The Price of Isolation: an Integrated Study of System Inefficiencies

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Abstract

We present in this paper an integrated study on the selfish behavior of the players participating in the system (modeled by a noncooperative game) *and* the myopic attitudes of the players (modeled by the 'rolling-horizon' type of strategies in the dynamic decision-making process), and introduce a solution termed *myopic Nash equilibrium* to characterize the equilibrium state when all the players are selfish and myopic. To characterize the loss of system efficiencies caused by such behaviors, we use a (worst possible) ratio between the total social value at the worst myopic Nash equilibrium and the optimal total social value, which is termed the *Price of Isolation* (PoI) in this paper. We consider some specific game models to bound this ratio. We first consider an extension of the static transportation game, with K players competing on a given set of routes to get their jobs done at minimum cost, over T stages. If the unit cost on each route in different stages are all affine linear in the total demand and the unit cost of the previous stage, then it is shown that the PoI of this game is precisely 4 as $T \to \infty$. We then turn to dynamic profit maximization games, including generalized settings of the Cournot and/or Bertrand competition games, with K producers competing to sell their products in the market over T stages. We prove that the PoI is in the order of $\frac{1}{KT}$ if the commodities are substitute in the Cournot game, and are complementary in the Bertrand game, when the unit costs of the resources used to manufacture the products and the selling prices of the finished products are assumed to be affine linear functions. Interpretations of these results are also discussed.

Keywords: noncooperative game, Nash equilibrium, price of isolation, dynamic optimization.

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1 Introduction

One of the most striking findings in the theory of economics is that the 'rational' behavior of each individual can collectively result in a much lowered overall performance of the entire system, eventually undermining the well-beings of all the individuals involved. This phenomenon is pervasive in social, political, economical, and even biological systems, and a suitable terrain for its study can be found in non-cooperative game theory. For instance, the notion of *Prisoner's Dilemma* is devised to illustrate precisely this point. Selfish behavior can hurt the self whose interest it sets out to promote in the first place. The wisdom contained in the previous statement may be profound; however, as such it appears to be no more than a pure philosophical thought. Selfish behavior is called rational for its own good reasons, not least because it is natural (and fair too in some sense), but also because it requires minimum communication cost, hence is attractive as a solution concept. Therefore, the subsequent inquiries, from OR/MS point of view, should go beyond any moral judgment. Rather, the issue becomes how to understand and quantify the loss of system efficiencies.

The seminal work of Koutsoupias and Papadimitriou [15] is a milestone along this direction. Essentially, the authors introduced the notion of *Price of Anarchy* (PoA), which is defined as the ratio between the social value of the worst Nash equilibrium solution and the best possible social value as a measurement for the degree of damage caused by the selfish behavior of the individuals. Interestingly, the first few pieces of news were not bad for the selfish (or anarchic) solutions. In fact, the traffic and routing game models (also known as the Wardrop model) were among the very first to be studied under this light, where infinite number of players each controlling infinitesimal amount of flow compete to use a network. When the latency function on each link is polynomial with non-negative coefficients and the degree less than or equal to p, Roughgarden [36, 37] showed that the PoA in that case is upper bounded by $\frac{(p+1)^{(p+1)/p}}{(p+1)^{(p+1)/p}-1} = \Theta(\frac{p}{\ln p})$, and this bound is independent of the network topology. When the latency functions are restricted to be affine linear, the PoA will be no more than $\frac{4}{3}$. Furthermore, Roughgarden and Tardos [38] established a so-called bicriteria bound to estimate the inefficiency of equilibria. The PoA bounds in these studies are reasonably close to 1, suggesting that selfish routing is simple and straightforward to implement, at a moderate cost of system efficiency loss.

Indeed there has been an intensive research on the PoA for various extensions of the traffic models recently. One extension, e.g., considers finitely many players in the game and assumes that each of them controls a strictly positive (as opposed to negligible) amount of traffic. If the flow is unsplittable, Mavronicolas and Spirakis [27] considered the mixed Nash equilibrium of the transportation game with a parallel network and K players, and proved that the expected PoA is upper bounded by $O(\ln K / \ln\ln K)$. For the general network, assuming the cost function on each link is affine linear, Awerbuch, Azar, Epstein [3], and Christodoulou and Koutsoupias [8] showed that the PoA is upper bounded by $\frac{5}{2}$. Gairing, Monien and Tiemann [16] considered the player specific latency functions and used some parameters characterizing the asymmetry among the players to bound the PoA. Later, Aland *et al.* [1] established the relationship between generalized golden ratio numbers and the PoA for both of unweighted and weighted congestion games with polynomial cost functions and improved the bounds in [8]. If the flow is splittable, Dumauf and Gairing [13] used the notion of the Wardrop equilibrium to obtain upper and lower bounds on the PoA provided that the cost functions are polynomial. Based on a geometric approach proposed by Correa, Schulz and Stier-Moses [10], Cominetti, Correa and Stier-Moses [9] obtained an upper bound for the PoA in this setting if the cost function satisfies some convexity conditions. Recent results on the PoA for cost minimization games can be found in Perakis [33] and Roughgarden [39].

The landscape changes dramatically when the study extends to the competition among the producers in the market, including the Cournot and Bertrand competition games ([12, 5]). In the Cournot game, players can choose their amounts of supply which affect the selling prices to maximize their own profits; in the Bertrand game, players are able to determine their selling prices which affect the demand quantity. There is a rich literature for both of these models. For the Cournot competition, Guo and Yang [18] used social surplus function as a benchmark and derived bounds on the PoA which are dependent on the market shares, demand and the number of players. Immorlica, Markakis and Pilouras [23] studied coalition formation in a dynamic setting of Cournot oligopolies and proved that the PoA under their notion of stability is bounded by $\Theta(1/K^{2/5})$, where K is the number of players. Kluberg and Perakis [24] extended the classic Cournot model and bounded the PoA by means of the market power parameters and the number of players and products. He, Wang and Zhang [21] considered competitions both for the sales and for the resources. They showed that if the unit costs and the unit prices are affine linear in the demand and supply, then the PoA is in the order of $O(1/K)$. For the Bertrand competition, Chawla and Roughgarden [6], and Chawla and Niu [7] extended the game to network settings and obtained bounds on the PoA; for more information, one is referred to Chapter 22 of [31].

Other from selfish behavior, myopic (or greedy) attitude also typically contributes to the system efficiency loss, in a dynamic decision process. There are a variety of models in the literature to characterize the situation. For instance, Sandal and Steinshamn [40] considered a harvesting game in a dynamic framework of Cournot competition and discussed several cases where losses occur due to the myopic behavior (cf. [26, 25] and the references therein). The so-called effort games in certain dynamic framework were studied as well; e.g. Bachrach, Zuckerman and Rosenschein [4] estimated the impact of myopia in effort games and proved some bounds for the measure introduced in the paper. Along a different line, the myopia issue is well studied in competitive analysis of online algorithms. Competitive analysis is to compare the relative performance of an online algorithm and the optimal offline algorithm, and the notion of *competitive ratio* is introduced by Sleator and Tarjan [41] to quantify the performance of the online algorithm. Since then the notion of competitive ratio has become a standard in analyzing the quality of online algorithms. The context, however, is entirely different from ours in this paper. Here we study the efficiency loss due to the greedy attitude, while the competitive analysis is mainly to measure the efficiency loss due to the lack of information. Closely related to our context is the work of Harks and Végh ([19]), where they investigated the selfish routing game with online demand and assumed that the game tends to a new Nash equilibrium whenever the new demand arises. They found that for nonatomic players and affine latency functions, the competitive ratio in this case is at most $\frac{4K}{K+2}$, where K denotes the number of commodities. Later, Harks, Heinz and Pfetsch [20] considered online multicommodity routing problems in networks, in which commodities had to be routed sequentially. It was shown that the competitive ratio of the greedy algorithm is $\frac{4K^2}{(1+K)^2}$ if the flow is splitable and the cost is affine. Farzad, Olver and Vetta [14] studied another variant of this model. They considered the players to be prioritized on links and proved that if the flow is splitable, then the worst-case PoA is exactly $(d+1)^{d+1}$ for polynomial latency functions of degree d (hence 4 for the linear cost functions).

Our Contribution

The aim of this paper is to study the *combined* impact of *selfish* behavior in noncooperative game and *myopic* (or *greedy*) attitude in dynamic decision making process in an integrated framework. In a literal sense, selfishness is a form of isolation (disconnectedness) in space, and myopia is a form of isolation (disconnectedness) in time. These two phenomena are intrinsically related; however, they are not identical (or symmetric) in the technical sense, as we shall see later in the analysis. We introduce the notion of *myopic Nash equilibrium* to characterize an equilibrium in the dynamic game, where each player in the game only focuses on the current status and optimizes his/her current utility value. If at any stage, given all the information about the previous stages, no player can improve his/her current status through changing his/her own current decision, then we say that the dynamic game attains a myopic Nash equilibrium. For ease of referencing, a new measure for the efficiency-loss of the myopic Nash equilibrium will be introduced and called the *Price of Isolation* (PoI for brevity). Specifically, for a game with multiple players played over multiple stages, suppose that the myopic Nash equilibria exist, then the PoI is defined to be the ratio between the *worst* total social value of the myopic Nash solutions and the optimal total social value. Note that when there are multiple myopic Nash equilibria in a specific game, we choose the worst one as a benchmark to characterize the possible largest loss of efficiency caused by *isolation*. One may still choose the *best* one as a benchmark to define a new measurement, corresponding to the notion of the Price of Stability. In a similar vein, we have a few similar measurements, including the price of anarchy/stability/isolation, together with the price of myopia which will be introduced in the subsequent section. The contexts of these ratios are shown in the following table:

Table 1: The Relationship of the Measurements

Framework	Solution Concept	Measurement
Noncooperative Game	the *Worst* Nash Equilibrium	Price of Anarchy
Noncooperative Game	the *Best* Nash Equilibrium	Price of Stability
Dynamic Decision Problem	the Greedy Solution	Price of Myopia
Dynamic Noncooperative Game	the Myopic Nash Equilibrium	Price of Isolation

To analyze the afore-introduced notion of PoI, we shall investigate several quite general forms of dynamic competitive game models, taking the splittable transportation game and economic competition games as the special cases.

The paper is organized as follows. As a start, in Section 2 we first focus on a pure dynamic transportation decision problem to study the inefficiency caused by myopia, introducing a measurement termed the *Price of Myopia* (PoM), which is effectively equivalent to the competitive ratio of the greedy online algorithm. In particular, we consider a dynamic process over T stages, whereby in each stage a prescribed throughput must be achieved. Assume that the unit cost on each link is affine linear in the workload on the link and the unit cost from the previous stage. The objective is to achieve the total throughput at minimum cost. In that case the PoM for the transportation problem is shown to be equal to 4 when T tends to infinity. Then, in Section 3 we extend the transportation problem to a multi-stage game framework involving a number of players. The existence of a *myopic Nash equilibrium* is proven, and a bound for the PoI is presented. Similar to the transportation problem case, the PoI for the transportation game is shown to be equal to 4 as T tends to infinity. In Section 4.1, an extended dynamic model of Cournot production competition is considered. Specifically, suppose that there are K noncooperative players competing for a set of shared resources, and competing to sell the products in the market to maximize their profit, over T stages. If the selling prices and the resource costs (respectively) at each stage are dependent on the supply and demand (respectively) in an affine linear fashion, then the PoI is in the magnitude of $\frac{1}{KT}$. The conclusion is extended to the Bertrand game in Section 4.2. Since the underlying assumptions are fairly simple and standard, the results can be viewed as a clear warning signal about the fast deterioration of the social value when many selfish and greedy players struggle to maximize their profits over an extended period of time. While the selfish and myopic decisions may be convenient, it really pays to be cooperative and forward looking in that case.

2 The Price of Myopia for a Dynamic Transportation Problem

2.1 The PoM for a Class of Dynamic Decision Problems

Consider a dynamic decision making process spanning over a period of stages, where the state in one stage depends on the action of the current stage and the state of the previous stage. We are interested in the two extremes of possible decision policies: the greedy policy and the dynamically optimal policy. To compare the difference of the outcomes, we introduce a quantity, to be called the *price of myopia* (PoM), which is defined as the ratio between the value of the myopic (or greedy) solution and the value of the dynamically optimal solution. Obviously this ratio will vary from one problem instance to the other. Following the same practice as for the definition of the worst-case approximation ratios, we shall call the PoM for a class of dynamic decision problem instances as the worst PoM in the instance belonging to this class. In other words, for a given class of dynamic decision problems $\mathscr{P}$, if the objective in the problem is nonnegative and is minimization, then the PoM for $\mathscr{P}$ is defined as:

$$\text{PoM}_{\mathscr{P}} = \sup_{P \in \mathscr{P}} \left\{ \frac{\text{the value of the greedy solution for } P}{\text{the value of optimal dynamic solution for } P} \right\},$$

which is always in the interval $[1, \infty]$: the larger value the heavier system efficiency loss. If the objective is nonnegative and is maximization, then

$$\text{PoM}_{\mathscr{P}} = \inf_{P \in \mathscr{P}} \left\{ \frac{\text{the value of the greedy solution for } P}{\text{the value of optimal dynamic solution for } P} \right\},$$

which is always in the interval $[0, 1]$: the smaller value the heavier system efficiency loss. Note that in many cases, PoM can be 0 (or ∞). Consider for instance the textbook example of dynamic fishing problem. Suppose the fish population in a pond is Q, and one may decide to take out half of the fish to sale or let the fish regenerate: in the first case the fish population in the next stage will regenerate back to Q and the sales profit will be λQ, while in the second case the fish population will be $2Q$ in the next stage. In this simple case, we assume that the initial fish population Q_0, the profit rate $\lambda > 0$, and the number of stages T are the parameters (or data) of the problem in class $\mathscr{P}$. Suppose that the objective is to maximize the total profit (ignoring the time discounting factor). Then, the PoM is at least $\frac{T}{2^T}$ for a given problem instance, and so $\text{PoM}_{\mathscr{P}}$ is $\frac{T}{2^T}$ if T is regarded as a fixed constant for all instances in $\mathscr{P}$, and $\text{PoM}_{\mathscr{P}}$ is 0 if the parameter T is a part of the input parameter in the problem class $\mathscr{P}$.

As we noted in Section 1, the numerical value of PoM is essentially the same as the competitive ratio of a greedy algorithm in the competitive analysis of online algorithms. The contexts however are completely different: in ours, the ratio is proposed to measure the loss of efficiency caused by greediness, while and competitive ratio is to capture the value of information in the online setting.

2.2 Bounding the PoM in a Dynamic Transportation Problem

In this subsection, we shall consider a nontrivial example to illustrate the afore introduced notion of PoM. Consider a transportation model involving T decision stages. In each stage, we are required to transport a given amount of commodity through a given network with a prescribed OD pair. Suppose that the commodity is splittable. We assume that the arcs[1] in the network are fixed, while the formation of the network may depend on the decision epoch t. Specifically, for stage t, denote the given network to be $G_t = (V_t, L; A_t)$, with the set of nodes V_t, the set of arcs L, and the node-to-arc incidence matrix[2] A_t. We assume multiple parallel links between nodes are allowed but no self-loop exists. Furthermore, we assume that $|V_t| = n_t$, $|L| = m$ and $A_t \in \mathbb{R}^{n_t \times m}$.

We denote the required origin-destination (OD) pair in stage t to be $\{s_t, d_t\}$. Let r_t denote the amount of commodity that needs to be transported in stage t. The transportation plan in stage t will be given by a vector $f_t \in \mathbb{R}^{m_t}$, whose components indicate the amount of the flows on the links. Let $f = (f_1^{\mathrm{T}}, \ldots, f_T^{\mathrm{T}})^{\mathrm{T}}$ denote the whole transportation plan consisting of the plans over the entire period of T stages. Clearly, a feasible plan is given by the constraints $A_t f_t = r_t \delta_{s_t} - r_t \delta_{d_t}$, where the notation δ_i signifies the unit vector in $\mathbb{R}^{n_t}$ whose i-th component is one while all others are zero.

Moreover, a transportation cost will be incurred for a flow on each link. We assume that the costs are dependent on the current flow as well as the cost of the previous stage. Specifically, let us denote the *unit* cost for the flow on link l in stage t to be a function $c_{lt}(f_{lt}, c_{l,t-1})$, where f_{lt} is the flow on link l in stage t and $c_{l,t-1}$ is the unit cost on link l in stage $t-1$. Therefore, the data (G, r, c) specifies an instance of the dynamic decision problem that is of interest to us in this section. Indeed, given the costs in previous stages, the transportation cost in stage t is defined as:

$$C_t = \sum_{l \in L} f_{lt} c_{lt}(f_{lt}, c_{l,t-1}).$$

Naturally, the total cost over the entire period considered is a simple summation: $\mathrm{TC}(f) = \sum_{t=1}^{T} C_t(f)$. To quantify the impact of the myopic attitude, we shall investigate two special transportation plans: the myopic one denoted by f^M, in which for all t, f_t^M minimizes the cost C_t in stage t given the costs in previous stages; the optimal one denoted by f^*, which minimizes the total cost TC over all feasible solutions. The so-called *Price of Myopia* (PoM) is now:

$$\mathrm{PoM} = \frac{\mathrm{TC}(f^M)}{\mathrm{TC}(f^*)}.$$

We shall consider the case where the unit cost function is affine linear in the current flow and the

[1] The terms *link* and *arc* are used interchangeably in this paper.

[2] Each row of A_t represents a node and each column of A_t represents an arc. For an arc connecting node i to node j, the corresponding column in A_t will have all 0 elements except for the i-th element, where it is $+1$, and the j-th element, where it is -1.

unit cost of the previous stage; that is, the unit cost function on link l is $c_{lt}(f_{lt}, c_{l,t-1}) = a_l f_{lt} + b_{lt} + \alpha_{t-1} c_{l,t-1}$, where $a_l, b_{lt} \geq 0$ and $c_{l0} = 0$ for all l and t. The parameter α_{t-1} represents the impact of c_{t-1} on c_t. We further assume that $0 \leq \alpha_t \leq 1$ for all t. Based on the dynamic recursion formula of the costs, it can be deduced that $c_{lt} = a_l f_{lt} + b_{lt} + \sum_{\tau=1}^{t-1} (\Pi_{h=\tau}^{t-1} \alpha_h)(a_l f_{l\tau} + b_{l\tau})$. Denote

$$L_\alpha := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \alpha_1 & 1 & 0 & \cdots & 0 \\ \alpha_1 \alpha_2 & \alpha_2 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Pi_{h=1}^{T-1} \alpha_h & \Pi_{h=2}^{T-1} \alpha_h & \Pi_{h=3}^{T-1} \alpha_h & \cdots & 1 \end{pmatrix} \in \mathbb{R}^{T \times T}.$$

Then the total transportation cost can be written in matrix notations:

$$\text{TC}(f) = f^{\text{T}} (L_\alpha \otimes \text{Diag}(a)) f + f^{\text{T}} (L_\alpha \otimes I_m) b,$$

where '$\otimes$' stands for the Kronecker product between two matrices, $\text{Diag}(a)$ is the diagonal matrix whose l-th diagonal entries are a_l, for $l = 1, ..., m$, and b is the long vector $(b_{11}, \ldots, b_{m1}, \ldots, b_{1T}, \ldots, b_{mT})^{\text{T}}$. Therefore the optimal transportation plan f^* should be the solution of the following optimization problem:

$$\begin{array}{lll} (P^*) & \min_f & f^{\text{T}} (L_\alpha \otimes \text{Diag}(a)) f + f^{\text{T}} (L_\alpha \otimes I_m) b \\ & \text{s.t.} & A_t f_t = r_t \delta_{s_t} - r_t \delta_{d_t}, \ \text{for } t = 1, \ldots, T, \\ & & f \geq 0. \end{array}$$

On the other hand, for the myopic transportation plan, f_t^M is the solution of the following optimization problem in stage t:

$$\begin{array}{lll} (P_t^M) & \min_{f_t} & \sum_{l \in L} (a_l f_{lt} + b_{lt} + \alpha_{t-1} c_{l,t-1}^M) f_{lt} \\ & \text{s.t.} & A_t f_t = r_t \delta_{s_t} - r_t \delta_{d_t}, \ f_t \geq 0. \end{array}$$

Here $c_{l,t-1}^M$ is the corresponding cost in stage $t-1$ on link l in the myopic plan. Since $a_l \geq 0$, the above problem for each stage is a convex quadratic program and hence the existence and the uniqueness (in the sense of costs on the links) of the myopic solution are implied. Regarding the cost in the solution, we have the following result:

Lemma 2.1 *In the myopic solution, suppose that we change the plan in stage t from f_t^M to any other feasible plan f_t, while the plans in all other stages remain unchanged. Then, we have*

$$C_t(f_t, c_{t-1}^M) - C_t(f_t^M, c_{t-1}^M) \geq \sum_{l \in L} a_l (f_{lt} - f_{lt}^M)^2.$$

Proof. Due to the properties of convex quadratic programming,

$$\begin{aligned} & C_t(f_t, c_{t-1}^M) - C_t(f_t^M, c_{t-1}^M) - \sum_{l \in L} a_l (f_{lt} - f_{lt}^M)^2 \\ = & \sum_{l \in L} \left[(a_l f_{lt} + b_{lt} + \alpha_{t-1} c_{l,t-1}) f_{lt} - \left(a_l f_{lt}^M + b_{lt} + \alpha_{t-1} c_{l,t-1}\right) f_{lt}^M - a_l (f_{lt} - f_{lt}^M)^2 \right] \\ = & \sum_{l \in L} \left(2 a_l f_{lt}^M + b_{lt} + \alpha_{t-1} c_{l,t-1}\right) (f_{lt} - f_{lt}^M) \geq 0. \end{aligned}$$

Note that the vector $2\,\mathrm{Diag}(a)\, f_t^M + b_t + \alpha_{t-1} \cdot c_{t-1}$ is the derivative of $C_t(\,\cdot\,, c_{t-1}^M)$ at the minimum point f_t^M. □

Following a similar argument as in Roughgarden [39] (with some necessary modifications), we get an upper bound for the price of myopia:

Theorem 2.2 *In the dynamic transportation decision problem, suppose that the unit cost on each link is affine linear in the total flow value and the previous cost on the link, then the price of myopia is upper bounded by* 4*.*

Proof. According to Lemma 2.1, to get an upper bound for the price of myopia it suffices to find a pair (λ, μ) with $\lambda > 0$ and $0 < \mu < 1$, such that

$$\sum_{t=1}^{T} \left[C_t(f_t^*, c_{t-1}^M) - \sum_{l \in L} a_l (f_{lt}^* - f_{lt}^M)^2 \right] \leq \lambda\, \mathrm{TC}(f^*) + \mu\, \mathrm{TC}(f^M). \tag{1}$$

Note that if (1) holds, then by Lemma 2.1 we have

$$\begin{aligned} \mathrm{TC}(f^M) &= \sum_{t=1}^{T} C_t(f_t^M, c_{t-1}^M) \leq \sum_{t=1}^{T} \left[C_t(f_t^*, c_{t-1}^M) - \sum_{l \in L} a_l (f_{lt}^* - f_{lt}^M)^2 \right] \\ &\leq \lambda\, \mathrm{TC}(f^*) + \mu\, \mathrm{TC}(f^M). \end{aligned}$$

Thus the price of myopia can be bounded as:

$$\mathrm{PoM} = \frac{\mathrm{TC}(f^M)}{\mathrm{TC}(f^*)} \leq \frac{\lambda}{1 - \mu}.$$

Let us now turn to the search of (λ, μ) to satisfy (1). Substituting the expression of the cost functions in (1), the intended inequality becomes:

$$
\begin{aligned}
& \sum_{t=1}^{T}\sum_{l\in L}\left[(a_l f_{lt}^* + b_{lt} + \alpha_{t-1}c_{l,t-1}^M) f_l^* - a_l(f_{lt}^* - f_{lt}^M)^2\right] \\
= & \sum_{t=1}^{T}\sum_{l\in L}\left[(2a_l f_{lt}^M + b_{lt} + \alpha_{t-1}c_{l,t-1}^M) f_l^* - a_l(f_{lt}^M)^2\right] \\
= & (f^*)^{\mathrm{T}}\left((I_T + L_\alpha)\otimes \mathrm{Diag}(a)\right) f^M + (f^*)^{\mathrm{T}}(L_\alpha \otimes I_m)b - (f^M)^{\mathrm{T}}\left(I_T \otimes \mathrm{Diag}(a)\right) f^M \\
\leq & \lambda\left[(f^*)^{\mathrm{T}}(L_\alpha \otimes \mathrm{Diag}(a)) f^* + (f^*)^{\mathrm{T}}(L_\alpha \otimes I_m)b\right] + \mu\left[(f^M)^{\mathrm{T}}(L_\alpha \otimes \mathrm{Diag}(a)) f^M + (f^M)^{\mathrm{T}}(L_\alpha \otimes I_m)b\right],
\end{aligned}
$$

where I_T is the T by T identity matrix. Regrouping the terms, the above is equivalent to

$$
\begin{aligned}
& \lambda(f^*)^{\mathrm{T}}(L_\alpha \otimes \mathrm{Diag}(a)) f^* + (f^M)^{\mathrm{T}}((I_T + \mu L_\alpha)\otimes \mathrm{Diag}(a)) f^M \\
& \quad -(f^*)^{\mathrm{T}}\left((I_T + L_\alpha)\otimes \mathrm{Diag}(a)\right) f^M + \left[(\lambda - 1) f^* + \mu f^M\right]^{\mathrm{T}}(L_\alpha \otimes I_m) b \geq 0. \qquad (2)
\end{aligned}
$$

By requiring $\lambda \geq 1, \mu \geq 0$, the linear part of the left hand side of (2) is obviously nonnegative. It will be sufficient to ensure that the quadratic term is also nonnegative. Denote $f_l = (f_{l1}, f_{l2}, \ldots, f_{lT})^{\mathrm{T}}$, and we have

$$
\begin{aligned}
& \lambda(f^*)^{\mathrm{T}}(L_\alpha \otimes \mathrm{Diag}(a)) f^* + (f^M)^{\mathrm{T}}((I_T + \mu L_\alpha)\otimes \mathrm{Diag}(a)) f^M - (f^*)^{\mathrm{T}}\left((I_T + L_\alpha)\otimes \mathrm{Diag}(a)\right) f^M \\
= & \sum_{l\in L} a_l\left[\lambda(f_l^*)^{\mathrm{T}} L_\alpha f_l^* + (f_l^M)^{\mathrm{T}}(I_T + \mu L_\alpha) f_l^M - (f_l^*)^{\mathrm{T}}(I_T + L_\alpha) f_l^M\right].
\end{aligned}
$$

Note that the above is a summation over index l. To ensure (2), we need only to establish the inequality for each link, namely,

$$
\lambda(f_l^*)^{\mathrm{T}} L_\alpha f_l^* + (f_l^M)^{\mathrm{T}}(I_T + \mu L_\alpha) f_l^M - (f_l^*)^{\mathrm{T}}(I_T + L_\alpha) f_l^M \geq 0.
$$

Since $f_l^* \geq 0$ and $f_l^M \geq 0$, we have $(f_l^*)^{\mathrm{T}} I_T f_l^* \leq (f_l^*)^{\mathrm{T}} L_\alpha^{\mathrm{T}} f_l^*$ and

$$
(f_l^y)^{\mathrm{T}} I_T f_l^y \geq \frac{1}{T+1}\left((f_l^y)^{\mathrm{T}}(I_T + E_T) f_l^y\right) \geq \frac{2}{T+1}(f_l^y)^{\mathrm{T}} L_\alpha f_l^y.
$$

It is sufficient to show that

$$
\lambda(f_l^*)^{\mathrm{T}} L_\alpha f_l^* + \left(\mu + \frac{2}{T+1}\right)(f_l^M)^{\mathrm{T}} L_\alpha f_l^M - (f_l^*)^{\mathrm{T}}(L_\alpha + L_\alpha^{\mathrm{T}}) f_l^M \geq 0. \qquad (3)
$$

Denote

$$
\Lambda_\alpha := \begin{pmatrix} \lambda(L_\alpha + L_\alpha^{\mathrm{T}}) & -(L_\alpha + L_\alpha^{\mathrm{T}}) \\ -(L_\alpha + L_\alpha^{\mathrm{T}}) & (\mu + \frac{2}{T+1})(L_\alpha + L_\alpha^{\mathrm{T}}) \end{pmatrix} = \begin{pmatrix} \lambda & -1 \\ -1 & \mu + \frac{2}{T+1} \end{pmatrix} \otimes (L_\alpha + L_\alpha^{\mathrm{T}}).
$$

Clearly, (3) is equivalent to $\begin{pmatrix} f^* \\ f^M \end{pmatrix}^{\mathrm{T}} \Lambda_\alpha \begin{pmatrix} f^* \\ f^M \end{pmatrix} \geq 0$. Observe that

$$L_\alpha^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -\alpha_1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\alpha_2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -\alpha_{T-1} & 1 \end{pmatrix}.$$

Therefore $L_\alpha^{-1} + (L_\alpha^{-1})^{\mathrm{T}}$ is positive semidefinite when $0 \leq \alpha_t \leq 1$ for $1 \leq t \leq T-1$. So L_α^{-1} (as well as L_α) is monotone. Thus, if $\lambda(\mu + \frac{2}{T+1}) \geq 1$, Λ_α will be a Kronecker product of two positive semidefinite matrices and hence positive semidefinite itself (cf. [22]), and then (3) holds for all f^* and f^M. Thus, the required conditions boils down to choosing (λ, μ) such that

$$\mu + \frac{2}{T+1} - \frac{1}{\lambda} \geq 0 \text{ with } \lambda \geq 1 \text{ and } \mu \geq 0.$$

While satisfying the above relations, we shall minimize $\frac{\lambda}{1-\mu}$, leading to the choice $\lambda = \frac{2T+2}{T+3}$ and $\mu = \frac{T-1}{2T+2}$. Summarizing, we have shown

$$\text{PoM} \leq \frac{(2T+2)/(T+3)}{1-(T-1)/(2T+2)} = \frac{4(T+1)^2}{(T+3)^2} < 4.$$

The theorem is proven. □

It is interesting to note that the above bound is asymptotically tight, as the following example shows, which is an adapted form of the example in Theorem 3.7 of [14] by Farzad, Olver and Vetta.

Example 2.3 Consider the network shown in Figure 1. The cost on the link from A_l to A_{l-1} is always 0, while the cost on the link from A_l to B at stage t is given by a function $c_{lt}(f_{lt}) = \frac{f_{lt}}{l} + c_{l,t-1}$, with $c_{l0} = 0$ for all $l = 1, \ldots, n$. In other words, $a_l = \frac{1}{l}, b_{lt} = 0$ and $\alpha_t = 1$ in this example.

Now we consider a dynamic transportation problem through this network over $T = n \times m$ stages, where m is a positive integer. Suppose that all the stages considered are divided into n periods and each of them contains m stages. At each stage in the i-th period, the decision maker is required to transport $1/m$ amount of commodity from A_{n-i+1} to B.

First, assume that the decision maker is myopic. We claim that at each stage in the i-th period; i.e. at the stage t satisfying $(i-1)m < t \leq im$ for some positive integer i, the myopic optimal solution has the following form:

$$f_{lt}^M = \frac{1}{m} \frac{l}{\sum_{j=1}^{n-i+1} j} = \frac{2l}{m} \left(\frac{1}{n-i+1} - \frac{1}{n-i+2} \right), \tag{4}$$

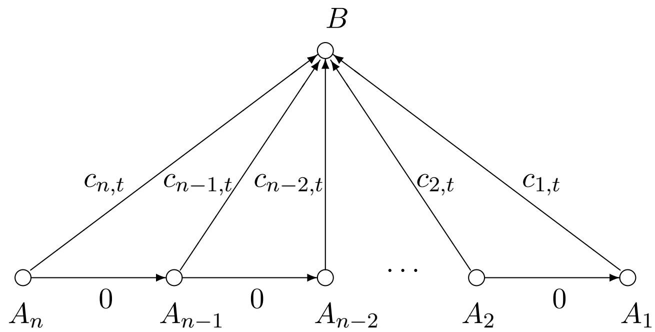

Figure 1: An example indicating a lower bound of PoM in transportation problem

for $l = 1, 2, ..., n - i + 1$.

Let us prove the above assertion by induction.

Consider the initial stage $t = 1$ and $i = 1$. The myopic transportation decision boils down to the following optimization problem:

$$\begin{array}{ll} \min & \sum_{l=1}^{n} \frac{x_l}{l} \times x_l \\ \text{s.t.} & \sum_{l=1}^{n} x_l = \frac{1}{m}, \;\; x_l \geq 0. \end{array}$$

The KKT optimality condition of the above model suggests that x_l^*/l must be all equal over the index l. Let $S_k := 1 + 2 + \cdots + k = \frac{k(k+1)}{2}$. The optimal solution of the above problem is therefore $x_l^* = \frac{l}{mS_n}$, $l = 1, 2, ..., n$. This proves (4) for $t = 1$ and $i = 1$.

Now we assume that (4) holds for all stages before the stage $(i-1)m + s$, where $1 \leq i \leq n$ and $0 < s \leq m$ are fixed integers. Let us consider what happens at the stage $t = (i-1)m + s$. Observe that at this decision point, all the available links for the decision maker are the ones from A_l to B with $l = 1, ..., n - i + 1$. Based on the induction hypothesis, we conclude that the unit cost on the link l ($l \leq n - i + 1$) at the stage $t - 1$ is given by

$$\begin{aligned} c_{l,t-1} &= \sum_{j=1}^{i-1} 2\left(\frac{1}{n-j+1} - \frac{1}{n-j+2}\right) + \frac{2(s-1)}{m}\left(\frac{1}{n-i+1} - \frac{1}{n-i+2}\right) \\ &= 2\left(\frac{1}{n-i+2} - \frac{1}{n+1}\right) + \frac{2(s-1)}{m}\left(\frac{1}{n-i+1} - \frac{1}{n-i+2}\right), \end{aligned}$$

which is independent of l. This means that the *accumulated* unit costs on all the available links at this point are equal; say, all are equal to c_{t-1}. Hence, the myopic decision problem at this point is

$$\begin{array}{ll} \min & \sum_{l=1}^{n-i+1} \left(\frac{x_l^t}{l} + c_{t-1}\right) \times x_l^t \\ \text{s.t.} & \sum_{l=1}^{n-i+1} x_l^t = \frac{1}{m}, \;\; x_l^t \geq 0, \end{array}$$

with the optimal solution being: $x_l^t = \frac{l}{mS_{n-i+1}} = \frac{2l}{m}\left(\frac{1}{n-i+1} - \frac{1}{n-i+2}\right)$, $l = 1, ..., n - i + 1$. This shows that (4) holds for $t = (i-1)m + s$ as well, and so (4) holds in general as a result of the induction.

Let us now calculate the total transportation cost. Note that when $t = (i-1)m + s$, the unit cost on link l is

$$
\begin{aligned}
c_{lt} &= \frac{1}{l}\frac{2l}{m}\left(\frac{1}{n-i+1} - \frac{1}{n-i+2}\right) + c_{l,t-1} \\
&= 2\left(\frac{1}{n-i+2} - \frac{1}{n+1}\right) + \frac{2s}{m}\left(\frac{1}{n-i+1} - \frac{1}{n-i+2}\right) \\
&> 2\left(\frac{1}{n-i+2} - \frac{1}{n+1}\right).
\end{aligned}
$$

The fact that the inequality holds for all $0 < s \le m$ implies that the total cost in the i-th period is at least

$$2\left(\frac{1}{n-i+2} - \frac{1}{n+1}\right) \times \frac{1}{m} \times m = 2\left(\frac{1}{n-i+2} - \frac{1}{n+1}\right).$$

Hence, the total cost of the myopic solution over the entire periods is at least

$$\mathrm{TC}(f^M) = 2\sum_{i=1}^{n}\left(\frac{1}{n-i+2} - \frac{1}{n+1}\right) = 2\left(H_n - \frac{2n}{n+1}\right)$$

where $H_n := 1 + 1/2 + \cdots + 1/n > \ln(n+1)$ denotes the harmonic series.

Now we consider another solution, in which only the link from A_i to B will be used in the i-th period. It is clear that the total cost for this solution is

$$
\begin{aligned}
\mathrm{TC}(f) &= \sum_{l=1}^{n}\left[\sum_{s=1}^{m} \frac{\frac{s-1}{m} + \frac{1}{m}}{l} \times \frac{1}{m}\right] \\
&= \sum_{l=1}^{n} \frac{1}{lm^2}\sum_{s=1}^{m} s = \frac{m+1}{2m} H_n.
\end{aligned}
$$

In summary, the PoM in this example is bounded below by:

$$\mathrm{PoM} \ge \frac{\mathrm{TC}(f^M)}{\mathrm{TC}(f)} \ge \frac{4m}{m+1} - \frac{8mn}{(m+1)(n+1)} \times \frac{1}{\ln n}.$$

If we choose $m = n = \sqrt{T}$, then $\mathrm{PoM} = 4 - O\left(\frac{1}{\ln T}\right)$.

Corollary 2.4 *Suppose T is taken as a fixed parameter of the dynamic transportation decision problem, then*

$$4 - O\left(\frac{1}{\ln T}\right) \le \mathrm{PoM} \le 4 - O\left(\frac{1}{T}\right)$$

and if T is regarded an input parameter of the dynamic transportation decision problem, then $\mathrm{PoM} = 4$.

3 The Myopic Nash Equilibrium and the Price of Isolation

In this section, we shall extend the dynamic transportation problem to a game framework, in which a finite number of self-interested players are facing the same dynamic decision processes and the decision of each player may affect the cost structures of each other. The network G_t in each stage is shared by all the players, and the tasks for the players may differ. First of all, the OD pair and required throughput of the players are different: for Player k in stage t the OD is $\{s_t^k, d_t^k\}$ and the required amount of transportation is r_t^k. Furthermore, we denote the transportation plan of player k to be a vector $x_t^k \in \mathbb{R}^m$ and a feasible flow is given by the constraints $A_t x_t^k = r_t^k \delta_{s_t^k} - r_t^k \delta_{d_t^k}$.

We denote the total flow on link l in stage t to be $f_{lt} = \sum_{k=1}^K x_{lt}^k$, and the corresponding unit cost is given by a function $c_{lt}(f_{lt}; c_{l,t-1})$, where $c_{l,t-1}$ is the unit cost on link l in stage $t-1$ and $c_{l0} = 0$ for all l and t. Then, given the decisions of other players in stage t (denoted as x_t^{-k}) and the cost in previous stage c_{t-1}, the *myopic* and *selfish* player k will face the following optimization problem at stage t:

$$
\begin{array}{lll}
(P_t^k) & \min & C_t^k(x_t^k, x_t^{-k}; c_{t-1}) = \sum_{l \in L} x_{lt}^k c_{lt}(f_{lt}; c_{l,t-1}) \\
& \text{s.t.} & A_t x_t^k = r_t^k \delta_{s_t^k} - r_t^k \delta_{d_t^k}, \\
& & x_t^k \geq 0.
\end{array}
$$

If each player at each stage makes the decision following the optimization problem above, the stable state that the game attains is called the *myopic Nash equilibrium*.

3.1 Myopic Nash Equilibrium

Let us elaborate on the *myopic Nash equilibrium* in this subsection. The conventional Nash equilibrium is a powerful solution concept for an n-person static game since its inception (Nash [30]). However, it is substantially more complex to rationalize the concept when it comes to dynamic games. Bizarre properties (in terms of the informational structure) of an unpolished Nash solution may occur in dynamic games. To mitigate this, Selten introduced the notion of subgame perfect Nash equilibrium: a Nash equilibrium that remains so on every *subgame*. However, it requires the existence of backward induction to ensure its existence and uniqueness, which cannot be guaranteed without perfect information. In reality, the parameters of the game in the future cannot be easily obtained or predicted with precision, let alone the possible courses of actions of other players. Moreover, it is all too often that the players are much more focusing on the immediate gains than the future potentials. As an example, consider the greenhouse gas emission control problem, which illustrates that the nations in the world can behave myopically and selfishly at the same time. To understand why decision makers may act myopically, an important reason is that most managers/politicians are evaluated by the performance within their tenure. Note that the current anti-Wall Street sentiment is triggered by the perceived *combined* myopic (greedy) and selfish (irresponsible) behaviors of the bankers in Wall

Street. Hence, one learned point is that if the decision-makers are left unchecked, then typically they will behave myopically and selfishly. If so is true for at least some players, then in a dynamic game setting, it is hard for any given player to count on the other players' adherence to their respective dynamically optimal strategies. This makes the straightforward notion of myopic Nash equilibrium all the more natural. At least it does not depend on any agreed behavior code about the courses of future actions. Consequently, in our framework, we assume that all the players in the game are selfish and myopic and each of them focuses only on his/her own immediate payoff. Then we say the game attains a myopic Nash equilibrium if no myopic player is able to improve his/her current utility only through changing his/her current strategy unilaterally. A stable solution concept in a dynamic game setting can be put in perspective as follows. Suppose that the competition only takes place among different stages, then the stable state will be the greedy solution; if the competition only takes place among the players, then the subgame perfect Nash equilibrium is an appropriate concept; if the information is imperfect and the number of players is large, then the myopic Nash equilibrium is applicable. One advantage of the myopic Nash equilibrium is that its existence can be easily established under mild conditions, as we shall see in the next subsection.

3.2 The Existence of Myopic Nash Equilibrium in General Games

Let us study the utility maximization game as an example to establish the existence of myopic Nash equilibrium. Consider K players in a dynamic game played over T stages. For each player k in each stage t, the payoff of the player will depend on the strategies of the player and the others in the current stage, as well as on the state inherited from the previous stage. Furthermore, the state in a stage is dependent on the strategies in the current and previous stages. Specifically, the payoff function is given by $u_t^k(x_t^k; x_t^{-k}; s_{t-1}^k)$, in which x_t^k denotes the strategy vector of player k in stage t, x_t^{-k} denotes the strategies of other players except k, and s_{t-1}^k represents the state in the previous stage and could be viewed as a function of (x_{-t}^k, x_{-t}^{-k}). We assume that u_t^k is continuous in all the variables and is concave in x_t^k for each fixed x_t^{-k} and s_{t-1}^k.

On the other hand, the strategy of player k is also restricted to be chosen from a set. We denote the strategy space of player k in stage t by S_t^k, which is assumed to be compact and convex. Now we want to characterize an equilibrium point, at which each player is selfish and greedy. It means that each player in the game only focuses on the current status and optimizes his/her current payoff; that is, the players in this type of game at each stage will focus on the following optimization problem:

$$\begin{array}{ll} \max_{x_t^k} & u_t^k(x_t^k; x_t^{-k}; s_{t-1}^k) \\ \text{s.t.} & x_t^k \in S_t^k \end{array}$$

for each x_t^{-k} and s_{t-1}^k; or rephrasing the dependence directly

$$\begin{array}{ll} \max_{x_t^k} & u_t^k(x_t^k; x_t^{-k}; x_{-t}^k, x_{-t}^{-k}) \\ \text{s.t.} & x_t^k \in S_t^k. \end{array}$$

Theorem 3.1 *Under the conditions stipulated above, a myopic Nash equilibrium exists.*

Proof. The proof is similar to the existence of Nash equilibrium (see e.g. [34]) in the static case. Note that the myopic Nash equilibrium is given by a point y such that

$$u_t^k(y) = \max_{x_t^k \in S_t^k} u_t^k(x_t^k; y_t^{-k}; y_{-t}^k, y_{-t}^{-k}).$$

Then we define the following function

$$\rho(y, x) = \sum_{k=1}^{K} \sum_{t=1}^{T} u_t^k(x_t^k; y_t^{-k}; y_{-t}^k, y_{-t}^{-k}).$$

We consider a point-to-set mapping Γ as follows:

$$\Gamma(y) = \{x \mid \rho(y, x) = \max_z \rho(y, z)\}.$$

It can be shown that the mapping Γ is upper semicontinuous. Furthermore, since the constraint set S_t^k for each player and each stage is convex and closed, $\Pi_{k=1}^K \Pi_{t=1}^T S_t^k$ is a compact set. Then by the Kakutani fixed point theorem, there exists a fixed point y such that $\rho(y, y) = \max_z \rho(y, z)$, which corresponds to a myopic Nash equilibrium. □

3.3 Potential Function for Myopic Nash Equilibrium in Dynamic Transportation Game

A potential game is a game whose Nash equilibrium is equivalent to an optimal solution of a function, called the *potential function*. One example of potential game is a static transportation game [9]. It is therefore interesting to know the role of potential functions in the context of myopic Nash equilibrium. Consider stage t of a dynamic game. Suppose that the inherited state from stage $t-1$ is s_{t-1}, and the action vector of all the players is x_t. The system dynamic is $s_t = f_t(s_{t-1}, x_t)$. Moreover, suppose that the game has a potential function $P_t(x_t; s_{t-1})$; i.e. a Nash equilibrium x_t corresponds to a minimizer of $P_t(x_t; s_{t-1})$. In that case, the myopic Nash equilibrium corresponds to sequentially minimizing $P_t(x_t; s_{t-1})$ subject to $x_t \in A_t$, and update $s_t = f_t(s_{t-1}, x_t)$, $t = 1, 2, ..., T$. Obviously, one may formulate this as a single lexicographically ordered optimization model, or as a limit of

$$\begin{array}{ll} \min & \sum_{t=1}^T \epsilon^{t-1} P_t(x_t; s_{t-1}) \\ \text{s.t.} & s_0 = \hat{s}_0 \\ & x_t \in A_t,\ s_t = f_t(s_{t-1}, x_t),\ t = 1, 2, ..., T, \end{array}$$

where $\hat{s}_0$ is the initial state.

However, there are other possible formulation of a single potential function for dynamic transportation games. To demonstrate this, we choose the affine linear functions as an example. In case c_{lt} is a nondecreasing affine linear function, i.e. $c_{lt} = a_l f_{lt} + b_{lt} + \alpha_{t-1} c_{l,t-1}$, where $a_l, b_{lt} \geq 0$, $0 \leq \alpha_t \leq 1$ and $c_{l0} = 0$ for all l and t. We know (e.g. from [9]) that a Nash equilibrium in the static game corresponds to the following optimization problem:

$$\begin{array}{lll} (QP_t) & \min & \frac{1}{2} x_t^{\mathrm{T}} ((I_K + E_K) \otimes \mathrm{Diag}(a)) x_t + (e_K \otimes (b_t + \alpha_{t-1} c_{t-1}))^{\mathrm{T}} x_t \\ & \text{s.t.} & A_t x_t^k = r_t^k \delta_{s_t^k} - r_t^k \delta_{d_t^k}, \text{ for all } k \\ & & x_t \geq 0. \end{array}$$

Here e_K is the (K by 1) all-one vector, E_K is the (K by K) all-one matrix, I_K are the K by K identity matrices, and notice that both b_t and $\alpha_{t-1} c_{t-1}$ are constant. This means that when the cost functions are affine linear, the state of the myopic Nash equilibrium in each stage is corresponding to a quadratic optimization problem (QP_t). Thus, the myopic Nash equilibrium exists, and if a_l is positive for all l, then the myopic Nash equilibrium will be unique.

Based on this, we further establish a global potential function whose optimal solutions correspond to myopic Nash equilibria of the dynamic transportation game with affine linear costs, i.e.,

$$c_{lt} = a_l f_{lt} + b_{lt} + \alpha_{t-1} c_{l,t-1}.$$

Let $z_t^k, s_t^k \in \mathbb{R}^n$ be the Lagrangian multipliers associated with the equality constraint $A_t x_t^k = r_t^k \delta_{s_t^k} - r_t^k \delta_{d_t^k}$ and $x_t^k \geq 0$, respectively. The Karush-Kuhn-Tucker optimality condition for (QP^t) is:

$$\begin{cases} A_t x_t^k = r_t^k \delta_{s_t^k} - r_t^k \delta_{d_t^k}, & \text{for all } k \\ ((I_K + E_K) \otimes \mathrm{Diag}(a)) x_t + e_K \otimes (b_t + \alpha_{t-1} c_{t-1}) + \sum_k A_t^{\mathrm{T}} z_t^k = s_t, & \\ (x_t^k)^{\mathrm{T}} s_t^k = 0, & \text{for all } k \\ x_t^k \geq 0, \; s_t^k \geq 0, & \text{for all } k. \end{cases}$$

Combining all the KKT conditions together with the inductive cost formulation, we have the following LCP formulation:

$$\begin{cases} c_{lt} = a_l \sum_k x_{lt}^k + b_{lt} + \alpha_{t-1} c_{l,t-1}, & \text{for all } l, t \\ A_t x_t^k = r_t^k \delta_{s_t^k} - r_t^k \delta_{d_t^k}, & \text{for all } k, t \\ ((I_K + E_K) \otimes \mathrm{Diag}(a)) x_t + e_K \otimes (b_t + \alpha_{t-1} c_{t-1}) + \sum_k A_t^{\mathrm{T}} z_t^k = s_t, & \text{for all } t \\ (x_t^k)^{\mathrm{T}} s_t^k = 0, & \text{for all } k, t \\ x_t^k \geq 0, \; s_t^k \geq 0, & \text{for all } k, t. \end{cases}$$

Therefore the solutions of this LCP correspond to the optimal solutions of the following minimization problem which has an optimal value of 0 (we already established the existence):

$$\begin{array}{lll} \min & \sum_{l,k,t} x_{lt}^k s_{lt}^k & \\ \text{s.t.} & c_{lt} = a_l \sum_k x_{lt}^k + b_{lt} + \alpha_{t-1} c_{l,t-1}, & \text{for all } l, t \\ & A_t x_t^k = r_t^k \delta_{s_t^k} - r_t^k \delta_{d_t^k}, & \text{for all } k, t \\ & ((I_K + E_K) \otimes \text{Diag}(a)) x_t + e_K \otimes (b_t + \alpha_{t-1} c_{t-1}) + \sum_k A_t^{\mathrm{T}} z_t^k = s_t, & \text{for all } t \\ & x_t^k \geq 0,\ s_t^k \geq 0, & \text{for all } k, t. \end{array}$$

It is easy to see that the constraint set is a polytope of (x, s, z, c), however the objective function is non-convex.

3.4 Bounding the PoI in a Dynamic Transportation Game

For a dynamic game, let us introduce here the notion of the *Price of Isolation* (PoI) as the ratio between the total objective values of the worst myopic Nash solution and the dynamically optimal social value. If the problem reduces to a single player (the dynamic decision model), then the PoI correspondingly reduces to the PoM as we discussed in Section 2; if the problem is static, then the PoI reduces to the PoA. In our discussion, let us denote y to be the solution where each player is myopic and selfish (i.e. y_t^k is optimal to C_t^k for all k and t), and denote x to be the dynamic optimal solution for the whole system (x is the minimum for SC over the feasible region). Then the price of isolation is $\text{PoI} = \frac{\text{SC}(y)}{\text{SC}(x)}$. Our subsequent analysis is similar to the pure dynamic case. First, we have the following result regarding the difference of the solutions at a myopic Nash equilibrium and at the optimum.

Lemma 3.2 *Suppose the game attains a myopic Nash Equilibrium and let us denote y to be the solution. Suppose that player k changes his/her strategy at stage t from y_t^k to any other feasible flow x_t^k, while the strategies of this player at the other stages and all other players' strategies remain unchanged. Then, the cost for player k will increase by at least $\sum_{l \in L} a_l (x_{lt}^k - y_{lt}^k)^2$; that is*

$$C_t^k(x_t^k, y_{-t}^k; y^{-k}) - C_t^k(y_t^k, y_{-t}^k; y^{-k}) \geq \sum_{l \in L} a_l (x_{lt}^k - y_{lt}^k)^2.$$

Proof. According to the definition of myopic Nash equilibrium, y_t^k is the optimal strategy of the myopic player k, whose cost function at stage t is $C_t^k(\,\cdot\,, y_{-t}^k; y^{-k})$. Then

$$\begin{aligned}
& C_t^k(x_t^k, y_{-t}^k; y^{-k}) - C_t^k(y_t^k, y_{-t}^k; y^{-k}) - \sum_{l\in L} a_l (x_{lt}^k - y_{lt}^k)^2 \\
= & \sum_{l\in L} \Bigg\{ \left[a_l(x_{lt}^k + \sum_{i\neq k} y_{lt}^i) + b_{lt} + \sum_{\tau=1}^{t-1} (\Pi_{h=\tau}^{t-1}\alpha_h)(b_{l\tau} + a_l f_{l\tau}^y) \right] x_{lt}^k \\
& - \left[a_l f_{lt}^y + b_{lt} + \sum_{\tau=1}^{t-1} (\Pi_{h=\tau}^{t-1}\alpha_h)(b_{l\tau} + a_l f_{l\tau}^y) \right] y_{lt}^k - a_l (x_{lt}^k - y_{lt}^k)^2 \Bigg\} \\
= & \sum_{l\in L} \left(2a_l y_{lt}^k + a_l \sum_{i\neq k} y_{lt}^i + b_{lt} + \sum_{\tau=1}^{t-1} (\Pi_{h=\tau}^{t-1}\alpha_h)(b_{l\tau} + a_l f_{l\tau}^y) \right) (x_{lt}^k - y_{lt}^k) \geq 0.
\end{aligned}$$

The last inequality is due to the convexity of function $C_t^k(\,\cdot\,, y_{-t}^k; y^{-k})$, and the fact that $2a_l y_{lt}^k + a_l \sum_{i\neq k} y_{lt}^i + b_{lt} + \sum_{\tau=1}^{t-1}(\Pi_{h=\tau}^{t-1}\alpha_h)(b_{l\tau} + a_l f_{l\tau}^y)$ is the derivative of $C_t^k(\,\cdot\,, y_{-t}^k; y^{-k})$ at the minimum point y_t^k. □

By applying the similar scheme as in Section 2, we have the following upper bound for the price of isolation:

Theorem 3.3 *For the dynamic transportation game, the price of isolation is upper bounded by* $\frac{4K^2(T+1)^2}{(K(T+1)+2)^2} < 4$.

Proof. Let x and y be the social optimum and a myopic Nash equilibrium respectively. We now set out to look for (λ, μ) with $\lambda > 0$ and $0 < \mu < 1$, such that

$$\sum_{k=1}^{K}\sum_{t=1}^{T} \left[C_t^k(x_t^k, y_{-t}^k; y^{-k}) - \sum_{l\in L}\sum_{t=1}^{T} a_l (x_{lt}^k - y_{lt}^k)^2 \right] \leq \lambda\, \mathrm{SC}(x) + \mu\, \mathrm{SC}(y). \tag{5}$$

If (5) is shown, then according to Lemma 3.2 we will have

$$\begin{aligned}
\mathrm{SC}(y) &= \sum_{k=1}^{K}\sum_{t=1}^{T} C_t^k(y^k; y^{-k}) \leq \sum_{k} \left[C^k(x_t^k, y_{-t}^k; y^{-k}) - \sum_{l\in L}\sum_{t=1}^{T} a_l (x_{lt}^k - y_{lt}^k)^2 \right] \\
&\leq \lambda\, \mathrm{SC}(x) + \mu\, \mathrm{SC}(y).
\end{aligned}$$

Thus the price of isolation can be bounded as:

$$\mathrm{PoI} = \frac{\mathrm{SC}(y)}{\mathrm{SC}(x)} \leq \frac{\lambda}{1-\mu}. \tag{6}$$

Substituting the exact forms of the cost functions in (5), the intended inequality becomes:

$$\begin{aligned}
&\sum_{k=1}^{K}\sum_{t=1}^{T}\sum_{l\in L}\left[a_l(x^k_{lt}+f^y_{lt}-y^k_{lt})x^k_{lt}+b_{lt}x^k_{lt}+\sum_{\tau=1}^{t-1}(\Pi_{h=\tau}^{t-1}\alpha_h)(b_{l\tau}+a_l f^y_{l\tau})x^k_{lt}-a_l(x^k_{lt}-y^k_{lt})^2\right]\\
\leq\ &\lambda\sum_{t=1}^{T}\sum_{l\in L}\left[a_l(f^x_{lt})^2+b_{lt}f^x_{lt}+\sum_{\tau=1}^{t-1}(\Pi_{h=\tau}^{t-1}\alpha_h)(b_{l\tau}+a_l f^x_{l\tau})f^x_{lt}\right]\\
&+\mu\sum_{t=1}^{T}\sum_{l\in L}\left[a_l(f^y_{lt})^2+b_{lt}f^y_{lt}+\sum_{\tau=1}^{t-1}(\Pi_{h=\tau}^{t-1}\alpha_h)(b_{l\tau}+a_l f^y_{l\tau})f^y_{lt}\right].
\end{aligned}$$

Notice that the order of summation can be interchanged, so the left hand side of the inequality above can be rewritten as:

$$LHS=\sum_{l\in L}\sum_{t=1}^{T}\left[\sum_{k=1}^{K}a_l(x^k_{lt}-y^k_{lt})y^k_{lt}+b_{lt}f^x_{lt}+a_l f^y_{lt}f^x_{lt}+\sum_{\tau=1}^{t-1}(\Pi_{h=\tau}^{t-1}\alpha_h)(b_{l\tau}+a_l f^y_{l\tau})f^x_{lt}\right].$$

We regroup the terms on both the left and the right hand sides and obtain

$$\begin{aligned}
&RHS-LHS\\
=\ &\sum_{l\in L}\sum_{t=1}^{T}a_l\left[\lambda(f^x_{lt})^2+\mu(f^y_{lt})^2-f^x_{lt}f^y_{lt}+\sum_{\tau=1}^{t-1}(\Pi_{h=\tau}^{t-1}\alpha_h)\left(\lambda f^x_{l\tau}f^x_{lt}+\mu f^y_{l\tau}f^y_{lt}-f^y_{l\tau}f^x_{lt}\right)\right.\\
&\left.-\sum_{k=1}^{K}y^k_{lt}(x^k_{lt}-y^k_{lt})\right]+\sum_{l\in L}\sum_{t=1}^{T}\left(b_{lt}+\sum_{\tau=1}^{t}(\Pi_{h=\tau}^{t-1}\alpha_h)b_{l\tau}\right)\left[(\lambda-1)f^x_{lt}+\mu f^y_{lt}\right]. \qquad (7)
\end{aligned}$$

By requiring $\lambda\geq 1,\mu\geq 0$, we have

$$(\lambda-1)f^x_{lt}+\mu f^y_{lt}\geq 0 \text{ for all } l \text{ and } t. \qquad (8)$$

Thus the second summation part of (7) is nonnegative because b_{lt}'s and α_τ's are nonnegative. Let us now pay attention to first summation part of (7). Again, observing that this is a summation over the index l, it suffices to establish the following inequality for each link:

$$\sum_{t=1}^{T}\left[\lambda(f^x_{lt})^2+\mu(f^y_{lt})^2-f^x_{lt}f^y_{lt}+\sum_{\tau=1}^{t-1}(\Pi_{h=\tau}^{t-1}\alpha_h)\left(\lambda f^x_{l\tau}f^x_{lt}+\mu f^y_{l\tau}f^y_{lt}-f^y_{l\tau}f^x_{lt}\right)-\sum_{k=1}^{K}y^k_{lt}(x^k_{lt}-y^k_{lt})\right]\geq 0.$$

Note that

$$\sum_{k=1}^{K}y^k_{lt}x^k_{lt}\leq f^x_{lt}f^y_{lt},\quad \text{and}\quad \sum_{k=1}^{K}(y^k_{lt})^2\geq\frac{1}{K}(f^y_{lt})^2.$$

Therefore, we only need to establish the following inequality:

$$\sum_{t=1}^{T}\left[\lambda(f^x_{lt})^2+(\mu+\frac{1}{K})(f^y_{lt})^2-2f^x_{lt}f^y_{lt}+\sum_{\tau=1}^{t-1}(\Pi_{h=\tau}^{t-1}\alpha_h)(\lambda f^x_{l\tau}f^x_{lt}+\mu f^y_{l\tau}f^y_{lt}-f^y_{l\tau}f^x_{lt})\right]\geq 0,$$

which can be rewritten in matrix notation:

$$\lambda(f_l^x)^{\mathrm{T}} L_\alpha f_l^x + (f_l^y)^{\mathrm{T}}(\mu L_\alpha + \frac{1}{K} I_T) f_l^y - (f_l^x)^{\mathrm{T}}(L_\alpha + I_T) f_l^y \geq 0. \tag{9}$$

Since $f_l^x \geq 0, f_l^y \geq 0$, we have $(f_l^x)^{\mathrm{T}} I_T f_l^y \leq (f_l^x)^{\mathrm{T}} L_\alpha^{\mathrm{T}} f_l^y$ and $(f_l^y)^{\mathrm{T}} I_T f_l^y \geq \frac{2}{T+1}(f_l^y)^{\mathrm{T}} L_\alpha f_l^y$. It suffices to ensure the matrix

$$\begin{pmatrix} \lambda(L_\alpha + L_\alpha^{\mathrm{T}}) & -(L_\alpha + L_\alpha^{\mathrm{T}}) \\ -(L_\alpha + L_\alpha^{\mathrm{T}}) & (\mu + \frac{2}{K(T+1)})(L_\alpha + L_\alpha^{\mathrm{T}}) \end{pmatrix} = \begin{pmatrix} \lambda & -1 \\ -1 & \mu + \frac{2}{K(T+1)} \end{pmatrix} \otimes (L_\alpha + L_\alpha^{\mathrm{T}})$$

to be positive semidefinite. Recall that L_α is monotone and thus we only need to make sure that $\begin{pmatrix} \lambda & -1 \\ -1 & \mu + \frac{2}{K(T+1)} \end{pmatrix}$ is positive semidefinite, which is implied by $\lambda\left(\mu + \frac{2}{K(T+1)}\right) \geq 1$ to serve that purpose. Combining the condition obtained from (8), we shall minimize $\frac{\lambda}{1-\mu}$, leading to the choice $\lambda = \frac{2K(T+1)}{K(T+1)+2}$ and $\mu = \frac{K(T+1)-2}{2K(T+1)}$. Finally, (6) leads to

$$\mathrm{PoI} \leq \frac{4K^2(T+1)^2}{(K(T+1)+2)^2} < 4.$$

□

Similar to PoM, we consider the price of isolation for a problem class $\mathscr{P}$ as the PoI of the worst instance in that problem class, and denote the value to be $PoI_{\mathscr{P}}$. Since the dynamic transportation decision problem is a special case of the dynamic transportation game, Example 2.3 also applies here. Therefore, we have:

Corollary 3.4 *If T is considered an input parameter of the dynamic transportation game, then we have* $\mathrm{PoI}_{\mathscr{P}} = 4$.

It is worth noting that the PoM and PoI coincide when T is large and are equal to 4, meaning that the worst case might occur even if all the players cooperate and form a grand coalition. Recall that the bound for the PoA for the static transportation game is $\frac{3}{2}$, as shown in [9]. The discrepancy somehow indicates the difference in nature between selfishness and greediness in this context, due to the asymmetry of the time dimension.

4 The Price of Isolation for the Dynamic Economic Competition Models

In this section we study the profit maximization game models. In such games the players compete to make profits, where the decisions available to them can be the quantity of the products to produce (the Cournot style model) which indirectly induces the selling price by the law of supply and demand, or it can be directly the price itself (the Bertrand style model). We investigate both settings in this section, while keeping in mind that our goal is to understand how the competition affects the system efficiency. For clarity, we present the study on the Cournot style model in Subsection 4.1, and the Bertrand style model in Subsection 4.2. Before delving into the details, a few words on the models are in order. To analyze the profit maximizing games we need to work with specific models regarding how the prices are determined. For technical reasons, in this paper we choose to work with an *affine linear* structure. This means that in the Cournot case, the price of a product is an affine linear function of its sales quantity, as well as the sales quantity of the substitute products; in the Bertrand case, the quantity sold is an affine linear function of the prices of the related products. While the affine linearity is certainly a restrictive and simplifying assumption, there are reasons to believe that it is a good starting point for the study. In the original works of Cournot and Bertrand, the linearity assumption was already in place, and one may argue that at least locally the changes are usually proportional, hence linear. It is therefore a reasonable approximation, in addition to its technical simplicity. Due to that reason, the linear price models (typically for the Bertrand case) are often seen in standard texts on game theory such as Gibbons [17], and Osborne and Rubinstein [32]. The linear price models can also be found in recent papers on the topic, e.g. Kluberg and Perakis [24].

4.1 Substitute Products in Cournot Oligopolistic Competition Game

Suppose that we have m given types of resources, which can be used to produce n types of products. As for how the producers actually deploy the resources and turn them into the final products depends on the infrastructure available to them, as well as the status of their knowhow's. In our model, this relationship is characterized by a pair of technology matrices, denoted by (M^k, N^k) for producer k. Namely, if producer k uses a nonnegative vector $x^k \in \mathbb{R}^m_+$ to denote the usage of resources and $v^k \in \mathbb{R}^n_+$ to denote the quantity of product produced from the resources, then the constraint $N^k v^k \le M^k x^k$ holds. The producers in the game use the resources to produce products and then attempt to sell the products in the market. The competition among the producers are twofold: first, the costs for the use of the shared resources are subject to competition; second, the products produced by different producers are substitute and the prices of the products are also subject to the competition in the market place. A static version of this model is considered in [21], where the price of anarchy is lower bounded by the inverse of the number of producers.

In this subsection, we consider a dynamic version of the game. Suppose there are K producers competing over T decision stages. The prices of the products and the costs of using the resources for each player are affected by two factors: the historical decisions of the previous stage and the decisions of the current stage. Specifically, we use p^k_{jt} and c^k_{lt} to denote the selling price of products j and the unit cost of using resource l for player k at stage t, respectively. Suppose that the amount of products

j produced by producer k at stage t is v_{jt}^k, and v_{jt}^{-k} is the decision of all other players except player k. Similarly, denote the usage of resource l for producer k at stage t to be x_{lt}^k and the other producers' decision to be x_{lt}^{-k}. For convenience, in the sequel we use the following vector notations for v (similar for all other variables and parameters with these three indices): $v_t^k := (v_{1t}^k, v_{2t}^k, \ldots, v_{nt}^k)^{\mathrm{T}}, v_{(j)}^k := (v_{j1}^k, v_{j2}^k, \ldots, v_{jT}^k)^{\mathrm{T}}$, and $v_{jt} := (v_{jt}^1, v_{jt}^2, \ldots, v_{jt}^K)^{\mathrm{T}}$. Then, the selling prices for producer k at stage t is assumed to be a vector $p_t^k(v_t^k, v_t^{-k}; p_{t-1}^k) \in \mathbb{R}_+^n$, which can be viewed as a function of the amount of current supply, v_t^k and v_t^{-k}, and the prices at the previous stage, p_{t-1}^k. Similarly, the unit cost of the resources is a vector $c_t^k(x_t^k, x_t^{-k}; c_{t-1}^k) \in \mathbb{R}_+^m$, which is dependent on the current total demand x_t and the cost level inherited from the previous stage.

Suppose that the technology-matrix pair for producer k is (M^k, N^k). To maximize the profit at stage t, the selfish and greedy producer k considers the following optimization problem:

$$
\begin{array}{lll}
(P_t^k) & \max & V_t^k(v_t^k, x_t^k; v_t^{-k}, x_t^{-k}) = (v_t^k)^{\mathrm{T}} p_t^k(v_t^k, v_t^{-k}; p_{t-1}^k) - (x_t^k)^{\mathrm{T}} c_t^k(x_t^k, x_t^{-k}; c_{t-1}^k) \\
& \text{s.t.} & N^k v_t^k \le M^k x_t^k, \\
& & x_t^k \ge 0, v_t^k \ge 0.
\end{array}
$$

Naturally, the social value is defined to be the summation of the profits of all the producers over the entire period: $\mathrm{SV}(v,x) = \sum_{k=1}^K \sum_{t=1}^T V_t^k(v_t^k, x_t^k; v_t^{-k}, x_t^{-k})$.

Let (w,y) denote a myopic Nash solution, and (v,x) denote the socially optimal solution. The PoI can be expressed as: $\mathrm{PoI} = \frac{\mathrm{SV}(w,y)}{\mathrm{SV}(v,x)}$.

As discussed at the beginning of the section, we now confine ourselves to the affine linear case where $p_t^k(v_t^k, v_t^{-k}; p_{t-1}^k)$ and $c_t^k(x_t^k, x_t^{-k}; c_{t-1}^k)$ are both affine linear functions. To be precise, we suppose that producer k produces v^k while other producers produce v^{-k}, then the price of product j for producer k at stage t can be written as $p_{jt}^k = q_{jt}^k - \sum_{i=1}^K \gamma_j^{ki} v_{jt}^i + \rho_{j,t-1} p_{j,t-1}^k$, where q_{jt}^k and γ_j^{ki} are positive parameters, $0 \le \rho_{j,t} \le 1$ and $p_{j,0}^k = 0$. The parameter γ_j^{ki} reflects the impact of the sales of product j by player i on the selling price for player k, and $\rho_{j,t}$ is a discounting factor to reflect the price dynamics. We assume $q_{jt}^k \ge 0$, and $\gamma_j^{ki} \ge 0$ since the price is deceasing in the amount of supply. Moreover, for each given product j, we assume that $(\gamma_j^{ki})^2 \le \gamma_j^{kk} \gamma_j^{ii}$ for all $1 \le i, k \le K$.

Also, for the unit cost of using resource l at stage t, $c_{lt}^k = b_{lt}^k + \sum_{i=1}^K \beta_l^{ki} x_{lt}^i + \alpha_{l,t-1} c_{l,t-1}^k$, where b_{lt}^k, β_l^{ki} are nonnegative constants and $0 \le \alpha_{l,t} \le 1$ for all l and t. Then producer k at stage t will face the following optimization problem:

$$
\begin{array}{lll}
(\tilde{P}_t^k) & \max & \sum_{j=1}^n (q_{jt}^k - \sum_{i=1}^K \gamma_j^{ki} v_{jt}^i + \rho_{j,t-1} p_{j,t-1}^k) v_{jt}^k - \sum_{l=1}^m (b_{lt}^k + \sum_{i=1}^K \beta_l^{ki} x_{lt}^i + \alpha_{l,t-1} c_{l,t-1}^k) x_{lt}^k \\
& \text{s.t.} & N^k v_t^k \le M^k x_t^k, \\
& & x_t^k \ge 0, v_t^k \ge 0.
\end{array}
$$

The existence of the myopic Nash equilibrium is guaranteed due to the boundedness of the solutions of the problem above (see also the argument in [21]). Letting z_t^k be the Lagrangian dual variable for the constraint $M^k x_t^k - N^k v_t^k \geq 0$, s_t^k be the dual variable for the constraint $x_t^k \geq 0$ and u_t^k be the dual variable for the constraint $v_t^k \geq 0$, the overall optimality condition (or the equilibrium condition) is the following monotone LCP system:

$$\begin{cases} b_{lt}^k + \sum_{i=1}^K \beta_l^{ki} x_{lt}^i + \alpha_{l,t-1} c_{l,t-1}^k + \beta_l^{kk} x_{lt}^k - ((M^k)^{\mathrm{T}} z_t^k)_l - s_{lt}^k = 0, \\ q_{jt}^k - \sum_{i=1}^K \gamma_j^{ki} v_{jt}^i + \rho_{j,t-1} p_{j,t-1}^k - \gamma_j^{kk} v_{jt}^k - ((N^k)^{\mathrm{T}} z_t^k)_j + u_{jt}^k = 0, \\ (z_t^k)^{\mathrm{T}} (M^k x_t^k - N^k v_t^k) = 0, \\ M^k x_t^k - N^k v_t^k \geq 0, z_t^k \geq 0, \\ (x_t^k)^{\mathrm{T}} s_t^k = 0,\ x_t^k \geq 0,\ s_t^k \geq 0, \\ (v_t^k)^{\mathrm{T}} u_t^k = 0,\ v_t^k \geq 0,\ u_t^k \geq 0. \end{cases}$$

Lemma 4.1 *At a myopic Nash equilibrium of an instance of the Cournot oligopolistic competition with solution (w, y), the profit of player k at stage t in the equilibrium equals to* $\sum_{j=1}^n \gamma_j^{kk} (w_{jt}^k)^2 + \sum_{l=1}^m \beta_l^{kk} (y_{lt}^k)^2$.

Proof. Since each player at the myopic Nash equilibrium attains optimality, given the strategies of the others, by the KKT condition we have

$$q_{jt}^k - \sum_{i=1}^K \gamma_j^{ki} w_{jt}^i + \rho_{j,t-1} p_{j,t-1}^k = \gamma_j^{kk} w_{jt}^k + ((N^k)^{\mathrm{T}} z_t^k)_j - u_{jt}^k,$$

and

$$b_{lt}^k + \sum_{i=1}^K \beta_l^{ki} y_{lt}^i + \alpha_{l,t-1} c_{l,t-1}^k = -\beta_l^{kk} y_{lt}^k + ((M^k)^{\mathrm{T}} z_t^k)_l + s_{lt}^k.$$

Hence the profit for producer k is

$$\begin{aligned} & \sum_{j=1}^n (q_{jt}^k - \sum_{i=1}^K \gamma_j^{ki} w_{jt}^i + \rho_{j,t-1} p_{j,t-1}^k) w_{jt}^k - \sum_{l=1}^m (b_{lt}^k + \sum_{i=1}^K \beta_l^{ki} y_{lt}^i + \alpha_{l,t-1} c_{l,t-1}^k) y_{lt}^k \\ = \; & \sum_{j=1}^n (\gamma_j^{kk} w_{jt}^k + ((N^k)^{\mathrm{T}} z_t^k)_j - u_{jt}^k) w_{jt}^k - \sum_{l=1}^m (-\beta_l^{kk} y_{lt}^k + ((M^k)^{\mathrm{T}} z_t^k)_l + s_{lt}^k) y_{lt}^k \\ = \; & \sum_{j=1}^n \gamma_j^{kk} (w_{jt}^k)^2 + \sum_{l=1}^m \beta_l^{kk} (y_{lt}^k)^2 + \sum_{j=1}^n w_{jt}^k (N^k)^{\mathrm{T}} z_t^k)_j - \sum_{l=1}^n y_{lt}^k ((M^k)^{\mathrm{T}} z_t^k)_l \\ = \; & \sum_{j=1}^n \gamma_j^{kk} (w_{jt}^k)^2 + \sum_{l=1}^m \beta_l^{kk} (y_{lt}^k)^2 + (z_t^k)^{\mathrm{T}} (N^k w_t^k - M^k y_t^k) = \sum_{j=1}^n \gamma_j^{kk} (w_{jt}^k)^2 + \sum_{l=1}^m \beta_l^{kk} (y_{lt}^k)^2. \end{aligned}$$

□

Lemma 4.2 *Denote (v, x) and (w, y) to be the solutions at the social optimum and at a myopic Nash equilibrium respectively. At the myopic Nash equilibrium (w, y), suppose that producer k switches* *his/her strategy at stage t to (v_t^k, x_t^k) while keeping his/her strategies in other stages and all other producers' strategies over the entire period unaltered. Then, the profit of producer k at stage t will decrease by at least an amount of $\sum_{j=1}^n \gamma_j^{kk}(v_{jt}^k - w_{jt}^k)^2 + \sum_{l=1}^m \beta_l^{kk}(x_{lt}^k - y_{lt}^k)^2$; that is,*

$$V_t^k(w_t^k, y_t^k; w_t^{-k}, y_t^{-k}) - V_t^k(v_t^k, x_t^k; w_t^{-k}, y_t^{-k}) \geq \sum_{j=1}^n \gamma_j^{kk}(v_{jt}^k - w_{jt}^k)^2 + \sum_{l=1}^m \beta_l^{kk}(x_{lt}^k - y_{lt}^k)^2.$$

Proof. By the definition of the myopic Nash equilibrium, (w_t^k, y_t^k) attains maximum for producer k's profit at stage t, assuming his/her strategy in other stages is (w_{-t}^k, y_{-t}^k) and all other producers' strategies are fixed as (w^{-k}, y^{-k}). Since the strategies of producer k in the previous stages have been fixed, p_{t-1} and c_{t-1} are not changed while producer k's strategy in stage t is changed, therefore

$$\begin{aligned}
& V_t^k(w_t^k, y_t^k; w_t^{-k}, y_t^{-k}) - V_t^k(v_t^k, x_t^k; w_t^{-k}, y_t^{-k}) - \sum_{j=1}^n \gamma_j^{kk}(v_{jt}^k - w_{jt}^k)^2 - \sum_{l=1}^m \beta_l^{kk}(x_{lt}^k - y_{lt}^k)^2 \\
= \; & (q_t^k + \mathrm{Diag}(\rho_{t-1}) p_{t-1}^k)^{\mathrm{T}}(w_t^k - v_t^k) \\
& \quad + \sum_{j=1}^n \left[\sum_{i \neq k} \gamma_j^{ki} w_{jt}^i v_{jt}^k + \gamma_j^{kk}(v_{jt}^k)^2 - \sum_{i \neq k} \gamma_j^{ki} w_{jt}^i w_{jt}^k - \gamma_j^{kk}(w_{jt}^k)^2 - \gamma_j^{kk}(v_{jt}^k - w_{jt}^k)^2 \right] \\
& -(b_t^k + \mathrm{Diag}(\alpha_{t-1}) c_{t-1}^k)^{\mathrm{T}}(y_t^k - x_t^k) \\
& \quad + \sum_{l=1}^m \left[\sum_{i \neq k} \beta_l^{ki} y_{lt}^i x_{lt}^k + \beta_l^{kk}(x_{lt}^k)^2 - \sum_{i \neq k} \beta_l^{ki} y_{lt}^i y_{lt}^k - \beta_l^{kk}(y_{lt}^k)^2 - \beta_l^{kk}(x_{lt}^k - y_{lt}^k)^2 \right] \\
= \; & \sum_{j=1}^n \left(q_{jt}^k - \sum_{i \neq k} \gamma_j^{ki} w_{jt}^i - 2\gamma_j^{kk} w_{jt}^k + \rho_{j,t-1} p_{j,t-1}^k \right)(w_{jt}^k - v_{jt}^k) \\
& \quad - \sum_{l=1}^m \left(b_{lt}^k + \sum_{i \neq k} \beta_l^{ki} y_{lt}^i + 2\beta_l^{kk} y_{lt}^k + \alpha_{l,t-1} c_{l,t-1}^k \right)(y_{lt}^k - x_{lt}^k) \\
= \; & \nabla V_t^k(w_t^k, y_t^k; w_t^{-k}, y_t^{-k})^{\mathrm{T}} \begin{pmatrix} w_t^k - v_t^k \\ y_t^k - x_t^k \end{pmatrix} \geq 0,
\end{aligned}$$

where the last inequality is due to the fact that (w_t^k, y_t^k) is maximal. □

Theorem 4.3 *Under our assumptions above, the PoI of the dynamic Cournot oligopolistic competition game is lower bounded by $\frac{1}{KT}$.*

Proof. Similar as our analysis for the transportation model, to get a lower bound for the price of isolation, it suffices to find a (λ, μ) pair with $\lambda > 0$ and $\mu > -1$, such that

$$\sum_{t=1}^{T}\sum_{k=1}^{K}\left[V_t^k(v_t^k, x_t^k; w_t^{-k}, y_t^{-k}) + \sum_{j=1}^{n}\gamma_j^{kk}(v_{jt}^k - w_{jt}^k)^2 + \sum_{l=1}^{m}\beta_l^{kk}(x_{lt}^k - y_{lt}^k)^2\right] \geq \lambda\,\mathrm{SV}(v,x) - \mu\,\mathrm{SV}(w,y). \tag{10}$$

Then, by Lemma 4.2, we can bound the price of isolation as well: $\mathrm{PoI} = \frac{\mathrm{SV}(w,y)}{\mathrm{SV}(v,x)} \geq \frac{\lambda}{1+\mu}$.

Here we use the similar matrix notations as before. Let

$$L_{\rho_j} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \rho_{j1} & 1 & \cdots & 0 \\ \rho_{j1}\rho_{j2} & \rho_{j2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \Pi_{h=1}^{T-1}\rho_{jh} & \Pi_{h=2}^{T-1}\rho_{jh} & \cdots & 1 \end{pmatrix}, \text{ and } L_{\alpha_l} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \alpha_{l1} & 1 & \cdots & 0 \\ \alpha_{l1}\alpha_{l2} & \alpha_{l2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \Pi_{h=1}^{T-1}\alpha_{lh} & \Pi_{h=2}^{T-1}\alpha_{lh} & \cdots & 1 \end{pmatrix}.$$

According to the dynamics of p_{jt}^k and c_{lt}^k, we may write the dynamic equation in the vector form as:

$$p_{(j)}^k = -\sum_{i=1}^{K}\gamma_j^{ki}L_{\rho_j}v_{(j)}^i + L_{\rho_j}q_{(j)}^k, \quad c_{(l)}^k = \sum_{i=1}^{K}\beta_l^{ki}L_{\alpha_l}x_{(l)}^i + L_{\alpha_l}b_{(l)}^k \tag{11}$$

Substitute the explicit form of each term in (10), and apply Lemma 4.1. Then the intended inequality (10) becomes:

$$\begin{aligned}
&\sum_{k=1}^{K}\sum_{j=1}^{n}\left[(v_{(j)}^k)^{\mathrm{T}}L_{\rho_j}q_{(j)}^k - \sum_{i=1}^{K}\gamma_j^{ki}(v_{(j)}^k)^{\mathrm{T}}L_{\rho_j}w_{(j)}^i - \gamma_j^{kk}(v_{(j)}^k)^{\mathrm{T}}(v_{(j)}^k - w_{(j)}^k) + \gamma_j^{kk}(v_{(j)}^k - w_{(j)}^k)^{\mathrm{T}}(v_{(j)}^k - w_{(j)}^k)\right] \\
&\quad - \sum_{k=1}^{K}\sum_{l=1}^{m}\left[(x_{(l)}^k)^{\mathrm{T}}L_{\alpha_l}b_{(l)}^k + \sum_{i=1}^{K}\beta_l^{ki}(x_{(l)}^k)^{\mathrm{T}}L_{\alpha_l}y_{(l)}^i + \beta_l^{kk}(x_{(l)}^k)^{\mathrm{T}}(x_{(l)}^k - y_{(l)}^k) - \beta_l^{kk}(x_{(l)}^k - y_{(l)}^k)^{\mathrm{T}}(x_{(l)}^k - y_{(l)}^k)\right] \\
\geq\ & \lambda\sum_{k=1}^{K}\sum_{j=1}^{n}\left[(v_{(j)}^k)^{\mathrm{T}}L_{\rho_j}q_{(j)}^k - \sum_{i=1}^{K}\gamma_j^{ki}(v_{(j)}^k)^{\mathrm{T}}L_{\rho_j}v_{(j)}^i\right] - \lambda\sum_{k=1}^{K}\sum_{l=1}^{m}\left[(x_{(l)}^k)^{\mathrm{T}}L_{\alpha_l}b_{(l)}^k + \sum_{i=1}^{K}\beta_l^{ki}(x_{(l)}^k)^{\mathrm{T}}L_{\alpha_l}x_{(l)}^i\right] \\
&\quad - \mu\sum_{t=1}^{T}\sum_{k=1}^{K}\left[\sum_{j=1}^{n}\gamma_j^{kk}(w_{jt}^k)^2 + \sum_{l=1}^{m}\beta_l^{kk}(y_{lt}^k)^2\right].
\end{aligned}$$

To simplify, we let $\lambda = 1$. Then, the desired inequality becomes:

$$\begin{aligned}
&\sum_{j=1}^{n}\sum_{k=1}^{K}\left[\sum_{i=1}^{K}\gamma_j^{ki}(v_{(j)}^k)^{\mathrm{T}}L_{\rho_j}(v_{(j)}^i - w_{(j)}^i) - \gamma_j^{kk}(w_{(j)}^k)^{\mathrm{T}}(v_{(j)}^k - w_{(j)}^k) + \mu\gamma_j^{kk}(w_{(j)}^k)^{\mathrm{T}}(w_{(j)}^k)\right] \\
&+ \sum_{l=1}^{m}\sum_{k=1}^{K}\left[\sum_{i=1}^{K}\beta_l^{ki}(x_{(l)}^k)^{\mathrm{T}}L_{\alpha_l}(x_{(l)}^i - y_{(l)}^i) - \beta_l^{kk}(y_{(l)}^k)^{\mathrm{T}}(x_{(l)}^k - y_{(l)}^k) + \mu\beta_l^{kk}(y_{(l)}^k)^{\mathrm{T}}y_{(l)}^k\right] \geq 0.
\end{aligned} \tag{12}$$

Next, we set $\mu = KT - 1$. Then, the first term on the left hand side of (12) can be rewritten as

$$\sum_{j=1}^{n}\sum_{k=1}^{K}\sum_{i=1}^{K}\gamma_j^{ki}(v_{(j)}^k - w_{(j)}^k)^{\mathrm{T}}L_{\rho_j}(v_{(j)}^i - w_{(j)}^i) + \sum_{j=1}^{n}\sum_{k=1}^{K}\sum_{i=1}^{K}\gamma_j^{ki}(w_{(j)}^k)^{\mathrm{T}}(L_{\rho_j} - I_T)v_{(j)}^i$$
$$+\sum_{j=1}^{n}\sum_{k=1}^{K}\left[KT\gamma_j^{kk}(w_{(j)}^k)^{\mathrm{T}}(w_{(j)}^k) - \sum_{i=1}^{K}\gamma_j^{ki}(w_{(j)}^k)^{\mathrm{T}}L_{\rho_j}w_{(j)}^i\right]. \tag{13}$$

Note that all of γ_j^{kk}, w_{jt}^k and v_{jt}^k are nonnegative, and $L_{\rho_j} - I_T \geq 0$, then

$$\sum_{j=1}^{n}\sum_{k=1}^{K}\sum_{i=1}^{K}\gamma_j^{ki}(w_{(j)}^k)^{\mathrm{T}}(L_{\rho_j} - I_T)v_{(j)}^i \geq 0.$$

Furthermore, $L_{\rho_j} - E_T \leq 0$ (i.e. $L_{\rho_j} - E_T$ is nonpositive componentwise), where E_T denotes the $T \times T$ all-one matrix. Thus,

$$\begin{aligned}
& \sum_{k=1}^{K}\left[KT\gamma_j^{kk}(w_{(j)}^k)^{\mathrm{T}}(w_{(j)}^k) - \sum_{i=1}^{K}\gamma_j^{ki}(w_{(j)}^k)^{\mathrm{T}}L_{\rho_j}w_{(j)}^i\right] \\
\geq\ & \sum_{k=1}^{K}\left[KT\gamma_j^{kk}(w_{(j)}^k)^{\mathrm{T}}(w_{(j)}^k) - \sum_{i=1}^{K}\gamma_j^{ki}(w_{(j)}^k)^{\mathrm{T}}E_T w_{(j)}^i\right] \\
=\ & (KT-1)\sum_{t=1}^{T}\sum_{k=1}^{K}\gamma_j^{kk}(w_{jt}^k)^2 - \sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{\tau=1}^{t-1}\sum_{i\neq k}\gamma_j^{ki}w_{j\tau}^i w_{jt}^k \\
\geq\ & \sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{\tau<t}\sum_{i<k}\left[\gamma_j^{kk}(w_{jt}^k)^2 - (\gamma_j^{ik} + \gamma_j^{ki})w_{j\tau}^i w_{jt}^k + \gamma_j^{ii}(w_{j\tau}^i)^2\right] \\
\geq\ & \sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{\tau<t}\sum_{i<k}\left(\sqrt{\gamma_j^{kk}}(w_{jt}^k) - \sqrt{\gamma_j^{ii}}(w_{j\tau}^i)\right)^2 \geq 0.
\end{aligned}$$

Therefore, we have

$$\text{The term (13)} \geq \sum_{j=1}^{n}\sum_{k=1}^{K}\sum_{i=1}^{K}\gamma_j^{ki}(v_{(j)}^k - w_{(j)}^k)^{\mathrm{T}}L_{\rho_j}(v_{(j)}^i - w_{(j)}^i).$$

Similarly, it can also be shown that

$$\text{The second term in (12)} \geq \sum_{l=1}^{m}\sum_{k=1}^{K}\sum_{i=1}^{K}\beta_l^{ki}(x_{(l)}^k - y_{(l)}^k)^{\mathrm{T}}L_{\alpha_l}(x_{(l)}^i - y_{(l)}^i).$$

To summarize, the left hand side of (12) is no less than

$$\sum_{k=1}^{K}\sum_{i=1}^{K}\left[\sum_{j=1}^{n}\gamma_j^{ki}(v_j^k - w_j^k)^{\mathrm{T}}L_{\rho_j}(v_j^i - w_j^i) + \sum_{l=1}^{m}\beta_l^{ki}(x_l^k - y_l^k)^{\mathrm{T}}L_{\alpha_l}(x_l^i - y_l^i)\right]. \tag{14}$$

Let

$$v_{(j)} = \left((v^1_{(j)})^{\mathrm{T}}, (v^2_{(j)})^{\mathrm{T}}, \ldots, (v^K_{(j)})^{\mathrm{T}}\right)^{\mathrm{T}}, v = \left((v_{(1)})^{\mathrm{T}}, (v_{(2)})^{\mathrm{T}}, \ldots, (v_{(m)})^{\mathrm{T}}\right)^{\mathrm{T}},$$

(similarly for w, x, y) and denote $\Gamma_j = (\gamma_j^{ki})_{K\times K}$, $\Xi_l = (\beta_l^{ki})_{K\times K}$, then the term (14) can be rewritten as

$$(v-w)^{\mathrm{T}}\mathrm{Diag}\left((L_{\rho_j} \otimes \Gamma_j)\right)(v-w) + (x-y)^{\mathrm{T}}\mathrm{Diag}\left((L_{\alpha_l} \otimes \Xi_l)\right)(x-y),$$

where $\mathrm{Diag}(\cdot)$ denotes the diagonal block matrix, whose diagonal block entries are the corresponding matrices. Furthermore, the second order derivative matrix for the social profit function is

$$\nabla^2 \mathrm{SV}(v,x) = \mathrm{sym}\left(\begin{pmatrix} \mathrm{Diag}\left((L_{\rho_j} \otimes \Gamma_j)\right) & 0 \\ 0 & \mathrm{Diag}\left((L_{\alpha_j} \otimes \Xi_l)\right) \end{pmatrix}\right).$$

Here 'sym' signifies the symmetric operation. Notice that (v,x) is the social maximum and that (w,y) is a feasible solution, and so

$$(v-w)^{\mathrm{T}}\mathrm{Diag}\left((L_{\rho_j} \otimes \Gamma_j)\right)(v-w) + (x-y)^{\mathrm{T}}\mathrm{Diag}\left((L_{\alpha_l} \otimes \Xi_l)\right)(x-y) \geq 0,$$

due to the second order optimality condition for (v,x). Putting all the pieces together, we have shown that (12) holds when $\lambda = 1$ and $\mu = KT - 1$. Consequently, we have $\mathrm{PoI} \geq \frac{1}{KT}$. □

This lower bound is essentially tight, as shown by the example below.

Example 4.4 Suppose there is only one kind of resource available to produce one kind of commodity. The technology-matrix-pairs for all the players are the same (M,N) and assume $M = N = (1)$. The unit cost function for each play to use the only resource is $c_t^k(x) = \sum_{i=1}^K x_t^i + c_{t-1}^k(x)$, with $c_0^k = 0$ and there is no competition for the price. Suppose that there are K identical players, with the price $p_t^k = 1 + p_{t-1}^k(v)$, with $p_1^k = 1 + \frac{1}{K}$ for all k. In this setting, there is only one myopic Nash equilibrium, at which each player will use $\frac{1}{K}$ amount of resources to produce the commodity to receive a profit of $\frac{1}{K^2}$ at each stage. Thus the total social profit over the whole period is $\frac{T}{K}$. On the other hand, consider a feasible strategy for the players: each player uses $\frac{1}{2K}$ amount of resources to produce the commodity at each stage. Then the profit for each player at stage t will be $\frac{t}{2} + \frac{1}{K}$. The total social profit will be larger than $K \cdot \sum_{t=1}^T \frac{t}{2} \cdot \frac{1}{2K} = \frac{T(T+1)}{8}$. Then the price of isolation in this game is at least $\frac{T/K}{T(T+1)/8} = \frac{8}{K(T+1)} = O(\frac{1}{KT})$.

Summarizing, we have:

Corollary 4.5 *Under our assumptions, if T and K are taken as fixed parameters of the dynamic Cournot oligopoly game, then*

$$\frac{1}{KT} \leq \mathrm{PoI}_{\mathscr{P}} \leq \frac{8}{K(T+1)}.$$

4.2 The Dynamic Bertrand Price Competition Game with Complemental Products

Unlike the assumptions on the Cournot competition model, in the Bertrand competition model it is assumed that the prices, rather than the production quantity, are the decision variables of the players. We show that if the products are complemental to each other, then the PoI is also in the magnitude of the number of players multiplied with the number of stages.

Our game setting is sketched as follows. Suppose that T stages are considered, and the production costs are ignored. Assume that there are K players and each of them sells one different product. Each player sets the selling price as the decision variable, and the demand for the product is dependent on its current price, the prices of the other products and its demand in the previous stage. We use p_t^k and d_t^k to denote the price and the demand for product k in stage t, respectively. Then d_t^k could be written as a function $d_t^k(p_t^k; p_t^{-k}; d_{t-1}^k)$.

Given the prices of the other products and the demand quantity at the previous stage, the selfish and greedy producer k maximizes the profit at stage t:

$$\max_{p_t^k \geq 0} \quad V_t^k(p_t^k; p_t^{-k}; p_{t-1}^k) = p_t^k d_t^k(p_t^k; p_t^{-k}; d_{t-1}^k).$$

Similarly, the social value is defined to be the summation of V_t^k: $\text{SV}(p) = \sum_{k=1}^K \sum_{t=1}^T V_t^k(p_t^k; p_t^{-k}; d_{t-1}^k)$. Let p^{NE} denote a myopic Nash solution, and p^* denote the socially optimal solution, then the PoI is $\text{PoI} = \frac{\text{SV}(p^{NE})}{\text{SV}(p^*)}$.

We reconsider the affine linear case where $d_t^k(p_t^k; p_t^{-k}; d_{t-1}^k)$ is an affine linear function. Namely, the demand function is in the following form

$$d_t^k(p_t^k; p_t^{-k}; d_{t-1}^k) = q_t^k - \gamma^{kk} p_t^k - \sum_{i \neq k} \gamma^{ki} p_t^i + \rho_{t-1} d_{t-1}^k,$$

where γ^{ki} and ρ_t are the factor-loading coefficients. Note that we assume the products are complemental, which is different from the Cournot case. In other words, we assume $\gamma^{ki} \geq 0$ for all k and i, which means that an increase in the prices of any other product does not result in an increase in the demand for the product. Furthermore, since the unfulfilled demand in the previous stages may be transferred to the subsequent stages, ρ_t should be nonnegative for all t. Furthermore, we again assume that $(\gamma^{ki})^2 \leq \gamma^{kk}\gamma^{ii}$ and $0 \leq \rho \leq 1$. Using the following vector notations: $p^k = (p_1^k, \ldots, p_T^k), q^k = (q_1^k, \ldots, q_T^k)$ and $d^k = (d_1^k, \ldots, d_T^k)$, we have

$$d^k = -\sum_{i=1}^K \gamma^{ki} L_\rho p^i + L_\rho q^k,$$

where L_ρ is defined similarly as in the previous subsection. Thus, a similar equation as (11) for the Cournot case results. Consequently, the same bound on the PoI is applicable in this case.

4.3 More Discussions

Note that we can bound the PoI in the Cournot game for the substitute products, and in the Bertrand game for the complementary products. How about the other combinations: complementary products in the Cournot game and substitute products in the Bertrand game? The following example shows that in both cases there is no bound on the PoI.

Example 4.6 Suppose we consider the Bertrand game over one stage. There are two players and two types of products. Note that we assume the products are substitutes, that is, for a given product, the increasing of the price of the other product will result in the increasing of the demand of the product. Therefore, γ^{ki} is nonpositive. We assume that both of them are -1. Then the demand function is given by $d^k = 1 - p^k + p^{3-k}$ for $k = 1, 2$. In this setting, there is only one myopic Nash equilibrium, at which each player will set the price to be $\frac{1}{3}$. Thus the total social profit is $\frac{2}{3}$. Nevertheless, $d^1 = d^2 = 1$ if $p^1 = p^2$. In other words, to optimize the social profit, we only need to ensure that the two prices are equal and then set the price arbitrarily high, making the social profit arbitrarily big as well, which means that the PoI has no bound in this case. Similar examples can be constructed for the Cournot game with complemental products.

5 Conclusions

In this paper, we study possible damages caused by the *selfish* and *greedy* nature of the decision makers. By a somewhat loose usage of the word 'isolation' we refer to some type of disconnectedness of a decision maker within the system, in terms of the lack of coordination with other players, as well as the lack of coordination with oneself over time. In plain language, the failure to coordinate with other players is also called *selfishness*; the failure to coordinate with oneself over time is known as *myopia* or *greediness*. Our aim here is to understand to what extent such usual behaviors *combined* can damage the performance of the whole system. The measurement for the loss of system efficiency is called the *Price of Isolation* (PoI) in this paper.

We investigated two types of framework for this study. The first one is a cost minimization model, where each player attempts to fulfill a desired throughput over time at a minimum cost, while competing for the deployment of a shared set of resources whose costs increase as the competition intensifies. A typical instance is a transportation game where the transporters compete to use the roads to complete their service orders. We proved in Section 3 that if the unit costs are affine linear functions then the PoI is precisely 4. The second framework describes a profit maximization game where K producers compete to sell their products in a common market, over a period of T stages. Two specific models, including Cournot and Bertrand oligopolistic competition games are considered. Assuming the unit costs of the resources and the unit prices of the products are all affine linear functions, if the products are substitutes, we proved in Section 4.1 that the PoI in Cournot game is in the order of $O\left(\frac{1}{KT}\right)$; if the products are complements, the same bound can be established in Bertrand game as well. The results can be summarized in Table 2.

Table 2: Summary of the Bounds

Models	PoM	PoI
Transportation Game	4	4
Cournot Game with Substitute Products	-	$O(1/KT)$
Bertrand Game with Complementary Products	-	$O(1/KT)$

Among the three models considered here, the first one shows that the percentage loss of the system efficiency is in a constant order, meaning that the cost of the selfish and greedy attitude will not deteriorate the system performance much further over time, even with many players and many stages played in the game. The second and the third model, however, depict a different picture. It says that as more profit-driven producers struggle to survive in the market, competing for a limited set of resources and on the sale prices as well, then the overall profit margin diminishes quickly (inversely proportional to the multiplicity of the number of players and the number of stages in the game). The insights gained from our study may help to shed some light to understand the nature of the competitive market: *How strong is the market force? What are the possible outcomes of a competitive market without regulation? When will the need for management and regulation arise?* Clearly, much more research efforts will be needed to fully understand those important issues.

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References

[1] S. Aland, D. Dumrauf, M. Gairing, B. Monien and F. Schoppmann, *Exact price of anarchy for polynomial congestion games*, In Proceedings of the 23rd Annual Symposium on Theoretical Aspects of Computer Science, 3884, pp. 218 – 229, 2006.

[2] E. Altman, T. Basar, T. Jimenez and N. Shimkin, *Competitive routing in networks with polynomial cost*, IEEE Trans. on Automatic Control, 47, pp. 92 – 96, 2002.

[3] B. Awerbuch, Y. Azar, and L. Epstein, *The price of routing unsplittable flow*, In *Proceedings of the thirty-seventh annual ACM Symposium on Theory of Computing*, pp. 57 – 66, 2005.

[4] Y. Bachrach, M. Zuckerman and J. Rosenschein, *Effort Games and the Price of Myopia*, Mathematical Logic Quarterly, 55, pp. 1 – 21, 2009.

[5] J. Bertrand, *Book review of theorie mathematique de la richesse sociale and of recherches sur les principles mathematiques de la theorie des richesses*, Journal de Savants, 67, pp. 499 – 508, 1883.

[6] S. Chawla and T. Roughgarden, *Bertrand competition in networks*, In Proceedings of the First International Symposium on Algorithmic Game Theory (SAGT), pp. 70 – 82, 2008.

[7] S. Chawla and F. Niu, *The price of anarchy in Bertrand games*, In Proceedings of ACM Conference on Electronic Commerce, pp. 305 – 314, 2009.

[8] G. Christodoulou and E. Koutsoupias, *The price of anarchy of finite congestion games.* In *Proceedings of the thirty-seventh annual ACM Symposium on Theory of Computing*, pp. 67 – 73, 2005.

[9] R. Cominetti, J. Correa and N. Stier-Moses, *The Impact of Oligopolistic Competition in Networks*, Operations Research, 57, pp. 1421 –1437, 2009.

[10] J. Correa, A. Schulz and N. Stier-Moses, *A Geometric Approach to the Price of Anarchy in Nonatomic Congestion Games*, Games and Economic Behavior, 64, pp. 457 – 469, 2008.

[11] R.W. Cottle, J.S. Pang, and R.E. Stone, *The Linear Complementarity Problem*, Academic Press, San Diego, 1992.

[12] A. Cournot, *Recherches sur les Principes Mathematiques de la Theorie des Richesses*, Paris: Hatchette,1838. English translation: Researches into the Mathematical Priciples of the Theory of Wealth, New York: Macmillan, 1897.

[13] D. Dumrauf and M. Gairing, *Price of Anarchy for Polynomial Wardrop Games*, In Proceedings of the Second International Workshop on Internet and Network Economics (WINE), 4286, pp. 319 – 330, 2006.

[14] B. Farzad, N. Olver and A. Vetta, *A Priority-Based Model of Routing*, Chicago Journal of Theoretical Computer Science, (1), 2008.

[15] E. Koutsoupias and C.H. Papadimitriou, *Worst-case equilibria.* In Proceedings of the 16th Annual Symposium on Theoretical Aspects of Computer Science, pp. 404 – 413, 1999.

[16] M. Gairing, B. Monien and K. Tiemann, *Routing (Un-)Splittable Flow in Games with Player-Specific Linear Latency Functions*, In Proceedings of the 33rd International Colloquium, ICALP, Part I, 4051, pp. 501 – 502, 2006.

[17] R. Gibbons, *A Primer in Game Theory*, Financial Times/Prentice Hall, 1992.

[18] X. Guo and H. Yang, *The Price of Anarchy of Cournot Oligopoly*, In Proceedings of the First International Workshop on Internet and Network Economics (WINE), 3828, pp. 246 – 257, 2005.

[19] T. Harks and L. Végh, *Nonadaptive Selfish Routing with Online Demands*, In Proceedings of Fourth Workshop on Combinatorial and Algorithmic Aspects of Networking, 4852, pp. 27 – 45, 2007.

[20] T. Harks, S. Heinz and M. Pfetsch. *Competitive Online Multicommodity Routing* Theory of Computing Systems 45, pp. 533 – 554, 2009.

[21] S. He, X. Wang and S. Zhang, *On a Generalized Cournot Oligopolistic Competition Game*, 2010. Accepted for publication in *Global Optimization*.

[22] R. Horn and C. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, 1991.

[23] N. Immorlica, E. Markakis and G. Piliouras, *Coalition Formation and Price of Anarchy in Cournot Oligopolies*, to be published in Proceedings of the Sixth International Workshop on Internet and Network Economics (WINE), 2010.

[24] J. Kluberg and G. Perakis, *Generalized quantity competition for multiple products and loss of efficiency*, In 46th Annual Allerton Conference on Communication, Control and Computing, pp. 930 – 936, 2008.

[25] M. Kopel and F. Szidarovszky, *Resource Dynamics under Partial Cooperation in an Oligopoly*, Journal of Optimization Theory and Applications, 128, pp. 393 – 410, 2006.

[26] G. Martín-Herrán and J. Rincón-Zapatero,*Efficient Markov perfect Nash equilibria: theory and application to dynamic fishery games*, Journal of Economic Dynamics and Control, 29, pp. 1073-1096, 2005.

[27] M. Mavronicolas and P. Spirakis, *The price of selfish routing*. In *Proceedings of the 33rd Annual ACM Symposium on the Theory of Computing*, pp. 510 – 519, 2001.

[28] D. Monderer and L.S. Shapley, *Potential Games*, Games and Economic Behavior, 14, pp. 124143, 1996.

[29] M. Mostagir, *Exploiting Myopic Learning*, to be published in Proceedings of the Sixth International Workshop on Internet and Network Economics (WINE), 2010.

[30] J.F. Nash, *Non-Cooperative Games*. Annals of Mathematics, 54 , pp. 286 – 295, 1951.

[31] N. Nisan, T. Roughgarden, É. Tardos and V.V. Vazirani *eds.*, *Algorithmic Game Theory*, Cambridge University Press, 2007.

[32] M. Osborne and A. Rubinstein, *A Course in Game Theory*, The MIT Press, 1994.

[33] G. Perakis, *The Price of Anarchy when Costs are Non-separable and Asymmetric*, Mathematics of Operations Research, Vol. 32, pp. 614 – 628, 2007.

[34] J.B. Rosen, *Existance and uniqueness of equilibrium points for concave N-person games*, Econometrica, 33, pp. 520 – 534, 1965.

[35] T. Roughgarden, *Selfish Routing and the Price of Anarchy*, MIT Press, 2005.

[36] T. Roughgarden, *The price of anarchy in networks with polynomial edge latency*, Technical Report TR2001-1847, Cornell University, 2001.

[37] T. Roughgarden, *The price of anarchy is independent of the network topology*, Journal of Computer and System Sciences, 67, pp. 341 – 364, 2003.

[38] T. Roughgarden and E. Tardos, *Bounding the inefficiency of equilibria in nonatomic congestion games*, Games and Economic Behavior, 47, pp. 389 – 403, 2002.

[39] T. Roughgarden, *Intrinsic robustness of the price of anarchy*. In *Proceedings of the 41st annual ACM Symposium on Theory of Computing*, pp. 513 – 522, 2009.

[40] L. Sandal and S. Steinshamn, *Dynamic Cournot-competitive harvesting of a common pool resource*, Journal of Economic Dynamics and Control, Volume 28, pp. 1781 – 1799, 2004.

[41] D. Sleator and R. Tarjan, *Amortized efficiency of list update and paging rules*, Communications of the ACM, 28, pp. 202 – 208, 1985.