On a Generalized Cournot Oligopolistic Competition Game

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Abstract

We consider a model whereby players compete for a set of shared resources to produce and sell substitute products in the same market, which can be viewed as a generalization of the classical Cournot oligopolistic competition model, or, from a different angle, the Wardrop type routing model. In particular, we suppose that there are K players, who compete for the usage of resources as well as the sales of the end-products. Moreover, the unit costs of the shared resources and the selling prices of the products are assumed to be affine linear functions in the consumption/production quantities. We show that the price of anarchy in this case is lower bounded by $1/K$, and this bound is essentially tight, which manifests the harsh nature of the competitive market for the producers.

Keywords: Cournot oligopoly competition, Nash equilibrium, price of anarchy.

1 Introduction

The research on the noncooperative/competitive games with congestion effect has formed a sizeable body of literature. Applications of such models range from computer networks, telecommunications, traffic networks, all the way to competitive economics. The core of the problem lies in the intrinsic externality, i.e. the conflict among the users of the resources: the action of one user (more conventionally, we shall adhere to the term *players* henceforth) affects the cost/profitability structure of all

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the others. For the background, we refer to [4, 20] for a survey, and [24] for a more updated and elaborated introduction.

A popular example of game where congestion plays a role is from traffic and routing. The first such model was due to Wardrop [32] (1952), and the aim was to study the traffic formation and congestions. In the Wardrop model each player controls an infinitesimal amount of flow, and the number of players is infinite. This introduces a natural notion of equilibrium, to be distinguished from the familiar Nash equilibrium.

Generally speaking, selfish behavior may socially be very inefficient. To explicitly quantify this matter, Koutsoupias and Papadimitriou [21] introduced the notion of the *Price of Anarchy* (PoA), which is the ratio between the social (total sum) value of the *worst* Nash equilibrium solution and the optimal social value. In the Wardrop case, Roughgarden [28] gave an upper bound for the PoA where the cost on each link is a polynomial function. Later, in [29] Roughgarden showed that the worst case occurs on a very simple network and hence the upper bound is tight, and in fact holds for virtually any network topology. Recently, Roughgarden [31] (also cf. Correa, Schulz and Stier-Moses [11]) proposed a new approach to obtain an upper bound for the PoA. The main point is that if a player adheres to the socially optimal strategy while other players adopt the selfish Nash strategies, then this player will of course be hurt due to the altruist behavior. However, if the damage can be controlled by a combination of the social values at the Nash equilibrium and at the true optimal social solution, then the PoA can be controlled as well. We shall use this technique to bound the PoA for our model to be introduced later.

An extension of the Wardrop model is to assume that there are only a finite number of players, each controlling a positive amount of traffic. In a sense, as the number of the players tends to infinity, the model then asymptotically assimilates the Wardrop one. In the case that the flow for each player is not splittable, Rosenthal [27] introduced a special class of the noncooperative game, known as the *congestion game*, and showed that the pure Nash equilibrium exists. Awerbuch, Azar, Epstein [5], and Christodoulou and Koutsouplas [7] showed that if the unit cost function is affine linear then the PoA is upper bounded by $\frac{5}{2}$. Gairing, Monien and Tiemann [14] considered the player specific latency functions and some potential functions, obtaining some upper and lower bounds on the PoA. If the flow is splittable, Dumauf and Gairing [13] used the notion of the Wardrop equilibrium to obtain upper and lower bounds on the PoA provided that the cost functions are polynomial. Cominetti, Correa and Stier-Moses [8] obtained an upper bound for the PoA in this setting if the cost function satisfies some convexity conditions. More references on the traffic routing problem can be found in [1, 2, 3, 6, 22, 23, 25, 30].

Another source of inspiration is from a fundamental economic model known as the Cournot oligopoly competition, which was proposed by Cournot in [12]. In the Cournot setting, the suppliers face the same market and choose their own production levels which affect the sales prices, to maximize their own profits. The competition among the producers may result in a congestion effect: the more commodities produced, the less unit revenue one can expect. Guo and Yang [15] employed the total surplus of consumers and producers as the social welfare and achieved bounds on the PoA which are dependent on the market shares and demand, and the number of players. Immorlica, Markakis and Pilouras [17] studied coalition formation in a dynamic setting of Cournot oligopolies and proved that the PoA under their notion of stability is bounded by $\Theta(K^{2/5})$, where K is the number of players. Based on the work [26] by Perakis (which was enhanced by Han, Sun and Ang [16] by incorporating the loss of efficiency under a framework of asymmetric costs), Kluberg and Perakis [19] extended the classic Cournot model to an asymmetric case for both substitute and complement products and bounded the PoA through the market power parameters and the number of players and products. One may find more information on the topic in [9, 10, 18].

This paper considers an extended model of Cournot. Since the resources to manufacture the products could be also viewed as upstream commodities in the real market, production cost increases due to more demand for the resources. Therefore, in addition to the competition in the market in terms of the sales, we shall also consider the competition on the resources. Furthermore, for each producer, we use a technology matrix to indicate the productivity of the producer and characterize the relationship between the resources and the products. Specifically, we consider the case where finitely many noncooperative players compete for a set of shared resources, and then attempt to sell the produced products in the market. We take the sum of the profit of all the players as the social value and our aim is to study the loss of the social efficiency caused by the competition. In particular, we investigate the Nash equilibrium of the Cournot game involving K players and obtain some bounds on the PoA in such games. The paper is organized as follows. In the next section, the model and the notations will be introduced. In Section 3, we show that the PoA is actually lower bounded by $\frac{1}{K}$, with K being the total number of players. Note that this bound is solely dependent on K, but not any other parameters in the model. We confine ourselves to the case where the competition will affect the unit cost of the shared resources, as well as the price of the products, in an affine linear manner. In Section 4, an example is given to show that this bound is essentially tight.

2 Model and Notations

Suppose there are m given types of resources that are used to produce n types of products. The relationship between the resources and the products can be characterized by a technology matrix, denoted by M. Namely, if one uses a nonnegative vector $x \in \Re^m_+$ to denote the usage of resources and $v \in \Re^n_+$ to denote the quantity of products being produced from the resources, we then have $Mv \leq x$. Suppose there are K producers (viewed as players in the framework of a game) who are in the business of making use of the resources to produce the products and earn their profits via the sales of the products. On the one hand, the cost of each type of resource increases as the usage of the resource increases. For resource l, we denote player k's usage of the resource as x_l^k, and the total usage of the resource as $f_l = \sum_{k=1}^K x_l^k$. Then, the *unit* cost for the usage of l is given by a function $c_l : f_l \hookrightarrow c_l(f_l)$. On the other hand, players earn profits from sales of the products. However, the price of each type of product is dependent on the amount of supply of similar products in the market. Suppose that the amount of products produced by player k is v^k, and v^{-k} is the decision of other players. The price vector of player k is assumed to be a vector $p^k(v^k, v^{-k}) \in \Re_+^n$. Note the technology matrix for player k is M_k. To maximize the profit, player k shall consider the following optimization problem:

$$
\begin{array}{lll}
(P_k) & \max & V^k(v^k, x^k; v^{-k}, x^{-k}) = (v^k)^{\mathrm{T}} p^k(v^k, v^{-k}) - (x^k)^{\mathrm{T}} c^k(x^k, x^{-k}) \\
 & \text{s.t.} & M_k v^k \le x^k,\ x^k \ge 0, v^k \ge 0.
\end{array}
$$

Naturally, given the decisions of all the players, the social value is a simple summation: $\mathrm{SV}(v, x) = \sum_{k=1}^K V^k(v^k, x^k; v^{-k}, x^{-k})$. Let us denote (w, y) to be the solution when the game reaches a Nash equilibrium; i.e. a solution at which no player will be able to improve his/her situation unilaterally. At the same time, let us denote (v, x) to be the socially optimal solution – the solution that maximizes the social function SV over all feasible solutions. The so-called *Price of Anarchy* (PoA) is defined as: $\mathrm{PoA} = \frac{\mathrm{SV}(w,y)}{\mathrm{SV}(v,x)}$.

3 Bounding the Price of Anarchy

In this section we shall consider the case where the unit costs for the usage of resources are affine linear in the total usages, and the unit selling prices are also affine linear in the total supply. In particular, suppose that the total usage of resource l is f_l, then the *unit cost* of resource l is $c_l(f_l) = a_l f_l + b_l$, where $a_l, b_l \ge 0$ are some constant parameters. Moreover, we assume the n types of products are uncorrelated, however the same type of products produced by different players are substitute to each other. Suppose player k produces v^k while other players produce v^{-k}, then the price for product j applicable to player k is $p^k(v^k, v^{-k})_j = q_j^k - \sum_{i=1}^K \gamma_j^{ki} v_j^i$, where q_j^k and γ_j^{ki} are nonnegative constants. In matrix notation we may write $p^k(v^k, v^{-k}) = q^k - \sum_{i=1}^K \Gamma^{ki} v^i$, where $\Gamma^{ki} = \mathrm{Diag}(\gamma_1^{ki}, \cdots, \gamma_n^{ki})$ for the fixed k, i. Here for a vector $v \in \Re^m$, $\mathrm{Diag}(v)$ denotes the $m \times m$ diagonal matrix whose l-th diagonal is v_l for $l = 1, ..., m$. Technically we assume that $(\gamma_j^{ki})^2 \le \gamma_j^{kk} \gamma_j^{ii}$ for all $1 \le i, k \le K$ and $1 \le j \le n$. (One may interpret this condition as to say that the effect on the price due to the actions of the other players is less significant than the effect of the action of oneself). Moreover, we also assume that $\gamma_j^{kk} > 0$ for all j, k, meaning that one's production of a certain product will affect his/her own sales of the same product. Player k will face the following optimization problem:

$$
\begin{array}{lll}
(\tilde{P}_k) & \max & \sum_{j=1}^n (q_j^k - \sum_{i=1}^K \gamma_j^{ki} v_j^i) v_j^k - \sum_{l=1}^m (a_l f_l + b_l) x_l^k \\
 & \text{s.t.} & M_k v^k \le x^k,\ x^k \ge 0, v^k \ge 0.
\end{array}
$$

Notice that the objective function for each player is concave and coercive in v when the actions of other players are fixed. The boundedness of the solutions follows from this and the fact that the feasible region is a polyhedron, which implies that Nash equilibrium exists. Letting z^k be the Lagrangian dual variable for the constraint $M_k v^k - x^k \le 0$, and s^k and t^k be the dual variables for the constraint $x^k \ge 0$ and $v^k \ge 0$ respectively, the overall optimality condition (or the equilibrium condition) is the following LCP system:

$$\begin{cases} M_k v^k - x^k \le 0,\ z^k \ge 0, \\ (z^k)^{\mathrm{T}}(M_k v^k - x^k) = 0, \\ \mathrm{Diag}(a)\sum_{i=1}^K x^i + \mathrm{Diag}(a)x^k - z^k - s^k = -b, \\ \sum_{i=1}^K \Gamma^{ki} v^i + \Gamma^{kk} v^k + M_k^{\mathrm{T}} z^k - t^k = q^k, \\ (x^k)^{\mathrm{T}} s^k = 0,\ x^k \ge 0,\ s^k \ge 0, \\ (v^k)^{\mathrm{T}} t^k = 0,\ v^k \ge 0,\ t^k \ge 0, \end{cases}$$

Lemma 3.1 *At a Nash equilibrium of the extended Cournot competition game with solution* (w, y)*, the profit of player* k *in the equilibrium equals to* $\sum_{j=1}^n \gamma_j^{kk}(w_j^k)^2 + \sum_{l=1}^m a_l (y_l^k)^2$.

Proof. Since each player at the Nash equilibrium attains optimality given the strategies of the others, by the KKT condition we have

$$(z^k)^{\mathrm{T}} M_k w^k = (z^k)^{\mathrm{T}} y^k,\ q^k - \sum_{i=1}^K \Gamma^{ki} w^i = \Gamma^{kk} w^k + M_k^{\mathrm{T}} z^k - t^k,\ \mathrm{Diag}(a)\sum_{i=1}^K y^i + \mathrm{Diag}(a) y^k + b - s^k = z^k.$$

Hence the profit for player k is

$$\begin{aligned} & \left(q^k - \sum_{i=1}^K \Gamma^{ki} w^i\right)^{\mathrm{T}} w^k - \left(\sum_{i=1}^K \mathrm{Diag}(a) y^i + b\right)^{\mathrm{T}} y^k \\ = \ & (\Gamma^{kk} w^k + M_k^{\mathrm{T}} z^k - t^k)^{\mathrm{T}} w^k - \left(\sum_{i=1}^K \mathrm{Diag}(a) y^i + b\right)^{\mathrm{T}} y^k \\ = \ & (w^k)^{\mathrm{T}} \Gamma^{kk} w^k + (z^k)^{\mathrm{T}} y^k - \left(\sum_{i=1}^K \mathrm{Diag}(a) y^i + b\right)^{\mathrm{T}} y^k \\ = \ & (w^k)^{\mathrm{T}} \Gamma^{kk} w^k + \left(\mathrm{Diag}(a)\sum_{i=1}^K y^i + \mathrm{Diag}(a) y^k + b - s^k\right)^{\mathrm{T}} y^k - \left(\mathrm{Diag}(a)\sum_{i=1}^K y^i + b\right)^{\mathrm{T}} y^k \\ = \ & (w^k)^{\mathrm{T}} \Gamma^{kk} w^k + (y^k)^{\mathrm{T}} \mathrm{Diag}(a) y^k = \sum_{j=1}^n \gamma_j^{kk} (w_j^k)^2 + \sum_{l=1}^m a_l (y_l^k)^2. \end{aligned}$$

□

Lemma 3.2 *Denote (v, x) and (w, y) to be the solutions at the social optimum and at a Nash equilibrium respectively. At the Nash equilibrium (w, y), suppose that player k switches to the strategy to (v^k, x^k) while all other players' strategies remain unchanged. Then, the profit of player k will decrease by at least an amount of $\sum_{j=1}^n \gamma_j^{kk}(v_j^k - w_j^k)^2 + \sum_{l=1}^m a_l(x_l^k - y_l^k)^2$; that is,*

$$V^k(w^k, y^k; w^{-k}, y^{-k}) - V^k(v^k, x^k; w^{-k}, y^{-k}) \geq \sum_{j=1}^n \gamma_j^{kk}(v_j^k - w_j^k)^2 + \sum_{l=1}^m a_l(x_l^k - y_l^k)^2.$$

Proof. By the definition of the Nash equilibrium, (w^k, y^k) is maximal for player k's profit function $V^k(\cdot, \cdot; w^{-k}, y^{-k})$, assuming the other players' strategies are fixed as (w^{-k}, y^{-k}). Therefore,

$$\begin{aligned}
& V^k(w^k, y^k; w^{-k}, y^{-k}) - V^k(v^k, x^k; w^{-k}, y^{-k}) - \sum_{j=1}^n \gamma_j^{kk}(v_j^k - w_j^k)^2 - \sum_{l=1}^m a_l(x_l^k - y_l^k)^2 \\
= \;& (q^k)^{\mathrm{T}}(w^k - v^k) + \sum_{j=1}^n \left[\sum_{i \neq k} \gamma_j^{ki} w_j^i v_j^k + \gamma_j^{kk}(v_j^k)^2 - \sum_{i \neq k} \gamma_j^{ki} w_j^i w_j^k - \gamma_j^{kk}(w_j^k)^2 - \gamma_j^{kk}(v_j^k - w_j^k)^2 \right] \\
& - \sum_{l=1}^m \left\{ \left[a_l(y_l^k + \sum_{i \neq k} y_k^i) + b_l \right] y_l^k - \left[a_l(x_l^k + \sum_{i \neq k} y_k^i) + b_l \right] x_l^k + a_l(x_l^k - y_l^k)^2 \right\} \\
= \;& \sum_{j=1}^n \left(q_j^k - \sum_{i \neq k} \gamma_j^{ki} w_j^i - 2\gamma_j^{kk} w_j^k \right)(w_j^k - v_j^k) - \sum_{l=1}^m \left(2a_l y_l^k + a_l \sum_{i \neq k} y_k^i + b_l \right)(y_l^k - x_l^k) \\
= \;& \nabla V^k(w^k, y^k; w^{-k}, y^{-k})^{\mathrm{T}} \begin{pmatrix} w^k - v^k \\ y^k - x^k \end{pmatrix} \geq 0,
\end{aligned}$$

where the last inequality is due to the fact that (w^k, y^k) is maximal. (Note that the inequality follows from the optimality condition; that is, if x^* is the maximal point of a concave function $q(x)$ over a convex set S, then $\nabla q(x^*)^{\mathrm{T}}(x^* - x) \geq 0$ for all $x \in S$). □

Before proceeding, let us note the following property concerning monotone matrices.

Lemma 3.3 *Consider a matrix $\Omega = (\alpha^{ij})_{K \times K}$. Suppose that the diagonal elements of Ω are non-negative, and $(\alpha^{ij})^2 \leq \alpha^{ii}\alpha^{jj}$ for all $1 \leq i, j \leq K$. Then,*

$$\bar{\Omega} := \begin{bmatrix} (K-1)\alpha^{11} & -|\alpha^{12}| & \dots & -|\alpha^{1K}| \\ \dots & \dots & \dots & \dots \\ -|\alpha^{K1}| & -|\alpha^{K2}| & \dots & (K-1)\alpha^{KK} \end{bmatrix}$$

is monotone.

Proof. Take any $\xi = (\xi^1, \dots, \xi^K)^{\mathrm{T}} \in \Re^K$. We have

$$\begin{aligned}
\xi^{\mathrm{T}}\bar{\Omega}\xi &= (\xi^1, \dots, \xi^K) \begin{bmatrix} (K-1)\alpha^{11} & -|\alpha^{12}| & \dots & -|\alpha^{1K}| \\ \dots & \dots & \dots & \dots \\ -|\alpha^{K1}| & -|\alpha^{K2}| & \dots & (K-1)|\alpha^{KK}| \end{bmatrix} \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^K \end{pmatrix} \\
&= \sum_{i=1}^{K} \left[(K-1)\alpha^{ii}(\xi^i)^2 - \sum_{j<i} (|\alpha^{ij}| + |\alpha^{ji}|)\, \xi^i \xi^j \right] \\
&= \sum_{i=1}^{K} \sum_{j<i} \left[\alpha^{ii}(\xi^i)^2 - (|\alpha^{ij}| + |\alpha^{ji}|)\, \xi^i \xi^j + \alpha^{jj}(\xi^j)^2 \right] \ge \sum_{i=1}^{K} \sum_{j<i} \left(\sqrt{\alpha^{ii}}\, |\xi^i| - \sqrt{\alpha^{jj}}\, |\xi^j| \right)^2 \ge 0.
\end{aligned}$$

Therefore, $\bar{\Omega}$ is a monotone matrix as asserted. □

Theorem 3.4 *The price of anarchy in the extended Cournot game with K players, is lower bounded by $\frac{1}{K}$.*

Proof. As before, let (v, x) and (w, y) denote the solutions at the social optimum and at a Nash equilibrium respectively. According to Lemma 3.2, to get a lower bound for PoA, it suffices to find a (λ, μ) pair with $\lambda > 0$ and $\mu > -1$, such that

$$\sum_{k=1}^{K} \left[V^k(v^k, x^k; w^{-k}, y^{-k}) + \sum_{j=1}^{n} \gamma_j^{kk}(v_j^k - w_j^k)^2 + \sum_{l=1}^{m} a_l (x_l^k - y_l^k)^2 \right] \ge \lambda\, \mathrm{SV}(v, x) - \mu\, \mathrm{SV}(w, y). \quad (1)$$

Note that if (1) holds, then by Lemma 3.2 we have

$$\begin{aligned}
\mathrm{SV}(w, y) &= \sum_{k=1}^{K} V^k(w^k, y^k; w^{-k} y^{-k}) \\
&\ge \sum_{k=1}^{K} \left[V^k(v^k, x^k; w^{-k}, y^{-k}) + \sum_{j=1}^{n} \gamma_j^{kk}(v_j^k - w_j^k)^2 + \sum_{l=1}^{m} a_l (x_l^k - y_l^k)^2 \right] \\
&\ge \lambda\, \mathrm{SV}(v, x) - \mu\, \mathrm{SV}(w, y).
\end{aligned}$$

Thus the PoA can be bounded as: $\mathrm{PoA} = \frac{\mathrm{SV}(w,y)}{\mathrm{SV}(v,x)} \ge \frac{\lambda}{1+\mu}$.

Let us now turn to searching (λ, μ) to satisfy (1). Denote f^x (respectively, f^y) to be the total usage of the resources when the game attains the social optimum (respectively, Nash equilibrium); i.e., $f_l^x = \sum_{k=1}^{K} x_l^k$, (respectively, $f_l^y = \sum_{k=1}^{K} y_l^k$). Substitute the explicit form of the cost functions in (1), and apply Lemma 3.1, then the intended inequality (1) becomes:

$$\sum_{k=1}^{K}\sum_{j=1}^{n}\left[q_j^k v_j^k - \sum_{i\neq k}\gamma_j^{ki} w_j^i v_j^k - \gamma_j^{kk}(v_j^k)^2 + \gamma_j^{kk}(v_j^k - w_j^k)^2\right]$$
$$-\sum_{k=1}^{K}\sum_{l=1}^{m}\left[a_l(x_l^k + f_l^y - y_l^k)x_l^k + b_l x_l^k - a_l(x_l^k - y_l^k)^2\right] - \lambda\sum_{k=1}^{K}\sum_{j=1}^{n}(q_j^k v_j^k - \sum_{i=1}^{K}\gamma_j^{ki} v_j^i v_j^k)$$
$$+\lambda\sum_{l=1}^{m}\left[a_l(f_l^x)^2 + b_l f_l^x\right] + \mu\sum_{k=1}^{K}\left[\sum_{j=1}^{n}\gamma_j^{kk}(w_j^k)^2 + \sum_{l=1}^{m}a_l(y_l^k)^2\right] \geq 0. \tag{2}$$

Observe that the left-hand side of the inequality (2) can be regrouped into two parts: Part I (see (3)) and Part II (see (6)) to be introduced below:

$$\text{'Part I'} = \lambda\sum_{l=1}^{m}\left[a_l(f_l^x)^2 + b_l f_l^x\right] + \mu\sum_{l=1}^{m}\sum_{k=1}^{K}a_l(y_l^k)^2 - \sum_{l=1}^{m}\left[a_l f_l^x f_l^y + a_l\sum_{k=1}^{K}[y_l^k(x_l^k - y_l^k)] + b_l f_l^x\right]. \tag{3}$$

Let us set $\lambda = 1$ and $\mu = K-1$, and the above can be further written as

$$\begin{aligned}\text{'Part I'} &= \sum_{l=1}^{m}a_l\left[(f_l^x)^2 + (\mu+1)\sum_{k=1}^{K}(y_l^k)^2 - f_l^x f_l^y - \sum_{k=1}^{K}x_l^k y_l^k\right]\\ &= \sum_{l=1}^{m}a_l(f_l^x - f_l^y)^2 + \sum_{l=1}^{m}a_l\left[(\mu+1)\sum_{k=1}^{K}(y_l^k)^2 - (f_l^y)^2 + f_l^x f_l^y - \sum_{k=1}^{K}x_l^k y_l^k\right].\end{aligned} \tag{4}$$

For any $1 \leq l \leq m$ we have $f_l^x f_l^y - \sum_{k=1}^K x_l^k y_l^k = (\sum_{k=1}^K x_l^k)(\sum_{k=1}^K y_l^k) - \sum_{k=1}^K x_l^k y_l^k \geq 0$, since all x_l^k's and y_l^k's are nonnegative. Moreover, $(\mu+1)\sum_{k=1}^K (y_l^k)^2 - (f_l^y)^2 \geq 0$ when $\mu = K-1$, due to the Cauchy-Schwartz inequality. Therefore,

$$\text{'Part I'} \geq \sum_{l=1}^{m}a_l(f_l^x - f_l^y)^2. \tag{5}$$

Now that $\lambda = 1$ and $\mu = K-1$, the second part of (2) becomes

$$\begin{aligned}\text{'Part II'} &= \sum_{k=1}^{K}\sum_{j=1}^{n}\left[-\sum_{i\neq k}\gamma_j^{ki}w_j^i v_j^k + \gamma_j^{kk}(w_j^k - v_j^k)^2\right] + \sum_{k=1}^{K}\sum_{j=1}^{n}\sum_{i\neq k}\gamma_j^{ki}v_j^i v_j^k + (K-1)\sum_{k=1}^{K}\sum_{j=1}^{n}\gamma_j^{kk}(w_j^k)^2\\ &= \sum_{k=1}^{K}\sum_{j=1}^{n}\left[\gamma_j^{kk}(v_j^k - w_j^k)^2 + \sum_{i\neq k}\gamma_j^{ki}(v_j^i - w_j^i)v_j^k + (K-1)\gamma_j^{kk}(w_j^k)^2\right],\end{aligned} \tag{6}$$

or equivalently,

$$\begin{aligned}\text{'Part II'} &= \sum_{j=1}^{n}\sum_{k=1}^{K}\sum_{i=1}^{K}\gamma_j^{ki}(v_j^i - w_j^i)(v_j^k - w_j^k) + \sum_{j=1}^{n}\sum_{k=1}^{K}\sum_{i\neq k}\gamma_j^{ki}v_j^i w_j^k\\ &\quad + \sum_{j=1}^{n}\left[(K-1)\sum_{k=1}^{K}\gamma_j^{kk}(w_j^k)^2 - \sum_{k=1}^{K}\sum_{i\neq k}\gamma_j^{ki}w_j^i w_j^k\right].\end{aligned} \tag{7}$$

Observing the independence among the commodities, the terms in (7) can be bounded for each j: the second term is nonnegative since $\gamma_j^{ki}, v_j^i, w_j^k \geq 0$; the sum of the third and forth terms, is nonnegative, thanks to Lemma 3.3. Thus, we have

$$\text{‘Part II’} \geq \sum_{j=1}^{n}\sum_{k=1}^{K}\sum_{i=1}^{K}\gamma_j^{ki}(v_j^i - w_j^i)(v_j^k - w_j^k). \tag{8}$$

Combining the two inequalities (5) and (8), we have

$$\begin{aligned}\text{‘Part I’} + \text{‘Part II’} \quad &\geq \quad \sum_{j=1}^{n}\sum_{k=1}^{K}\sum_{i=1}^{K}\gamma_j^{ki}(v_j^i - w_j^i)(v_j^k - w_j^k) + \sum_{l=1}^{m} a_l (f_l^x - f_l^y)^2 \\ &= \quad -\begin{pmatrix} v - w \\ x - y \end{pmatrix}^{\mathrm{T}} \nabla^2 \mathrm{SV}(v,x) \begin{pmatrix} v - w \\ x - y \end{pmatrix}.\end{aligned}$$

Notice that (v, x) is the social maximum and that (w, y) is a feasible solution; therefore, this term is nonnegative due to the second order optimality condition. It means that if we let $\lambda = 1$ and $\mu = K - 1$, then (2) holds (consequently (1) holds), implying PoA $\geq \frac{1}{K}$. □

4 Tightness of the Bound

The lower bound in the previous section may at first appear to be quite loose. However, it is essentially tight, as our next example shows.

Example 4.1 Assume all the resources are free and there is only one kind of commodity to produce. Suppose that there are K identical players, with the price $p^k = 1 + \frac{1}{K} - \sum_{i=1}^{K} v^i$ for all k, where v^i is the production level of player i. Note that in the Nash equilibrium, each player will produce $\frac{1}{K}$. The profit for each player is $\left(1 + \frac{1}{K}\right) \cdot \frac{1}{K} - \frac{1}{K} = \frac{1}{K^2}$, and the total social profit is $\frac{1}{K}$. One can easily compute that the social optimal solution is to produce a total of $\frac{1}{2}\left(1 + \frac{1}{K}\right)$ units of product. This yields the total social value of $\frac{1}{4}\left(1 + \frac{1}{K}\right)^2$. Thus, the corresponding PoA $= \frac{\frac{1}{K}}{\frac{1}{4}\left(1+\frac{1}{K}\right)^2} = \frac{4K}{K^2+2K+1}$. This example shows that the bound on PoA in Theorem 3.4 is essentially tight.

The tightness of the bound suggests that the outcome at the Nash equilibrium gets increasingly inefficient as the number of players increases in the game: the PoA is inversely proportional to the number of players in the game. Our numerical simulation tests confirm this result. In a certain sense, the result also helps to illustrate how the ‘market force’ comes into existence, from the consumers’ point of view.

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