Polymatroid Optimization, Submodularity, and Joint Replenishment Games

Simai He* Jiawei Zhang† Shuzhong Zhang‡

June 30, 2011

Abstract

In this paper we consider the problem of maximizing a separable concave function over a polymatroid. More specifically, we study the submodularity of its optimal objective value in the parameters of the objective function. This question is interesting in its own right and is encountered in many applications. But our research has been mainly motivated by a cooperative game associated with the well-known joint replenishment model. By applying our general results on polymatroid optimization, we prove that this cooperative game is submodular (i.e. its characteristic cost function is submodular), if the joint setup cost is a normalized and non-decreasing submodular function. Furthermore, the same result holds true for a more general one-warehouse multiple retailer game, which affirmatively answers an open question posed by Anily and Haviv [1].

Key Words: Polymatroid Optimization, Separable Concave Function, Cooperative Games, Joint Replenishment Problem

*Department of Management Sciences, City University of Hong Kong, Kowloon Tong, Hong Kong. Email: simaihe@cityu.edu.hk. This work was partly supported by Hong Kong RGC Earmarked Grants (GRF) 143711.

†Department of Information, Operations, and Management Sciences, Stern School of Business, New York University, New York, USA. Email: jzhang@stern.nyu.edu. This work was partly supported by National Science Foundation Grant CMMI-0654116.

‡Industrial and Systems Engineering Program, University of Minnesota, Minneapolis, MN 55455. Email: zhangs@umn.edu (On leave from Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Shatin, Hong Kong.)

1 Introduction

In recent years, many companies have come to realize that their performance can be improved significantly by exploring innovative collaborative strategies in supply chain management. Companies can collaborate in many different ways. For example, shippers that make small, frequent less-than-truckload shipments can collaborate and consolidate their orders into full truckloads. It has been reported that such collaboration among shippers leads to significant reduction in transportation cost as well as inventory cost. It is also known that inventory pooling is an effective way to reduce safety stock and increase customer service [4, 13]. Thus, some companies collaborate by sharing their inventories. The cooperation usually takes the form of lateral transshipment from a location with a surplus of on-hand inventory to a location that faces a stockout.

One issue in such collaboration is keeping different parties motivated to collaborate. The willingness to collaborate often depends on the existence of mechanisms that allocate the cost or gain (from the collaboration) in such a way that is considered advantageous by all the participants. Even though collaboration often leads to overall cost reduction, it is not always the case that such mechanisms exist. Indeed, getting all parties to agree on how to share costs and benefits was identified by some as one of the major barriers to collaborative commerce in practice (see [4, 13]).

It is natural to apply cooperative game theory to analyze cost allocation issues. Indeed, supply chain collaborations have motivated more and more studies on cooperative games in the last few years; see Nagarajan and Sosic [14] for an excellent review in this area.

Our paper is motivated by a cooperative game that is associated with the well-known joint replenishment model. In this model, there are multiple retailers which sell a single product. Constant customer demand occurs at each retailer over an infinite time horizon. The retailers replenish their inventories by ordering from an external supplier. There are two types of costs: a holding cost charged against each unit of inventory per unit time at each retailer, and a setup cost charged against each order that is a *submodular* function of the set of retailers that places the order together.[1] The lead times are assumed to be zero, i.e., orders are delivered instantaneously. The goal of the model is to find an inventory replenishment policy for the system that minimizes the

[1] We shall define submodularity in Section 2. Roughly speaking, it captures the notion of decreasing marginal cost. For examples of submodular setup cost functions, we refer interested readers to Federgruen and Zheng [7].

long-run average cost over an infinite time horizon. The optimal policy for this joint replenishment problem is unknown. However, it is well-known that a class of easy-to-implement policies, called power-of-two policies, are 98% effective; see Roundy [19] and Federgruen and Zheng [7].

We assume that the retailers follow an optimal power-of-two policy to replenish their inventories. We are interested in the question of how the system-wide cost should be allocated among the retailers. A proper cost allocation scheme is important particularly when the retailers belong to different firms or are decentralized divisions of an organization. For this purpose, we formulate a cooperative game (in coalitional form) denoted by (N, V) where the grand coalition N is the set of all retailers, and for any subset $S \subseteq N$, the characteristic cost function $V(S)$ is the system-wide cost under an optimal power-of-two policy when the system consists only of retailers in S[2]. We call this cooperative game the joint replenishment game.

The theoretical question that we would like to address regarding the joint replenishment game is whether the characteristic cost function $V(\cdot)$ is submodular or not. If the answer is yes, then the joint replenishment game is submodular. This question is of particular importance since a submodular game has many nice properties. We mention a few of them below. *First*, if $V(\cdot)$ is submodular, then the grand coalition is stable (Shapley [21]). That is, there exists a cost allocation under which no group of retailers would be better off by deviating from the grand coalition and acting alone. Such an allocation is often called a core allocation. *Second*, if $V(\cdot)$ is submodular, then there exist efficient (polynomial time) algorithms to find a core allocation and check whether a given allocation is a core allocation or not (Topkis [24], page 227). This is important because for a non-submodular game, it is possible that finding a core allocation can be done in polynomial time, but the problem of deciding whether a given allocation is a core allocation or not may be NP-hard. *Finally*, for a submodular game, its nucleolus can be computed in polynomial time (Faigle et al. [5]), it has a large core (Sharkey [22]), and its stable set coincides with the core (Sharkey [22]). See Peleg and Sudholter [16] for the definition of the aforementioned important concepts in cooperative game theory.

In a recent paper, Anily and Haviv [1] show that the joint replenishment game is submodular when the joint setup cost function, denoted by $K(\cdot)$, has the so-called first order interaction

[2]Roughly speaking, a cooperative game is given by specifying a cost for every coalition. This is in contrast to non-cooperative games which are defined by the set of players, their strategies, and the payoffs for the set of outcomes.

structure, i.e., there exist K_0 and K_i for $i \in N$ such that $K(S) = K_0 + \sum_{i \in S} K_i$ for any $S \subseteq N$. However, the submodularity of the joint replenishment game with general submodular setup cost function $K(\cdot)$ has been posted as an open question in [1]. Zhang [28] shows that the joint replenishment game admits a population monotonic allocation scheme, which typically is an indication that a game may be submodular. The population monotonicity implies that no retailer would be worse off when a new retailer joins the coalition. As we shall see in Section 3, the function $V(S)$ can be expressed as

$$\begin{aligned} V(S) = \quad & \max \quad \sum_{i \in S} f_i(k_i) \\ & \text{s.t.} \quad \sum_{i \in A} k_i \leq K(A) \quad \forall A \subseteq S \\ & \qquad\quad k \in \mathbb{R}_+^{|S|}, \end{aligned} \tag{1}$$

where $k \in \mathbb{R}_+^{|S|}$ is the decision variable and for each $i \in N$, $f_i(k_i)$ is a concave function of k_i. Also, given our assumptions on the joint replenishment model, the feasible set of (1) turns out to be a polymatroid. Our goal is to show that the function $V(\cdot)$ defined in (1) is submodular.

This motivates us to consider the class of optimization problems of maximizing a separable concave function (or minimizing a separable convex function) over a polymatroid. Besides the joint replenishment model described above, this class of problems has many important applications in combinatorial optimization, resource allocation [10], dynamic scheduling [26], information theory [23], and many other areas. These problems can be solved by greedy algorithms; see Edmonds [3] and Federgruen and Groenevelt [6] and the references therein. We mention that this class of problems is a special case of the polynomially solvable problems studied by Hochbaum and Shanthikumar [11].

The main contributions of this paper are the following. First, we show that the optimal objective value (of the polymatroid maximization problem with a separable concave objective function) as a function of the index set is submodular. This immediately implies that the joint replenishment game is submodular. We also prove the submodularity of the optimal objective value with respect to certain parameters of the objective function. This can be used to prove the submodularity of the one-warehouse multiple retailer game studied in [27], which is a generalization of the joint replenishment game.

The remainder of the paper is organized as follows. In Section 2, we present our result regarding maximizing a separable concave function over a polymatroid. This result is applied, in Section 3, to derive the submodularity of the joint replenishment game and the one-warehouse multiple retailer game. We conclude the paper in Section 4.

2 A Structural Result on Polymatroid Optimization

In this section, we consider the problem of maximizing a separable concave function over a polymatroid. We study the submodularity of the optimal objective value with respect to the parameters of the objective function and the index set. In order to present our key results, we first formally introduce the necessary concepts and notations below.

Given a finite set E, let $2^E = \{A : A \subseteq E\}$ be its power set. A function $z : 2^E \to \mathbb{R}$ is said to be *submodular* if for all $A, B \subseteq E$,

$$z(A \cup B) + z(A \cap B) \leq z(A) + z(B).$$

A function $z : 2^E \to \mathbb{R}$ is said to be *supermodular* if $-z$ is submodular.

A function $z : 2^E \to \mathbb{R}$ is called a *rank function*, if it satisfies the following conditions:

- z is normalized, i.e., $z(\emptyset) = 0$;
- z is nondecreasing, i.e., $z(A) \leq z(B)$ whenever $A \subseteq B \subseteq E$;
- z is submodular.

For a given finite set E, and a function $z : 2^E \to \mathbb{R}$, the polyhedron

$$P(z, E) = \{x \in \mathbb{R}_+^{|E|} : \sum_{i \in A} x_i \leq z(A) \quad \text{for all} \quad A \subseteq E\}$$

is called a *polymatroid*, if z is a rank function. Throughout this paper, we let $P(z, E)$ denote the polymatroid defined by the finite set E and the rank function z.

A set $\mathcal{X} \subseteq \mathbb{R}^n$ is a *sublattice* if for any $x, y \in \mathcal{X}$, we have $x \vee y \in \mathcal{X}$ and $x \wedge y \in \mathcal{X}$, where $x \vee y$ and $x \wedge y$ denote, respectively, the coordinatewise maximum and minimum of x and y, i.e., $x \vee y = (\max(x_1, y_1), \cdots, \max(x_n, y_n))$ and $x \wedge y = (\min(x_1, y_1), \cdots, \min(x_n, y_n))$. If $\mathcal{X} \subseteq \mathbb{R}^n$ is a sublattice, then a function $f : \mathcal{X} \to \mathbb{R}$ is said to be *submodular*, if for all $x, y \in \mathcal{X}$,

$$f(x \vee y) + f(x \wedge y) \leq f(x) + f(y)$$

A function $f : \mathcal{X} \to \mathbb{R}$ is said to be *supermodular*, if $-f$ is submodular. Several supermodular and submodular functions that we shall refer to in this paper are listed below.

Example 1. Let $\mathcal{X} \subseteq \mathbb{R}^n, \mathcal{Y} \subseteq \mathbb{R}^n$ be two sublattices. Then the function $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ defined by $f(x, y) = x^{\mathrm{T}} y$ is supermodular.

Example 2. Let $\mathcal{X} \subseteq \mathbb{R}^n, \mathcal{Y} \subseteq \mathbb{R}^n$ be two sublattices. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a separable convex function. Then the function $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ defined by $f(x, y) = g(x - y)$ is submodular. In particular, the function $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ defined by $f(x, y) = \sum_{i=1}^n (x_i - y_i)^+$ is submodular.

Here and throughout the paper, we denote $x^+ = x \vee 0$ for any $x \in \mathbb{R}^n$.

Example 3. If function $f : \mathbb{R}^n \to \mathbb{R}$ is submodular, and function $g_i : \mathbb{R} \to \mathbb{R}$ is monotonic for each $i = 1, 2, \ldots, n$, then the function $h : \mathbb{R}^n \to \mathbb{R}$ defined by $h(x) = f(y)$ with $y_i = g_i(x_i)$ is submodular too.

2.1 Optimizing a Linear Function

In this subsection, we start with a special case: maximizing a linear function over a polymatroid. More specifically, for any vector $a \in \mathbb{R}^n$ where $n = |E|$, consider

$$\begin{array}{ll} \max & \sum_{i \in E} a_i x_i \\ \text{s.t.} & x \in P(z, E). \end{array} \tag{2}$$

The following is a well-known result [3] concerning an optimal solution of linear program (2). We shall refer to this result in several places of this paper.

Lemma 1. *Assume $a \in \mathbb{R}^n_+$ and let $\tilde{\pi} = (\tilde{\pi}_1, \cdots, \tilde{\pi}_n)$ be a permutation of set E, so that $a_{\tilde{\pi}_1} \geq a_{\tilde{\pi}_2} \geq \cdots \geq a_{\tilde{\pi}_n} \geq 0$. Define $x_{\tilde{\pi}} = (x_{\tilde{\pi}_i} : i \in E)$ as follows.*

$$\begin{aligned} x_{\tilde{\pi}_1} &= z(\{\tilde{\pi}_1\}) \\ x_{\tilde{\pi}_i} &= z(\{\tilde{\pi}_1, \tilde{\pi}_2, \cdots, \tilde{\pi}_i\}) - z(\{\tilde{\pi}_1, \tilde{\pi}_2, \cdots, \tilde{\pi}_{i-1}\}), \quad i = 2, 3, \cdots, n. \end{aligned} \tag{3}$$

Then $x_{\tilde{\pi}}$ is an optimal solution to (2). Furthermore, if for any $i, j \in E$ with $i \neq j$, $a_i \neq a_j$, then $x_{\tilde{\pi}}$ is the unique optimal solution to (2).

We next study the property of the optimal objective value of linear program (2). Let g be the optimal objective value of (2), i.e.,

$$g(a) \quad := \quad \max \quad a^{\mathrm{T}} x \qquad \text{s.t.} \quad x \in P(z, E). \tag{4}$$

Clearly, g is a convex function of a. There are actually two different ways to interpret the function g. First, g is a natural extension of the set function z: for any $S \subseteq E$ and denoting 1_S to be the indicator vector of S, we always have $g(1_S) = z(S)$. Second, for any $a \in \mathbb{R}^n_+$ there is a unique decomposition

$$a = \lambda_1 1_{S_1} + \cdots + \lambda_m 1_{S_m}$$

where $\lambda_i > 0$, $i = 1, ..., m$, and $S_1 \supset S_2 \supset \cdots \supset S_m$. Then,

$$g(a) = \lambda_1 z(S_1) + \cdots + \lambda_m z(S_m). \tag{5}$$

Lovász [12] showed that the definitions (4) and (5) are equivalent if and only if z is submodular.

For any $a \in \mathbb{R}^n_+$, an explicit way to write g is to introduce a permutation of set E, denoted by $\pi(a)$, so that

$$a_{\pi_1(a)} \geq a_{\pi_2(a)} \geq \cdots \geq a_{\pi_n(a)}.$$

Furthermore, we define the index sets

$$\Pi_i(a) = \{\pi_1(a), ..., \pi_i(a)\}, \; i = 1, 2, ..., n.$$

As a convention, denote $\Pi_0(a) := \emptyset$. Then, for any $a \in \mathbb{R}^n_+$

$$g(a) = \sum_{i=1}^{n} a_{\pi_i(a)} \left(z(\Pi_i(a)) - z(\Pi_{i-1}(a)) \right). \tag{6}$$

Our main result of this section is to show the submodularity of the optimal objective value of problem (2) with respect to the objective parameter vector, even if there are lower and upper bounds on the decision variables.

Theorem 1. *Consider the problem*

$$\begin{aligned} g(a) := \quad \max \quad & \textstyle\sum_{i \in E} a_i x_i \\ \text{s.t.} \quad & x \in P(z, E), \\ & \underline{\omega} \leq x \leq \overline{\omega}. \end{aligned} \tag{7}$$

Then, (i) g is homogeneous, i.e. $g(\lambda a) = \lambda g(a)$ for any $\lambda \geq 0$; (ii) $g(a)$ is a convex function; (iii) $g(a)$ is submodular, i.e.

$$g(a \vee b) + g(a \wedge b) \leq g(a) + g(b) \tag{8}$$

for all $a, b \in \mathbb{R}^n$, if (7) is feasible.

Proof. The properties (i) and (ii) are rather straightforward; they follow directly from the definition of g in (4). Let us now focus on (iii).

We first show that it is sufficient to prove the submodularity without the box constraint: $\underline{\omega} \leq x \leq \overline{\omega}$. Without loss of generality, we assume that $\underline{\omega} \geq 0$. Define another set function $z' : 2^E \to \mathbb{R}$, such that for any $S \subseteq E$,

$$z'(S) = \min_{R \subseteq S} \left\{ z(S \setminus R) + \sum_{i \in R} \overline{\omega}_i \right\}.$$

It is known from [3] that z' is a rank function, i.e., $P(z', E)$ is a polymatroid, and that furthermore,

$$P(z', E) = P(z, E) \cap \{x : x \leq \overline{\omega}\}.$$

Therefore, linear program (7) is equivalent to

$$\begin{array}{ll} \max & \sum_{i \in E} a_i x_i \\ \text{s.t.} & x \in P(z', E) \\ & x \geq \underline{\omega}, \end{array}$$

which in turn is equivalent to

$$\begin{array}{ll} \max & \sum_{i \in E} a_i y_i + \sum_{i \in E} a_i \underline{\omega}_i \\ \text{s.t.} & \sum_{i \in S} y_i \leq \overline{z}(S) \quad \forall S \subseteq E \\ & y \geq 0, \end{array} \tag{9}$$

where $\overline{z}(S) = z'(S) - \sum_{i \in S} \underline{\omega}_i$ for all $S \subseteq E$.

Define $\hat{z}(S) = \min_{S' \supseteq S} \overline{z}(S')$. Let y be a feasible solution to problem (9). Notice that for any $S' \supseteq S$,

$$\sum_{i \in S} y_i \leq \sum_{i \in S'} y_i \leq \bar{z}(S').$$

Thus,

$$\sum_{i \in S} y_i \leq \hat{z}(S).$$

On the other hand, if $\sum_{i\in S} y_i \le \hat{z}(S)$, then $\sum_{i\in S} y_i \le \overline{z}(S)$. Thus, problem (9) is equivalent to

$$\begin{aligned} \max \quad & \sum_{i\in E} a_i y_i + \sum_{i\in E} a_i \underline{w}_i \\ \text{s.t.} \quad & \sum_{i\in S} y_i \le \hat{z}(S) \quad \forall S \subseteq E \\ & y \ge 0. \end{aligned}$$

Now we show that $\hat{z}$ is a rank function. It is clear that $\hat{z}$ is non-decreasing and $\hat{z}(\emptyset) = 0$. We need only to show that $\hat{z}$ is submodular.

Notice that $\overline{z}$ is submodular. For any sets S_1 and S_2, there exist $S_1' \supseteq S_1$ and $S_2' \supseteq S_2$ such that $\hat{z}(S_1) = \overline{z}(S_1')$ and $\hat{z}(S_2) = \overline{z}(S_2')$. Therefore, we have that

$$\begin{aligned} & \hat{z}(S_1) + \hat{z}(S_2) \\ = \quad & \overline{z}(S_1') + \overline{z}(S_2') \\ \ge \quad & \overline{z}(S_1' \cap S_2') + \overline{z}(S_1' \cup S_2') \\ \ge \quad & \hat{z}(S_1 \cap S_2) + \hat{z}(S_1 \cup S_2) \end{aligned}$$

where the first inequality follows from the fact that $\overline{z}$ is submodular, and the second inequality follows from the fact that $S_1' \cap S_2' \supseteq S_1 \cap S_2$ and $S_1' \cup S_2' \supseteq S_1 \cup S_2$.

We next show that it is sufficient to prove the submodularity of g in $\mathbb{R}^n_+$. Assume that g is submodular in $\mathbb{R}^n_+$. For any $a \notin \mathbb{R}^n_+$, notice that $g(a) = g(a^+)$. Thus, for any $a, b \in \mathbb{R}^n$, we have

$$\begin{aligned} & g(a \vee b) + g(a \wedge b) \\ = \quad & g((a \vee b)^+) + g((a \wedge b)^+) \\ = \quad & g(a^+ \vee b^+) + g(a^+ \wedge b^+) \\ \le \quad & g(a^+) + g(b^+) \\ = \quad & g(a) + g(b) \end{aligned}$$

where the inequality follows from the submoduarity of g in $\mathbb{R}^n_+$.

Therefore, in the rest of the proof, we prove (8) for $a, b \in \mathbb{R}^n_+$. It suffices to show: for any $a, u, v \in \mathbb{R}^n_+$, and $u^{\mathrm{T}} v = 0$, it holds that

$$g(a+v) - g(a) \ge g(a+u+v) - g(a+u). \tag{10}$$

Notice that $u^{\mathrm{T}}v = 0$ means that $\mathrm{supp}\,(u) \cap \mathrm{supp}\,(v) = \emptyset$ where we denote $\mathrm{supp}\,(u)$ to be the support of u, i.e., $\mathrm{supp}\,(u) = \{i \in E : u_i > 0\}$. Furthermore, since g is clearly a continuous function, we need only to show that (10) holds for the vectors a, u, v whose positive parts are in general geometric positions, i.e., the positive coordinates of $a, u, v, a+u, a+v, u+v$ are all different.

To that end, we define, for any $i \in E$,

$$s_{\pi_i(a)} = z(\Pi_i(a)) - z(\Pi_{i-1}(a)).$$

By (6), it is clear that $s(a) \in \partial g(a)$, i.e., $s(a)$ is a subgradient for the convex function g at a. Clearly, as long as $\pi(a)$ remains unchanged, $s(a)$ is a constant vector.

Furthermore, we note that since a, u, v are generally positioned on their supports, the permutations $\pi(a+tv)$ and $\pi(a+u+tv)$ are uniquely determined on $\mathrm{supp}\,(a) \cup \mathrm{supp}\,(v)$ and $\mathrm{supp}\,(a) \cup \mathrm{supp}\,(u) \cup \mathrm{supp}\,(v)$ respectively, for almost all t in $[0,1]$ except for no more than $O(n^2)$ discrete values of t. By (6), it follows that, $g(a+tv)$ and $g(a+u+tv)$ as functions of t are everywhere differentiable, except for at most $O(n^2)$ points.

Therefore,

$$g(a+v) - g(a) = \int_0^1 s(a+tv)^{\mathrm{T}} v dt$$

and

$$g(a+u+v) - g(a+u) = \int_0^1 s(a+u+tv)^{\mathrm{T}} v dt,$$

(see e.g. Corollary 24.2.1 of Rockafellar [18]). It follows that, in order to prove (10), it will be sufficient to show

$$s(a+tv)^{\mathrm{T}} v \geq s(a+u+tv)^{\mathrm{T}} v \tag{11}$$

for almost all $t \in [0,1]$ (except for at most $O(n^2)$ points).

Now, consider a general t value such that $\pi(a+tv)$ and $\pi(a+u+tv)$ are uniquely determined for the parts $\mathrm{supp}\,(a) \cup \mathrm{supp}\,(v)$ and $\mathrm{supp}\,(a) \cup \mathrm{supp}\,(u) \cup \mathrm{supp}\,(v)$ respectively, and consider a given $i \in \mathrm{supp}\,(v)$. Since $\mathrm{supp}\,(u) \cap \mathrm{supp}\,(v) = \emptyset$, for any j, if $(a+tv)_j > (a+tv)_i$, then $(a+u+tv)_j > (a+u+tv)_i$. This is to say, if

$$i = \pi_k(x+tv) = \pi_{k'}(x+u+tv),$$

then $k \leq k'$, and

$$\Pi_{k-1}(x+tv) \subseteq \Pi_{k'-1}(x+u+tv).$$

Consequently,

$$\begin{aligned} s_i(x+tv) &= z(\Pi_{k-1}(x+tv)\cup\{i\}) - z(\Pi_{k-1}(x+tv)) \\ &\geq z(\Pi_{k'-1}(x+u+tv)\cup\{i\}) - z(\Pi_{k'-1}(x+u+tv)) \\ &= s_i(x+u+tv), \end{aligned}$$

where the inequality is due to the submodularity of z. Thus, (11) holds for almost all $t \in [0,1]$, which proves (10), hence the submodularity. $\square$

Theorem 1 immediately implies the following result, which shall be useful in the next subsection.

Corollary 1. *Consider the problem:*

$$\begin{aligned} \max \quad & \textstyle\sum_{i\in E} \alpha_i a_i x_i + \beta_i a_i + \gamma_i x_i + \delta_i \\ \text{s.t.} \quad & x \in P(z,E) \\ & \underline{\omega} \leq x \leq \overline{\omega}. \end{aligned} \tag{12}$$

Let $g : \mathbb{R}^{|E|} \to \mathbb{R}$ denote the optimal objective value, as a function of a. Then function g is submodular if $\alpha_i \geq 0$ for all $i \in E$.

Proof. Since the sum of submodular functions is still submodular, we can safely assume that $\beta_i = \delta_i = 0$ for all $i \in E$. Let

$$\begin{aligned} h(b) \quad = \quad \max \quad & \textstyle\sum_{i\in E} b_i x_i \\ \text{s.t.} \quad & x \in P(z,E) \\ & \underline{\omega} \leq x \leq \overline{\omega}. \end{aligned} \tag{13}$$

Then by Theorem 1 we conclude that $h(b)$ is submodular with respect to b. Since $b(a)_i := \alpha_i a_i + \gamma_i$ is a monotonically increasing function of a_i for all $i \in E$, we know that function $g(a) = h(b(a))$ is submodular with respect to a. $\square$

The application of Theorem 1 to the joint replenishment game will be described in Section 3. Here we provide two simple examples where Theorem 1 can be applied.

Example 3. Let $c \in \mathbb{R}^n$ and p be an integer such that $1 \leq p \leq n$. Denote the sum of the p largest coordinates of c by $\sigma(p,c)$. It is shown in [25] (Proposition 4) that $\sigma(p,c)$ as a function of

c is submodular in $\mathbb{R}^n$. This can be seen by applying Theorem 1 directly. Notice that

$$\begin{aligned} \sigma(p,c) = \quad \max \quad & \sum_{i=1}^{n} c_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_i = p \\ & 0 \le x_i \le 1. \end{aligned}$$

It is clear that the linear program above can be cast in the form of problem (7), using the rank function of the so-called uniform matroid of rank p. Thus it follows from Theorem 1 that $\sigma(p,c)$ is submodular in c.

Example 4. Let $\lambda \in \mathbb{R}^n$ so that $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$. For any $c \in \mathbb{R}^n$, let $c_{[i]}$ be the ith largest component of c. Define $f(\lambda, c) = \sum_{i=1}^{n} \lambda_i c_{[i]} : \mathbb{R}^n \to \mathbb{R}$. It is shown in [17] (Theorem 4.1) that $f(\lambda, c)$ is a submodular function in c. This can be seen by noticing that

$$f(\lambda, c) = \sum_{k=1}^{n} (\lambda_k - \lambda_{k+1}) k(c)$$

where $\lambda_{n+1} = 0$ and $k(c)$ is the sum of the k largest coordinates of c. By the result of Example 3, we know that $f(\lambda, c)$ is submodular in c.

2.2 Maximizing a Separable Concave Function

In this subsection, we generalize the result in the previous subsection to the case where the objective function is separable concave. The key idea underlying the proof is to linearize the objective function.

Theorem 2. *Fix a finite set E and a polymatroid $P(z,E)$. For any $i \in E$, let $f_i : \mathbb{R}^2 \to \mathbb{R}$ be a supermodular function. For any $a \in \mathbb{R}^{|E|}$, define*

$$\begin{aligned} g(a) = \quad \max \quad & \textstyle\sum_{i \in E} f_i(x_i, a_i) \\ \text{s.t.} \quad & x \in P(z,E). \end{aligned} \tag{14}$$

Then $g : \mathbb{R}^{|E|} \to \mathbb{R}$ is submodular if $f_i(x_i, a_i)$ is concave in both x_i and a_i, for all $i \in E$.

To see that Theorem 2 generalizes Theorem 1, we notice that $f_i(x_i, a_i) = a_i x_i : \mathbb{R}^2 \to \mathbb{R}$ is supermodular in (a_i, x_i) and concave in both a_i and x_i. In order to prove Theorem 2, we need to carefully linearize the functions $f_i(x_i, a_i)$. To that end, we need the following lemma.

Lemma 2. *If $\psi(y,b)$ is supermodular in $(y,b) \in \mathbb{R}^2$ and concave in both y and b, then for any $y_1 < y_2$ and $b_1 < b_2$, there exists a function*

$$L(y,b) = \alpha by + \beta b + \gamma y + \delta$$

such that

- $\alpha \geq 0$;
- $L(y,b) \leq \psi(y,b)$ *for any* $(y,b) \in [y_1,y_2] \times [b_1,b_2]$; *and*
- $L(y_i,b_j) = \psi(y_i,b_j)$ *for any* $i,j \in \{1,2\}$.

Proof. Define

$$\begin{cases} \alpha' &= \frac{\psi(y_2,b_2)+\psi(y_1,b_1)-\psi(y_2,b_1)-\psi(y_1,b_2)}{(b_2-b_1)(y_2-y_1)}, \\ \beta' &= \frac{\psi(y_1,b_2)-\psi(y_1,b_1)}{b_2-b_1}, \\ \gamma' &= \frac{\psi(y_2,b_1)-\psi(y_1,b_1)}{y_2-y_1}, \\ \delta' &= \psi(y_1,b_1) \end{cases}$$

and

$$L(y,b) = \alpha'(y-y_1)(b-b_1) + \beta'(b-b_1) + \gamma'(y-y_1) + \delta'.$$

By supermodularity of $\psi(y,b)$, we know that

$$\psi(y_2,b_2) + \psi(y_1,b_1) = \psi((y_2,b_1) \vee (y_1,b_2)) + \psi((y_2,b_1) \wedge (y_1,b_2)) \geq \psi(y_2,b_1) + \psi(y_1,b_2)$$

and thus $\alpha = \alpha' \geq 0$. It is also easy to verify that $L(y_i,b_j) = \psi(y_i,b_j)$ for any $i,j \in \{1,2\}$. Finally, for any $(y,b) \in [y_1,y_2] \times [b_1,b_2]$, (y,b) can be expressed as a convex combination of two points (y_1,b) and (y_2,b), i.e., there exists $\lambda_i \geq 0, i = 1,2$ such that $\lambda_1 + \lambda_2 = 1$ and $(y,b) = \lambda_1(y_1,b) + \lambda_2(y_2,b)$. Since $\psi(y,b)$ is concave in y, we have

$$\psi(y,b) \geq \lambda_1\psi(y_1,b) + \lambda_2\psi(y_2,b).$$

Similarly, there exists $\mu_i \geq 0, i = 1,2$ such that $\mu_1 + \mu_2 = 1$ and $b = \mu_1 b_1 + \mu_2 b_2$. Since $\psi(y,b)$ is concave in b, we have

$$\psi(y_1,b) \geq \mu_1\psi(y_1,b_1) + \mu_2\psi(y_1,b_2)$$

and

$$\psi(y_2,b) \geq \mu_1\psi(y_2,b_1) + \mu_2\psi(y_2,b_2).$$

It then follows that

$$\psi(y,b) \geq \lambda_1\mu_1\psi(y_1,b_1) + \lambda_1\mu_2\psi(y_1,b_2) + \lambda_2\mu_1\psi(y_2,b_1) + \lambda_2\mu_2\psi(y_2,b_2).$$

On the other hand, one can verify that

$$L(y,b) = \lambda_1\mu_1 L(y_1,b_1) + \lambda_1\mu_2 L(y_1,b_2) + \lambda_2\mu_1 L(y_2,b_1) + \lambda_2\mu_2 L(y_2,b_2).$$

Therefore, by the fact that $L(y_i,b_j) = \psi(y_i,b_j)$ for any $i,j \in \{1,2\}$, we get $L(y,b) \leq \psi(y,b)$ for any $(y,b) \in [y_1,y_2] \times [b_1,b_2]$. $\square$

It can be easily seen from the proof that the supermodularity of $\psi(y,b)$ is used to guarantee the non-negativity of the coefficient of yb in the function $L(y,b)$. The concavity of $\psi(y,b)$ is used to ensure that $L(y,b)$ is a lower bound of $\psi(y,b)$. Now we are ready to prove Theorem 2.

Proof of Theorem 2. For any $a \in \mathbb{R}_+^{|E|}$, let $x(a)$ denote an optimal solution to problem (14). Now for any $b,d \in \mathbb{R}_+^{|E|}$, by Lemma 2, for any $i \in E$, there exists a linear function $L_i(a_i,x_i) = \alpha_i a_i x_i + \beta_i x_i + \gamma_i a_i + \delta_i$ such that $\alpha_i \geq 0$, and for any

$$(a_i,x_i) \in [b_i \wedge d_i, b_i \vee d_i] \times [x(b\vee d)_i \vee x(b\wedge d)_i, x(b\vee d)_i \wedge x(b \wedge d)_i],$$

we have $f_i(a_i,x_i) \geq L_i(a_i,x_i)$, and the inequality holds as an equality when (a_i,x_i) is an extreme point of the set $[b_i \wedge d_i, b_i \vee d_i] \times [x(b\vee d)_i \wedge x(b\wedge d)_i, x(b\vee d)_i \vee x(b\wedge d)_i]$. We further denote

$$\Omega(b,d) := \left\{x \in \mathbb{R}_+^{|E|} : x_i \in [x(b\vee d)_i \wedge x(b\wedge d)_i, x(b\vee d)_i \vee x(b\wedge d)_i], \quad \forall i \in E\right\}$$

$$F(a,x) = \sum_{i\in E} f_i(a_i,x_i), \quad \text{and} \quad L(a,x) = \sum_{i\in E} L_i(a_i,x_i).$$

These constructions and definitions, together with Theorem 1, lead to

$$\begin{aligned}
& g(b\vee d) + g(b\wedge d) && (15)\\
= \;& F(b\vee d, x(b\vee d)) + F(b\wedge d, x(b\wedge d)) && (16)\\
= \;& L(b\vee d, x(b\vee d)) + L(b\wedge d, x(b\wedge d)) && (17)\\
\leq \;& \max_{x\in P(z,E)\cap\Omega(b,d)} L(b\vee d, x) + \max_{x\in P(z,E)\cap\Omega(b,d)} L(b\wedge d, x) && (18)\\
\leq \;& \max_{x\in P(z,E)\cap\Omega(b,d)} L(b, x) + \max_{x\in P(z,E)\cap\Omega(b,d)} L(d, x) && (19)\\
\leq \;& \max_{x\in P(z,E)\cap\Omega(b,d)} F(b, x) + \max_{x\in P(z,E)\cap\Omega(b,d)} F(d, x) && (20)\\
\leq \;& \max_{x\in P(z,E)} F(b, x) + \max_{x\in P(z,E)} F(d, x) && (21)\\
= \;& g(b) + g(d). && (22)
\end{aligned}$$

Equality (16) holds because of the definition of $x(b \vee d)$ and $x(b \wedge d)$. Equality (17) and inequality (20) hold by the construction of function L. Inequality (18) holds because $x(b \vee d)$ and $x(b \wedge d)$ are in $P(z,E) \cap \Omega(b,d)$. Inequality (19) follows from Corollary 1. Inequality (21) holds since $P(z,E) \cap \Omega(b,d) \subseteq P(z,E)$. $\square$

2.3 Submodularity Results on Sets

In this subsection, we show that our submodularity results on lattices imply the analogous results on sets.

Theorem 3. *Fix a finite set E and a polymatroid $P(z,E)$. For each $i \in E$, let $f_i : \mathbb{R} \to \mathbb{R}$ be a concave function. For any $A \subseteq E$, define*

$$\begin{aligned} h(A) = \quad \max \quad & \textstyle\sum_{i \in A} f_i(x_i) \\ \text{s.t.} \quad & x \in P(z,E). \end{aligned} \tag{23}$$

Then $h : 2^E \to \mathbb{R}$ is submodular.

Proof. For each $i \in E$, we define a function $\tilde{f}_i$ as follows. Recall that f_i is concave. If f_i is non-decreasing, then let $\tilde{f}_i = f_i$. Otherwise, there must exist $x_i^* \in \mathbb{R}$ such that $f_i(x_i^*) = \max_{x_i \in \mathbb{R}} f_i(x_i)$. In this case, define

$$\tilde{f}_i(x_i) = \begin{cases} f_i(x_i) & \text{if } x_i \le x_i^* \\ f_i(x_i^*) & \text{otherwise.} \end{cases}$$

It is clear that $\tilde{f}_i$ is non-decreasing and concave. Furthermore, problem (23) is equivalent to

$$\begin{aligned} h(A) = \quad \max \quad & \textstyle\sum_{i \in A} \tilde{f}_i(x_i) \\ \text{s.t.} \quad & x \in P(z,E). \end{aligned} \tag{24}$$

For each $i \in E$, define $\bar{f}_i : \mathbb{R}^2 \to \mathbb{R}$ such that $\bar{f}_i(x_i, a_i) = \tilde{f}_i(x_i) a_i$. It is straightforward to verify that $\bar{f}_i$ is supermodular in (x_i, a_i) and concave in both x_i and $a_i \ge 0$. Therefore, by Theorem 2, if we define, for each $a \in \mathbb{R}_+^{|E|}$,

$$\begin{aligned} g(a) = \quad \max \quad & \sum_{i \in E} \bar{f}_i(x_i, a_i) \\ \text{s.t.} \quad & x \in P(z,E), \end{aligned}$$

then $g : \mathbb{R}_+^{|E|} \to \mathbb{R}$ is submodular. Let $a^A \in \mathbb{R}_+^{|E|}$ such that for each $i \in E$,

$$a_i^A = \begin{cases} 1 & \text{if } \ i \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then $g(a^A) = h(A)$, and for any $A, B \subseteq E$, we have $a^{A \cup B} = a^A \vee a^B$ and $a^{A \cap B} = a^A \wedge a^B$. Therefore,

$$\begin{aligned} h(A \cup B) + h(A \cap B) &= g(a^{A \cup B}) + g(a^{A \cap B}) \\ &= g(a^A \vee a^B) + g(a^A \wedge a^B) \\ &\leq g(a^A) + g(a^B) \\ &= h(A) + h(B), \end{aligned}$$

which implies that $h : 2^E \to \mathbb{R}$ is submodular. This completes the proof. □

We notice that Schulz and Uhan [20] proved Theorem 3 for polymatroid optimization with linear objective functions. They use this result to show that certain scheduling games are supermodular.

3 One-Warehouse Multiple Retailer Game

In this section, we consider the one-warehouse multiple retailer game studied by Zhang [27]; it is a generalization of the joint replenishment game studied by Anily and Haviv [1] and Zhang [28]. The presentation of the model follows closely to that in [27]. In this model, we are given a set of n retailers, denoted by $N = \{1, 2, \cdots, n\}$. The demand that retailer i faces is continuous and deterministic at a fixed rate $d_i > 0$. The retailers place orders to a single warehouse to satisfy customer demands. These orders generate demands at the warehouse, which holds inventory and is replenished from an external supplier. Backlogging is not allowed in this model. The lead time is assumed to be zero, i.e., orders arrive instantaneously[3].

[3] This simplifying assumption is common in the literature of continuous-time joint replenishment problems; see, for example, [19] and [7]. We are not aware of any worst-case analysis of inventory policies when retailers have arbitrary non-zero lead times.

For ease of presentation, the warehouse is denoted by 0. Also, any $i \in N \cup \{0\}$ is called a facility, i.e., a facility can be a warehouse or a retailer.

For each $i \in N \cup \{0\}$, there is a per unit holding cost rate h_i. For simplicity we denote, $H_i = \frac{1}{2} h_i d_i$ and $H_i^w = \frac{1}{2} h_0 d_i$, for each $i \in N$. We also assume that $0 < h_0 < h_i$ and thus $0 < H_i^w < H_i$, for any $i \in N$. This assumption is common in the literature; see, e.g., Roundy [19], and Federgruen et al. [8]. When a subset $S \subseteq N \cup \{0\}$ of facilities places an order together, a joint setup cost is incurred, which is denoted by $K(S)$ with $K(\emptyset) = 0$. We assume that $K(S)$ is a non-decreasing submodular function. Therefore, $K(S)$ is a rank function.

We restrict ourselves to the so-called power-of-two inventory policies, which can be characterized by an $(n+1)$-tuple, $(T_0, T_i : i \in N)$, where T_i is the replenishment interval at facility i for $i \in N \cup \{0\}$. That is, the replenishment epoches of facility i are $0, T_i, 2T_i, \cdots$. Furthermore, we require that for all $i \in N \cup \{0\}$, $T_i = 2^{m_i}\mathcal{L}$ where $\mathcal{L} > 0$ is a constant called the base planning period, and m_i is an integer that can be negative. Denote

$$\Gamma_{\mathcal{L}} = \{t : t > 0 \quad \text{and} \quad t = 2^m \mathcal{L} \quad \text{for some} \quad m \in \mathbb{Z}\}.$$

The effectiveness of power-of-two policies has been discussed in Federgruen et al. [8]. If the base planning period $\mathcal{L}$ is chosen arbitrarily, then the optimal power-of-two policy yields an average cost that is at most 6% higher than the optimal cost, and thus is 94% effective. By choosing the best $\mathcal{L}$, the optimal power-of-two policy is 98% effective.

Now we consider a cooperative game associated with this inventory model. We denote this game by $(N, V_{\Gamma_{\mathcal{L}}})$. Here N is the grand coalition of n retailers and $V_{\Gamma_{\mathcal{L}}}$ is the characteristic cost function defined for every coalition $S \subseteq N$. In particular, $V_{\Gamma_{\mathcal{L}}}(\emptyset) = 0$ and for $\emptyset \neq S \subseteq N$, $V_{\Gamma_{\mathcal{L}}}(S)$ is the long-run average cost, under an optimal power-of-two policy, of the system that consists of the warehouse and the retailers in S.

Federgruen et al. [8] have shown that

$$V_{\Gamma_{\mathcal{L}}}(S) := \min_{\mathbb{T}_S \in \Gamma_{\mathcal{L}}^{S \cup \{0\}}} \max_{k \in P(K, S \cup \{0\})} \left\{ \frac{k_0}{T_0} + \sum_{i \in S} \left(\frac{k_i}{T_i} + H_i T_i + H_i^w \max\{T_0 - T_i, 0\} \right) \right\} \tag{25}$$

where $\mathbb{T}_S = (T_0, T_i : i \in S)$ and $\Gamma_{\mathcal{L}}^{S \cup \{0\}} = \{t = (t_i : i \in S \cup \{0\})$ with $t_i \in \Gamma_{\mathcal{L}}, \forall\, i \in S \cup \{0\}\}$.

The game (N, V_Γ) is called a concave game if the set function $V_\Gamma(\cdot)$ is submodular.

3.1 Submodularity of the Joint Replenishment Game

Now we consider a special case of the one-warehouse multiple retailer game when there is no warehouse. This reduces to the joint replenishment game studied in Zhang [28]. We denote the game by $(N, V_{J\Gamma_{\mathcal{L}}})$ where the characteristic cost function $V_{J\Gamma_{\mathcal{L}}}(\cdot)$ is defined as, for any $S \subseteq N$,

$$V_{J\Gamma_{\mathcal{L}}}(S) := \min_{T_i \in \Gamma_{\mathcal{L}}: i \in S} \max_{k \in P(K,S)} \sum_{i \in S} \left(\frac{k_i}{T_i} + H_i T_i \right). \tag{26}$$

(This can be obtained by setting $k_0 = 0$ and $H_i^w = 0$ in (25).) It is known that we can change the order of the optimization of (26) from min-max to max-min without changing the optimal objective value [27]. That is,

$$V_{J\Gamma_{\mathcal{L}}}(S) := \max_{k \in P(K,S)} \sum_{i \in S} \min_{T_i \in \Gamma_{\mathcal{L}}} \left(\frac{k_i}{T_i} + H_i T_i \right). \tag{27}$$

In [28], an analytical solution to problem (27) was derived, which is in turn used to propose a population monotonic allocation scheme for the joint replenishment game. As most of the cooperative games that admit a population monotonic allocation scheme are submodular, Zhang [28] conjectured that the joint replenishment game is submodular. Here we show that this is indeed the case.

Theorem 4. *The joint replenishment game $(N, V_{J\Gamma_{\mathcal{L}}})$ is submodular.*

Proof. For each fixed T_i, the function $\frac{k_i}{T_i} + H_i T_i$ is linear in k_i. Then it is clear that, for any $i \in S$,

$$\min_{T_i \in \Gamma_{\mathcal{L}}} \left(\frac{k_i}{T_i} + H_i T_i \right)$$

is a concave function of k_i, which we denote by $f_i(k_i)$. Thus, from (27),

$$V_{J\Gamma_{\mathcal{L}}}(S) := \max_{k \in P(K,S)} \sum_{i \in S} f_i(k_i).$$

By Theorem 3, $V_{J\Gamma_{\mathcal{L}}}(S)$ is submodular. This completes the proof. $\square$

Anily and Haviv [1] proved that Theorem 4 holds for a special case of the joint replenishment game where the joint setup cost function has the first order interaction structure. Theorem 4 generalizes their main result.

3.2 Submodularity of the One-Warehouse Multiple Retailer Game

Now we consider the submodularity of the one-warehouse multiple retailer game $(N, V_{\Gamma_{\mathcal{L}}})$, where the function $V_{\Gamma_{\mathcal{L}}}(S)$ is defined by (25). It is tempting to prove the submodularity of $V_{\Gamma_{\mathcal{L}}}(S)$ by following the approach used in the proof of Theorem 4. The dual problem of (25) can be formulated as follows [27]:

$$\max_{k \in P(K, S\cup\{0\}), 0 \le u_i \le H_i^w : i \in S} \quad \min_{\mathbb{T}_S \in \Gamma_{\mathcal{L}}^{S\cup\{0\}}} \left\{ \frac{k_0}{T_0} + \left(\sum_{i \in S} u_i \right) T_0 + \sum_{i \in S} \left((H_i - u_i) T_i + \frac{k_i}{T_i} \right) \right\}. \qquad (28)$$

It is known that this pair of primal-dual problems (25) and (28) do not have a duality gap. However, the objective function of (28) is not separable. Therefore, the results developed in Section 2 are not directly applicable to problem (28). In order to prove the submodularity of $V_{\Gamma_{\mathcal{L}}}(S)$, we focus on the primal formulation (25) and apply Theorem 1 to the inner maximization problem of (25).

Theorem 5. *The one-warehouse multiple retailer game $(N, V_{\Gamma_{\mathcal{L}}})$ is submodular.*

Proof. For any $S \subseteq N$, let $\mathbb{T}_S^*$ be an optimal solution to the outer minimization problem of (25). Denote $\bar{\Gamma}_{\mathcal{L}} = \Gamma_{\mathcal{L}} \cup \{+\infty\}$. Then for any $S \subseteq N$, we have

$$\begin{aligned} V_{\Gamma_{\mathcal{L}}}(S) &= \min_{\mathbb{T}_S \in \Gamma_{\mathcal{L}}^{S\cup\{0\}}} \quad \max_{k \in P(K, S\cup\{0\})} \left\{ \frac{k_0}{T_0} + \sum_{i \in S} \left(\frac{k_i}{T_i} + H_i T_i + H_i^w \max\{T_0 - T_i, 0\} \right) \right\} && (29) \\ &= \min_{\mathbb{T}_N \in \Gamma_{\mathcal{L}}^{N\cup\{0\}}} \quad \max_{k \in P(K, N\cup\{0\})} \left\{ \frac{k_0}{T_0} + \sum_{i \in N} \frac{k_i}{T_i} + \sum_{i \in S} \left(H_i T_i + H_i^w \max\{T_0 - T_i, 0\} \right) \right\}. && (30) \end{aligned}$$

In particular, if we define $\mathbb{T}_{S,N}^*$ such that $(\mathbb{T}_{S,N}^*)_i = (\mathbb{T}_S^*)_i$ for $i \in S$ and $(\mathbb{T}_{S,N}^*)_i = +\infty$ otherwise, then $\mathbb{T}_{S,N}^*$ is an optimal solution to the outer minimization problem of (30).

Notice that the feasible set of the outer minimization problem of (30), $\bar{\Gamma}_{\mathcal{L}}^{n+1}$, is a sublattice of $\mathbb{R}^{n+1}$. That is, for any coalitions $A, B \subseteq N$, and for any i, $(\mathbb{T}_{A,N}^* \vee \mathbb{T}_{B,N}^*)_i \in \bar{\Gamma}_{\mathcal{L}}$ and $(\mathbb{T}_{A,N}^* \wedge \mathbb{T}_{B,N}^*)_i \in \bar{\Gamma}_{\mathcal{L}}$. Then for any coalitions $A, B \subseteq N$, we define $\mathbb{T}_N^{A\cap B} = \mathbb{T}_{A,N}^* \vee \mathbb{T}_{B,N}^*$ and $\mathbb{T}_N^{A\cup B} = \mathbb{T}_{A,N}^* \wedge \mathbb{T}_{B,N}^*$, which must be feasible solutions to problem (30) for $S = A \cap B$ and $S = A \cup B$ respectively. For any $\mathbb{T}_N = (T_i : i \in N \cup \{0\})$ and $S \subseteq N$, we let $(\mathbb{T}_N)_S$ denote the restriction of $\mathbb{T}_N$ to the subset

$S \cup \{0\}$, i.e., $(\mathbb{T}_N)_S = (T_i : i \in S \cup \{0\})$. Furthermore, we define

$$
\begin{aligned}
g_N(\mathbb{T}_N) &= \max_{k \in P(K, N\cup\{0\})} \left\{ \frac{k_0}{T_0} + \sum_{i\in N} \frac{k_i}{T_i} \right\}, \\
h_S((\mathbb{T}_N)_S) &= \sum_{i\in S} \left(H_i T_i + H_i^w \max\{T_0 - T_i, 0\}\right), \\
G_S(\mathbb{T}_N) &= g_N(\mathbb{T}_N) + h_S((\mathbb{T}_N)_S).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
& V_{\Gamma_{\mathcal{L}}}(A \cup B) + V_{\Gamma_{\mathcal{L}}}(A \cap B) \\
\le\ & G_{A\cup B}(\mathbb{T}_N^{A\cup B}) + G_{A\cap B}(\mathbb{T}_N^{A\cap B}) \\
=\ & G_{A\cup B}(\mathbb{T}^*_{A,N} \wedge \mathbb{T}^*_{B,N}) + G_{A\cap B}(\mathbb{T}^*_{A,N} \vee \mathbb{T}^*_{B,N}) \\
=\ & g_N(\mathbb{T}^*_{A,N} \wedge \mathbb{T}^*_{B,N}) + g_N(\mathbb{T}^*_{A,N} \vee \mathbb{T}^*_{B,N}) + h_{A\cup B}((\mathbb{T}^*_{A,N} \wedge \mathbb{T}^*_{B,N})_{A\cup B}) + h_{A\cap B}((\mathbb{T}^*_{A,N} \vee \mathbb{T}^*_{B,N})_{A\cap B}).
\end{aligned}
$$

By Theorem 1, $g_N(\mathbb{T}_N)$ is submodular in $(1/T_i : i \in N \cup \{0\})$, and thus submodular in $\mathbb{T}_N$. Therefore,

$$
g_N(\mathbb{T}^*_{A,N} \wedge \mathbb{T}^*_{B,N}) + g_N(\mathbb{T}^*_{A,N} \vee \mathbb{T}^*_{B,N}) \le g_N(\mathbb{T}^*_{A,N}) + g_N(\mathbb{T}^*_{B,N}). \tag{31}
$$

Also,

$$
\begin{aligned}
h_{A\cup B}((\mathbb{T}^*_{A,N} \wedge \mathbb{T}^*_{B,N})_{A\cup B}) &= h_{A\cap B}((\mathbb{T}^*_{A,N} \wedge \mathbb{T}^*_{B,N})_{A\cap B}) \\
&\quad + \sum_{i\in A\setminus B} \left(H_i(\mathbb{T}^*_{A,N})_i + H_i^w \max\{(\mathbb{T}^*_{A,N})_0 - (\mathbb{T}^*_{A,N})_i, 0\}\right) \\
&\quad + \sum_{i\in B\setminus A} \left(H_i(\mathbb{T}^*_{B,N})_i + H_i^w \max\{(\mathbb{T}^*_{B,N})_0 - (\mathbb{T}^*_{B,N})_i, 0\}\right).
\end{aligned}
$$

We know that $\max\{T_0 - T_i, 0\}$ is submodular in (T_0, T_i) for any $i \in S$. Therefore, $h_S(\mathbb{T}_S) = \sum_{i\in S} (H_i T_i + H_i^w \max\{T_0 - T_i, 0\})$ is submodular in $\mathbb{T}_S$. Thus,

$$
h_{A\cap B}((\mathbb{T}^*_{A,N} \wedge \mathbb{T}^*_{B,N})_{A\cap B}) + h_{A\cap B}((\mathbb{T}^*_{A,N} \vee \mathbb{T}^*_{B,N})_{A\cap B}) \le h_{A\cap B}((\mathbb{T}^*_{A,N})_{A\cap B}) + h_{A\cap B}((\mathbb{T}^*_{B,N})_{A\cap B}).
$$

It follows that

$$
\begin{aligned}
& h_{A\cup B}((\mathbb{T}^*_{A,N} \wedge \mathbb{T}^*_{B,N})_{A\cup B}) + h_{A\cap B}((\mathbb{T}^*_{A,N} \vee \mathbb{T}^*_{B,N})_{A\cap B}) \\
\le\ & h_{A\cap B}((\mathbb{T}^*_{A,N})_{A\cap B}) + \sum_{i\in A\setminus B} \left(H_i(\mathbb{T}^*_{A,N})_i + H_i^w \max\{(\mathbb{T}^*_{A,N})_0 - (\mathbb{T}^*_{A,N})_i, 0\}\right) \\
& + h_{A\cap B}(\mathbb{T}^*_{B,N})_{A\cap B}) + \sum_{i\in B\setminus A} \left(H_i(\mathbb{T}^*_{B,N})_i + H_i^w \max\{(\mathbb{T}^*_{B,N})_0 - (\mathbb{T}^*_{B,N})_i, 0\}\right) \\
=\ & h_A((\mathbb{T}^*_{A,N})_A) + h_B((\mathbb{T}^*_{B,N})_B).
\end{aligned}
$$

This, together with (31), implies that

$$\begin{aligned} & V_{\Gamma_{\mathcal{L}}}(A \cup B) + V_{\Gamma_{\mathcal{L}}}(A \cap B) \\ \leq \quad & g_N(\mathbb{T}^*_{A,N}) + h_A((\mathbb{T}^*_{A,N})_A) + g_N(\mathbb{T}^*_{B,N}) + h_B((\mathbb{T}^*_{B,N})_B) \\ = \quad & V_{\Gamma_{\mathcal{L}}}(A) + V_{\Gamma_{\mathcal{L}}}(B) \end{aligned}$$

which shows that $V_{\Gamma_{\mathcal{L}}}(S)$ is submodular. $\square$

We remark that Theorem 5 can be generalized to the case where there are upper and lower bounds on the replenishment intervals of the retailers and the warehouse. The reason is that, with this additional constraint, the feasible set for the replenishment intervals is still a sublattice, and so the proof of Theorem 5 will still go through.

4 Concluding Remarks

In this paper, we have obtained some structural results regarding polymatroid optimization. We identify conditions so that the optimal objective function is a submodular function in the index set and the objective parameters. In the most general version, for each $a \in \mathbb{R}^{|E|}$, we consider the following problem,

$$\max_{x \in P(z,E)} \sum_{i \in E} f_i(x_i, a_i)$$

where $P(z, E)$ is a polymatroid, and for each $i \in E$, $f_i : \mathbb{R}^2 \to \mathbb{R}$ is submodular in (x_i, a_i), and concave in both x_i and a_i. We prove that the optimal objective value as a function of parameter a is submodular. This result and its variants have been applied to analyze the joint replenishment game and the one-warehouse multiple retailer game.

The submodularity results regarding polymatroid optimization may find other applications as well, given the wide range of applications of polymatroid optimization. One possible area is for problems of scheduling multiclass queueing systems that satisfy strong conservation laws; see for example Garbe and Glazebrook [9], where the objective function of the optimization problem is linear, but the feasible set is slightly more general than a polymatroid.

Acknowledgements. The authors are extremely grateful to two anonymous referees and the associate editor for their detailed and constructive comments that considerably improved the exposition of this paper. The authors also would like to express their gratitude to Shoshana Anily and Arie Tamir for their valuable comments.

References

[1] Anily, S. and Haviv, M. (2007). The Cost Allocation Problem for the First Order Interaction Joint Replenishment Model. *Operations Research*, 55 (2), pp. 292–302.

[2] Dror, M. and Hartman, B. (2007). Shipment Consolidation: Who pays for it and how much. *Management Science*, 53, pp. 78–87.

[3] Edmonds, J. (1970). Submodular functions, matroid, and certain polyhedra. In *Combinatorial Structures and Their Applications*. Gordon & Breach, New York, pp. 69–87.

[4] Cooperation and Collaboration between Logistic Service Providors (LSP's): Threat or Opportunity? European Chemical Transport Association White Paper, 2006.

[5] Faigle, U., Kern, W. and Kuipers, J. (2001). On the computation of the nucleolus of a cooperative game. *International Journal of Game Theory*, 30, pp. 79–98.

[6] Federgruen, A. and Groenevelt, H. (1986). The Greedy Procedure for Resource Allocation Problems: Necessary and Sufficient Conditions for Optimality. *Operations Research*, 34(6), pp. 909–918.

[7] Federgruen, A. and Zheng, Y.S. (1992). The joint replenishment problem with general joint cost structures. *Operations Research*, 40, pp. 384–403.

[8] Federgruen, A., Queyranne, M. and Zheng, Y.S. (1992). Simple power-of-two policies are close to optimal in general class of production/distribution networks with general joint setup costs. *Mathematics of Operations Research*, 17, pp. 951–963.

[9] Garbe, R. and Glazebrook, K. D. (1998). Submodular Returns and Greedy Heuristics for Queueing Scheduling Problems. *Operations Research*, 46, pp. 336–346.

[10] Hochbaum, D.S. and Hong, S.-P. (1995). About strongly polynomial time algorithms for quadratic optimization over submodular constraints, *Mathematical Programming*, 69, pp. 269–309.

[11] Hochbaum, D.S., and Shanthikumar, J.G. (1990). Convex separable optimization is not much harder than linear optimization. *Journal of ACM*, 37, pp. 843–862.

[12] Lovász, L. (1983). Submodular functions and convexity. In: Bachem, A., Grötschel, M., and B. Korte (eds.), *Mathematical Programming: The State of the Art*.

[13] Collaborative Commmerce: Complelling Benefits, Significant Obstacles. NerveWire Inc. White Paper, 2002.

[14] Nagarajan, M. and Sošić, G. (2006). Game-Theoretical Analysis of Cooperation Among Supply Chain Agents: Review and Extensions. *European Journal of Operational Research*, to appear.

[15] Nemhauser, G.L., Wolsey, L.A., and Fisher, M.L. (1978). An Analysis of Approximations for Maximizing Submodular Set Functions -I. *Mathematical Programming*, 14, pp. 265–294.

[16] Peleg, B. and Sudholter, P. (2003). Introduction to the Theory of Cooperative Games. Kluwer Academic Publishers, Boston.

[17] Puerto, J. and Tamir, A. (2005). Locating tree-shaped facilities using the ordered median objective. *Mathematical Programming*, 102, pp. 313–338.

[18] Rockafellar, R.T. (1970). *Convex Analysis*. Princeton University Press.

[19] Roundy, R.O. (1985). 98% effective integer-ratio lot-sizing for one warehouse multi-retailer systems. *Management Science*, 31, pp. 1416–1430.

[20] Schulz, A.S. and Uhan N.A. (2007). Encouraging Cooperation in Sharing Supermodular Costs. *Lecture Notes in Computer Science*, Vol. 4627, pp. 271–285.

[21] Shapley, L. S. Cores of Convex Games. (1971). *International Journal of Game Theory*, 1, pp. 11–26.

[22] Sharkey, W. W. (1982). Cooperative games with large cores. *International Journal of Game Theory*, 11, pp. 175-182.

[23] Tse, D. and Hanly, V. (1998). Multiaccess fading channels—-Part I: Polymatroid structure, optimal resource allocation, and throughput capacities. *IEEE Trans. Inform. Theory*, 44, pp. 2796–2815.

[24] Topkis, D. M. (1998). *Supermodularity and Complementarity.* Princeton University Press, Princeton, NJ.

[25] Wolsey, L.A. (1982). Maximising Real-Valued Submodular Functions: Primal and Dual Heuristics for Location Problems. *Mathematics of Operations Research*, 7(3), pp. 410–425.

[26] Yao, D.D. (2002). Dynamic Scheduling via Polymatroid Optimization. *Lecture Notes In Computer Science*, Vol. 2459, pp. 89–113.

[27] Zhang, J. (2009). Cost Allocation for Joint Replenishment Models. *Operations Research*, 57, pp. 146–156.

[28] Zhang, J. (2008). Joint Replenishement Game and Maximizing an H-Schur Concave Function Over a Polymatroid. Working paper.