi t}$, $z \leq \sqrt{y}$, $w \leq \sqrt{z}$ and $\xi \leq \sqrt{w}$.
Assume $p = 16$. Then $l = 4$, $2^4 - p = 0$, and $p - 1 = 15$. It is verified that for $\xi \geq 0$, the constraint $\xi^{16} \leq t$ is equivalent to $\xi \leq t^{1/16}$, which in turn is equivalent to $\exists y \geq 0, z \geq 0, w \geq 0$, such that $y \leq \sqrt{t}$, $z \leq \sqrt{y}$, $w \leq \sqrt{z}$ and $\xi \leq \sqrt{w}$.

In general, other values of $p$ can be dealt with the same reformulation procedure and the resulting optimization problem is a standard SOCP.

REFERENCES

Figure 1: Filter Output modulus versus the tap number. Minimum ISL filter (plus-dashed green curve); minimum PSL filter (circle-solid black curve); matched filter (cross-dashed blue curve); minimum $L_p$-norm filter (3) with $p \in \{2, 3, 10, 17, 30\}$ (solid red curves).

Figure 2: Output modulus of the minimum ISNR filter (14) versus the tap number for $p = 3$ and $\eta \in \{0.006, 0.099, 0.0139, 0.0179, 0.0219, 0.0259, 0.0280\}$; $\eta = \eta_{\text{max}} = 0.0280$ matched filter (plus-solid blue curve); $\eta = \eta_{\text{min}} = 0.006$ minimum $L_p$ filter (3) (circle-dashed green curve); minimum ISNR filter (14) for other values of $\eta$ (dot-dashed red curves).

Table I: ISL, PSL and ISNR in dB for the minimum $L_p$-norm filter (3), $P = 68$ and $p \in \{1, 2, 3, 10, 17, 30, +\infty\}$.

<table>
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<th>$p$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>10</th>
<th>17</th>
<th>30</th>
<th>$+\infty$</th>
</tr>
</thead>
</table>

Table II: ISL, PSL and ISNR in dB for the minimum ISNR filter (14), $P = 68$, $p = 3$, $\eta \in \{\eta_{\text{min}} = 0.006, 0.010, 0.014, 0.018, 0.022, 0.026, \eta_{\text{max}} = 0.028\}$. The last column refers to the matched filter.

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>0.006</th>
<th>0.001</th>
<th>0.014</th>
<th>0.018</th>
<th>0.022</th>
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<td>-6.90</td>
<td>-6.11</td>
<td>-5.47</td>
<td>-4.92</td>
<td>-4.70</td>
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