On Cones of Nonnegative Quartic Forms

Bo JIANG * Zhening LI † Shuzhong ZHANG ‡

September 2, 2011

Abstract

Historically, much of the theory and practice in nonlinear optimization has revolved around the quadratic models. Though quadratic functions are nonlinear polynomials, they are well structured and easy to deal with. Limitations of the quadratics, however, become increasingly binding as higher degree nonlinearity is imperative in modern applications of optimization. In the recent years, one observes a surge of research activities in polynomial optimization, and modeling with quartic or higher order polynomial functions has been more commonly accepted. On the theoretical side, there are also major recent progresses on polynomial functions and optimization. For instance, Ahmadi *et al.* [2] proved that checking the convexity of a quartic polynomial function is strongly NP-hard in general, which settles a long-standing open question. In this paper we proceed to studying six fundamentally important convex cones of quartic functions in the space of symmetric quartic tensors, including the cone of nonnegative quartic polynomials, the sum of squared polynomials, the convex quartic polynomials, and the sum of fourth powered polynomials. It turns out that these convex cones coagulate into a chain in decreasing order. The complexity status of these cones is sorted out as well. Finally, potential applications of the new results to solve highly nonlinear and/or combinatorial optimization problems are discussed.

Keywords: cone of polynomial functions; nonnegative quartic forms; sum of squares; sos-convexity.

Mathematics Subject Classification 2010: 15A69, 08A40, 90C25, 49N15.

---

*Industrial and Systems Engineering Program, University of Minnesota, Minneapolis, MN 55455. Email: jiang373@umn.edu

†Department of Mathematics, Shanghai University, Shanghai, 200444, China. Email: zheningli@gmail.com

‡Industrial and Systems Engineering Program, University of Minnesota, Minneapolis, MN 55455. Email: zhangs@umn.edu. (On leave from Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Shatin, Hong Kong. Email: zhang@se.cuhk.edu.hk)

1 Introduction

Checking the convexity of a quadratic function boils down to testing the positive semidefiniteness of its Hessian matrix in the domain. Since the Hessian matrix is constant, the test can be done easily. A natural question thus arises:

> Given a fourth degree polynomial function in n variables, can one still easily tell if the function is convex or not?

This simple-looking question was first put forward by Shor [28] in 1992, which turned out later to be a very challenging question to answer. For almost two decades, the question remained open. Only until recently Ahmadi *et al.* [2] proved that checking the convexity of a general quartic polynomial function is actually strongly NP-hard. Their result not only settled this particular open problem, but also helped to highlight a crucial difference between quartic and quadratic polynomials, which makes the study of quartic polynomials all the more compelling and interesting.

On the practical side, quartic polynomial optimization has a wide spectrum of applications, including sensor network localization [7], MIMO radar waveform optimization [12], portfolio management with high moments information [20], quantum entanglement problem [13], among many others. This has stimulated a burst of recent research activities with regard to quartic polynomial optimization. Due to the NP-hardness of quartic polynomial optimization models (see e.g. [25, 15, 23]), there is a considerable amount of recent research work devoted to the approximation algorithms for solving various quartic polynomial optimization models. Luo and Zhang [25] proposed an approximation algorithm for optimization of a quartic polynomial with quadratic constraints. Ling *et al.* [24] considered a special quartic optimization model, which is to maximize a bi-quadratic function over two spheres. He *et al.* [15, 16] extended the study to arbitrary degree polynomials. So [35] improved some of the approximation bounds presented in [15]. For a comprehensive survey on the topic, one may refer to the recent Ph.D. thesis of Li [23]. Another well studied approach to cope with general polynomial optimization problems is the so-called SOS method proposed by Lasserre [21] and Parillo [29]. Theoretically, it can solve any general polynomial optimization model to any given accuracy through resorting to a sequence of Semidefinite Programs (SDP). A main drawback of this approach is that the size of those (SDP) problems may grow intolerably large quickly. Interested readers may find more information in a recent survey [22] and the references therein.

There is an intrinsic connection between optimizing a polynomial function and the description of *all* the polynomial functions that are nonnegative over a given domain. For the case of quadratic polynomials, this connection was explored by Sturm and Zhang in [36], and later for the bi-quadratic case in Luo *et al.* [26]. Such investigations can be traced back to the 19th century when the relationship between nonnegative polynomial functions and the sum of squares (SOS) of polynomials was explicitly studied. One concrete question of interest was: *Given a multivariate polynomial function that takes only nonnegative values over the real numbers, can it be represented as a sum of squares of polynomial functions?* Hilbert [19] in 1888 showed that the only three classes of polynomial functions where this is generically true can be explicitly identified: (1) univariate polynomials; (2) multivariate quadratic polynomials; (3) bivariate quartic polynomials. Since polynomial functions with a fixed degree form a vector space, and the *nonnegative* polynomials and the *SOS* polynomials form two convex cones respectively within that vector space, the afore-mentioned results can be understood as a specification of three particular cases where these two convex cones coincide, while in general of course the cone of nonnegative polynomials is larger. There are certainly other interesting convex cones in the same vector space. For instance, the convex polynomial functions form yet another convex cone in that vector space. Helton and Nie [18] introduced the notion of sos-convex polynomials, to indicate the polynomials whose Hessian matrix can be decomposed as a sum of squares of polynomial matrices. All these classes of convex cones are important in their own rights. They are also important for the sake of optimization of polynomial functions. There are some substantial recent progresses along this direction. As we mentioned earlier, e.g., the question of Shor [28] regarding the complexity of deciding the convexity of a quartic polynomial was nicely settled by Ahmadi *et al.* [2]. It is also natural to inquire if the Hessian matrix of a convex polynomial is sos-convex. Ahmadi and Parrilo [3] gave an example to show that this is not the case in general. Blekherman proved that a convex polynomial is not necessary a sum of squares [6] if the degree of the polynomial is larger than two. However, Blekherman's proof is not constructive, and it remains an open problem to construct a concrete example of convex polynomial which is not a sum of squares. Reznick [34] studied the sum of m-th powers of real linear forms.

In view of the cones formed by the polynomial functions (e.g. the cones of nonnegative polynomials, the convex polynomials, the SOS polynomials and the sos-convex polynomials), it is natural to inquire about their relational structures. We shall set out to explore this in the paper. In a way there is a 'phase transition' in terms of complexity when the scope of polynomials goes beyond quadratics. Compared to the quadratic case (cf. Sturm and Zhang [36]), the structure of the quartic forms is far from being clear. We believe that the class of quartic polynomial functions (or the class of quartic forms) is an appropriate subject of study on its own right, beyond quadratic functions (or matrices). There are at least three immediate reasons to elaborate on the quartic polynomials, rather than polynomial functions of other degrees. First of all, nonnegativity is naturally associated with even degree polynomials, and the quartic polynomial is next to quadratic polynomials in that hierarchy. Second, quartic polynomials represent a landscape *after* the 'phase transition' takes place. However, dealing with quartic polynomials is still manageable, as far as notations are concerned. Finally, from an application point of view, quartic polynomial optimization is by far the most relevant polynomial optimization model beyond quadratic polynomials. The afore-mentioned examples such as kurtosis risks in portfolio management ([20]), the bi-quadratic optimization models ([24]), and the nonlinear least square formulation of sensor network localization ([7]) are all such examples. In this paper we aim to present the relational structure of several important convex cones in the vector space of quartic forms. We shall also motivate the study by some examples from applications.

2 Preparations

2.1 Notations

Throughout this paper, we use the lower-case letters to denote vectors (e.g., $x \in \mathbf{R}^n$), the capital letters to denote matrices (e.g., $A \in \mathbf{R}^{n^2}$), and the capital calligraphy letters to denote fourth order tensors (e.g., $\mathcal{F} \in \mathbf{R}^{n^4}$), with subscriptions of indices being their entries (e.g., $x_1, A_{ij}, \mathcal{F}_{ijk\ell} \in \mathbf{R}$). The boldface capital letters are reserved for sets in the Euclidean space, e.g., various sets of quatric forms to be introduced later, as well as $\mathbf{R}^{n^4}$, the space of n-dimensional fourth order tensors.

A generic quartic form is a fourth degree homogeneous polynomial function in n variables, or specifically the function

$$f(x) = \sum_{1 \le i \le j \le k \le \ell \le n} \mathcal{G}_{ijk\ell}\, x_i x_j x_k x_\ell, \tag{1}$$

where $x = (x_1, \cdots, x_n)^{\mathrm{T}} \in \mathbf{R}^n$. Closely related to a quartic form is a fourth order super-symmetric tensor $\mathcal{F} \in \mathbf{R}^{n^4}$. A tensor is said to be *super-symmetric* if its entries are invariant under all permutations of its indices. The set of n-dimensional super-symmetric fourth order tensors is denoted by $\mathbf{S}^{n^4}$. In fact, super-symmetric tensors are bijectively related to forms. In particular, restricting to fourth order tensors, for a given super-symmetric tensor $\mathcal{F} \in \mathbf{S}^{n^4}$, the quartic form in (1) can be uniquely determined by the following operation:

$$f(x) = \mathcal{F}(x, x, x, x) := \sum_{1 \le i,j,k,\ell \le n} \mathcal{F}_{ijk\ell}\, x_i x_j x_k x_\ell, \tag{2}$$

where $x \in \mathbf{R}^n$, $\mathcal{F}_{ijk\ell} = \mathcal{G}_{ijk\ell}/|\Pi(ijk\ell)|$ and $\Pi(ijk\ell)$ is the set of all distinctive permutations of the indices $\{i, j, k, \ell\}$, and vice versa. (This is the same as the one-to-one correspondence between a symmetric matrix and a quadratic form.) In the remainder of the paper, we shall more frequently use a super-symmetric tensor $\mathcal{F} \in \mathbf{S}^{n^4}$ to indicate a quartic form $f(x)$ or $\mathcal{F}(x, x, x, x)$, i.e., the notion of "super-symmetric fourth order tensor" and "quartic form" are used interchangeably in this paper.

Given a quartic form $\mathcal{F} \in \mathbf{S}^{n^4}$ and a matrix $X \in \mathbf{R}^{n^2}$, we may also define the following operation (in the same spirit as (2)):

$$\mathcal{F}(X, X) := \sum_{1 \le i,j,k,\ell \le n} \mathcal{F}_{ijk\ell}\, X_{ij} X_{k\ell}.$$

We call a fourth order tensor $\mathcal{G} \in \mathbf{R}^{n^4}$ *partial-symmetric*, if

$$\mathcal{G}_{ijk\ell} = \mathcal{G}_{jik\ell} = \mathcal{G}_{ij\ell k} = \mathcal{G}_{k\ell ij} \quad \forall\, 1 \le i,j,k,\ell \le n.$$

Essentially this means that the tensor form is symmetric for the first and the last two indices respectively, and is also symmetric by swapping the first two and the last two indices. The set of all partial-symmetric fourth order tensors in $\mathbf{R}^{n^4}$ is denoted by $\overrightarrow{\mathbf{S}}^{n^4}$. Obviously $\mathbf{S}^{n^4} \subsetneq \overrightarrow{\mathbf{S}}^{n^4} \subsetneq \mathbf{R}^{n^4}$ if $n \ge 2$.

For any fourth order tensor $\mathcal{G} \in \mathbf{R}^{n^4}$, we introduce a *symmetrization* mapping $\mathrm{sym}\colon \mathbf{R}^{n^4} \mapsto \mathbf{S}^{n^4}$, which is $\mathcal{F} = \mathrm{sym}\,\mathcal{G}$ with

$$\mathcal{F}_{ijk\ell} = \frac{1}{|\Pi(ijk\ell)|} \sum_{\pi \in \Pi(ijk\ell)} \mathcal{G}_\pi \quad \forall\, 1 \le i,j,k,\ell \le n.$$

The symbol '$\otimes$' represents the outer product of vectors or matrices. If $\mathcal{F} = x \otimes x \otimes x \otimes x$ for some $x \in \mathbf{R}^n$, then $\mathcal{F}_{ijk\ell} = x_i x_j x_k x_\ell$; and if $\mathcal{G} = X \otimes X$ for some $X \in \mathbf{R}^{n^2}$, then $\mathcal{G}_{ijk\ell} = X_{ij} X_{k\ell}$. The symbol '$\bullet$' denotes the operation of inner product. As a result, we have $\mathcal{F}(x,x,x,x) = \mathcal{F} \bullet (x \otimes x \otimes x \otimes x)$.

2.2 Introducing the Quartic Forms

In this subsection we shall formally introduce the definitions of the quartic forms (or equivalently, super-symmetric fourth order tensors) discussed in this paper. Let us start with the well known notion of positive semidefinite (PSD) and the sum of squares (SOS) of polynomials.

Definition 2.1 *A quartic form $\mathcal{F} \in \mathbf{S}^{n^4}$ is called quartic PSD if*

$$\mathcal{F}(x,x,x,x) \ge 0 \quad \forall\, x \in \mathbf{R}^n. \tag{3}$$

The set of all quartic PSD forms in $\mathbf{S}^{n^4}$ is denoted by $\mathbf{S}^{n^4}_+$.

If a quartic form $\mathcal{F} \in \mathbf{S}^{n^4}$ can be written as a sum of squares of polynomial functions, then these polynomials must be quadratic forms, i.e.,

$$\mathcal{F}(x,x,x,x) = \sum_{i=1}^m \left(x^{\mathrm{T}} A^i x\right)^2 = (x \otimes x \otimes x \otimes x) \bullet \sum_{i=1}^m A^i \otimes A^i,$$

where $A^i \in \mathbf{S}^{n^2}$, the set of symmetric matrices. However, $\sum_{i=1}^m \left(A^i \otimes A^i\right) \in \overrightarrow{\mathbf{S}}^{n^4}$ is only partial-symmetric, and may not be exactly $\mathcal{F}$, which must be super-symmetric. To place it in the family $\mathbf{S}^{n^4}$, a symmetrization operation is required. Since $x \otimes x \otimes x \otimes x$ is super-symmetric, we still have $(x \otimes x \otimes x \otimes x) \bullet \mathrm{sym}\left(\sum_{i=1}^m A^i \otimes A^i\right) = (x \otimes x \otimes x \otimes x) \bullet \sum_{i=1}^m A^i \otimes A^i$.

Definition 2.2 *A quartic form $\mathcal{F} \in \mathbf{S}^{n^4}$ is called quartic SOS if $\mathcal{F}(x,x,x,x)$ is a sum of squares of quadratic forms, i.e., there exist m symmetric matrices $A^1, \cdots, A^m \in \mathbf{S}^{n^2}$ such that*

$$\mathcal{F} = \mathrm{sym}\left(\sum_{i=1}^m A^i \otimes A^i\right) = \sum_{i=1}^m \mathrm{sym}\left(A^i \otimes A^i\right).$$

The set of quartic SOS forms in $\mathbf{S}^{n^4}$ is denoted by $\mathbf{\Sigma}^2_{n,4}$.

As all quartic SOS forms constitute a convex cone, we have

$$\mathbf{\Sigma}^2_{n,4} = \mathrm{sym}\,\mathrm{cone}\left\{A \otimes A \,|\, A \in \mathbf{S}^{n^2}\right\}.$$

Usually, for a given $\mathcal{F} = \mathrm{sym}\left(\sum_{i=1}^m A^i \otimes A^i\right)$ it maybe a challenge to write it explicitly as a sum of squares, although the construction can in principle be done in polynomial-time by semidefinte programming (SDP), which however is costly. In this sense, having a quartic SOS tensor in super-symmetric form may not always be beneficial, since the super-symmetry can destroy the SOS structure.

Since $\mathcal{F}(X,X)$ is a quadratic form, the usual sense of nonnegativity carries over. Formally we introduce this notion below.

Definition 2.3 *A quartic form $\mathcal{F} \in \mathbf{S}^{n^4}$ is called quartic matrix PSD if*

$$\mathcal{F}(X,X) \geq 0 \quad \forall\, X \in \mathbf{R}^{n^2}.$$

The set of quartic matrix PSD forms in $\mathbf{S}^{n^4}$ is denoted by $\mathbf{S}_+^{n^2 \times n^2}$.

Related to the sum of squares for quartic forms, we now introduce the notion to the *sum of quartics* (SOQ): If a quartic form $\mathcal{F} \in \mathbf{S}^{n^4}$ is SOQ, then there are m vectors $a^1, \cdots, a^m \in \mathbf{R}^n$ such that

$$\mathcal{F}(x,x,x,x) = \sum_{i=1}^m \left(x^{\mathrm{T}} a^i\right)^4 = (x \otimes x \otimes x \otimes x) \bullet \sum_{i=1}^m a^i \otimes a^i \otimes a^i \otimes a^i.$$

Definition 2.4 *A quartic form $\mathcal{F} \in \mathbf{S}^{n^4}$ is called quartic SOQ if $\mathcal{F}(x,x,x,x)$ is a sum of fourth powers of linear forms, i.e., there exist m vectors $a^1, \cdots, a^m \in \mathbf{R}^n$ such that*

$$\mathcal{F} = \sum_{i=1}^m a^i \otimes a^i \otimes a^i \otimes a^i.$$

The set of quartic SOQ forms in $\mathbf{S}^{n^4}$ is denoted by $\mathbf{\Sigma}^4_{n,4}$.

As all quartic SOQ forms also constitute a convex cone, we denote

$$\mathbf{\Sigma}^4_{n,4} = \text{cone}\,\{a \otimes a \otimes a \otimes a \,|\, a \in \mathbf{R}^n\} \subseteq \mathbf{\Sigma}^2_{n,4}.$$

In the case of quadratic functions, it is well known that for a given homogeneous form (i.e., a symmetric matrix, for that matter) $A \in \mathbf{S}^{n^2}$ the following two statements are equivalent:

- A is positive semidefinite (PSD): $A(x,x) := x^{\mathrm{T}}Ax \geq 0$ for all $x \in \mathbf{R}^n$.
- A is a sum of squares (SOS): $A(x,x) = \sum_{i=1}^m (x^{\mathrm{T}}a^i)^2$ (or equivalently $A = \sum_{i=1}^m a^i \otimes a^i$) for some $a^1, \cdots, a^m \in \mathbf{R}^n$.

It is therefore clear that the four types of quartic forms defined above are actually different extensions of the nonnegativity. In particular, quartic PSD and quartic matrix PSD forms are extended from quadratic PSD, while quartic SOS and SOQ forms are in the form of summation of nonnegative polynomials, and are extended from quadratic SOS. We will show later that there is an interesting hierarchical relationship:

$$\mathbf{\Sigma}^4_{n,4} \subsetneq \mathbf{S}^{n^2 \times n^2}_{+} \subsetneq \mathbf{\Sigma}^2_{n,4} \subsetneq \mathbf{S}^{n^4}_{+}. \tag{4}$$

Recently, a class of polynomials termed the *sos-convex polynomials* (cf. Helton and Nie [18]) has been brought to attention, which is defined as follows (see [4] for three other equivalent definitions of the sos-convexity):

> A multivariate polynomial function $f(x)$ is sos-convex if its Hessian matrix $H(x)$ can be factorized as $H(x) = (M(x))^{\mathrm{T}} M(x)$ with a polynomial matrix $M(x)$.

The reader is referred to [3] for applications of the sos-convex polynomials. In this paper, we shall focus on $\mathbf{S}^{n^4}$ and investigate sos-convex quartic forms with the hierarchy (4). For a quartic form $\mathcal{F} \in \mathbf{S}^{n^4}$, it is straightforward to compute its Hessian matrix $H(x) = 12\mathcal{F}(x,x,\cdot,\cdot)$, i.e.,

$$(H(x))_{ij} = 12\mathcal{F}(x,x,e^i,e^j) \quad \forall\, 1 \leq i,j \leq n,$$

where $e^i \in \mathbf{R}^n$ is the vector whose i-th entry is 1 and other entries are zeros. Therefore $H(x)$ is a quadratic matrix of x. If $H(x)$ can be decomposed as $H(x) = (M(x))^{\mathrm{T}} M(x)$ with $M(x)$ being a polynomial matrix, then $M(x)$ must be linear with respect to x.

Definition 2.5 *A quartic form $\mathcal{F} \in \mathbf{S}^{n^4}$ is called quartic sos-convex, if there exists a linear matrix $M(x)$ of x, such that its Hessian matrix*

$$12\mathcal{F}(x,x,\cdot,\cdot) = (M(x))^{\mathrm{T}} M(x).$$

The set of quartic sos-convex forms in $\mathbf{S}^{n^4}$ is denoted by $\mathbf{\Sigma}^2_{\nabla^2_{n,4}}$.

Helton and Nie [18] proved that a nonnegative polynomial is sos-convex, then it must be SOS. In particular, if the polynomial is a quartic form, by denoting the i-th row of the linear matrix $M(x)$ to be $x^{\mathrm{T}}A^i$ for $i = 1, \dots, m$ and some matrices $A^1, \cdots, A^m \in \mathbf{R}^{n^2}$, then $(M(x))^{\mathrm{T}}M(x) = \sum_{i=1}^m (A^i)^{\mathrm{T}} x x^{\mathrm{T}} A^i$. Therefore

$$\mathcal{F}(x,x,x,x) = x^{\mathrm{T}}\mathcal{F}(x,x,\cdot,\cdot)x = \frac{1}{12}x^{\mathrm{T}}(M(x))^{\mathrm{T}}M(x)x = \frac{1}{12}\sum_{i=1}^m \left(x^{\mathrm{T}}A^i x\right)^2 \in \mathbf{\Sigma}^2_{n,4}.$$

In addition, the Hessian matrix for a quartic sos-convex form is obviously positive semidefinite for any $x \in \mathbf{R}^n$. Hence sos-convexity implies convexity. Combining these two facts, we conclude that a quartic sos-convex form is both SOS and convex, which motivates us to study the last quartic forms in this paper.

Definition 2.6 *A quartic form $\mathcal{F} \in \mathbf{S}^{n^4}$ is called convex and SOS, if it is both quartic SOS and convex. The set of quartic convex and SOS forms in $\mathbf{S}^{n^4}$ is denoted by $\mathbf{\Sigma}^2_{n,4} \bigcap \mathbf{S}^{n^4}_{cvx}$.*

Here $\mathbf{S}^{n^4}_{cvx}$ is denoted to be the set of all convex quartic forms in $\mathbf{S}^{n^4}$.

2.3 Main Results and the Organization of the Paper

All the sets of the quartic forms defined in Section 2.2 are clearly convex cones. The remainder of this paper is organized as follows. In Section 3, we start by studying the cones: $\mathbf{S}^{n^4}_+$, $\mathbf{\Sigma}^2_{n,4}$, $\mathbf{S}^{n^2 \times n^2}_+$, and $\mathbf{\Sigma}^4_{n,4}$. First of all, we show that they are all closed, and that they can be presented in different formulations. As an example, the cone of quartic SOQ forms is

$$\mathbf{\Sigma}^4_{n,4} = \operatorname{cone}\{a \otimes a \otimes a \otimes a \,|\, a \in \mathbf{R}^n\} = \operatorname{sym} \operatorname{cone}\left\{A \otimes A \,|\, A \in \mathbf{S}^{n^2}_+,\, \operatorname{rank}(A) = 1\right\},$$

which can also be written as

$$\operatorname{sym} \operatorname{cone}\left\{A \otimes A \,|\, A \in \mathbf{S}^{n^2}_+\right\},$$

meaning that the rank-one constraint can be removed without affecting the cone itself. Moreover, among these four cones there are two primal-dual pairs: $\mathbf{S}^{n^4}_+$ is dual to $\mathbf{\Sigma}^4_{n,4}$, and $\mathbf{\Sigma}^2_{n,4}$ is dual to $\mathbf{S}^{n^2 \times n^2}_+$. The hierarchical relationship $\mathbf{\Sigma}^4_{n,4} \subsetneq \mathbf{S}^{n^2 \times n^2}_+ \subsetneq \mathbf{\Sigma}^2_{n,4} \subsetneq \mathbf{S}^{n^4}_+$ will also be shown in the same section.

In Section 4, we further study two more cones: $\mathbf{\Sigma}^2_{\nabla^2_{n,4}}$ and $\mathbf{\Sigma}^2_{n,4} \bigcap \mathbf{S}^{n^4}_{cvx}$. Interestingly, these two new cones can be nicely placed in the hierarchical scheme (4):

$$\mathbf{\Sigma}^4_{n,4} \subsetneq \mathbf{S}^{n^2 \times n^2}_+ \subsetneq \mathbf{\Sigma}^2_{\nabla^2_{n,4}} \subseteq \left(\mathbf{\Sigma}^2_{n,4} \bigcap \mathbf{S}^{n^4}_{cvx}\right) \subsetneq \mathbf{\Sigma}^2_{n,4} \subsetneq \mathbf{S}^{n^4}_+. \tag{5}$$

The complexity status of all these cones are discussed in Section 5, and in particular we show that $\mathbf{\Sigma}^2_{\nabla^2_{n,4}} \subsetneq \left(\mathbf{\Sigma}^2_{n,4} \bigcap \mathbf{S}^{n^4}_{cvx}\right)$ unless $P = NP$, completing the picture presented in (5), on the premise that $P \neq NP$. The low dimensional cases of these cones are also discussed in Section 5. Specially, for the case $n = 2$, all the six cones reduce to only two distinctive ones, and for the case $n = 3$, they reduce to exactly three distinctive cones. Finally, in Section 6 we discuss applications of quartic conic programming, including bi-quadratic assignment problems and eigenvalues of super-symmetric tensors.

3 Quartic PSD Forms, Quartic SOS Forms, and the Dual Cones

Let us now consider the first four cones of quartic forms introduced in Section 2.2: $\mathbf{\Sigma}^4_{n,4}$, $\mathbf{S}^{n^2\times n^2}_+$, $\mathbf{\Sigma}^2_{n,4}$, and $\mathbf{S}^{n^4}_+$.

3.1 Closedness

Proposition 3.1 $\mathbf{\Sigma}^4_{n,4}$*,* $\mathbf{S}^{n^2\times n^2}_+$*,* $\mathbf{\Sigma}^2_{n,4}$*, and* $\mathbf{S}^{n^4}_+$ *are all closed convex cones.*

While $\mathbf{S}^{n^4}_+$ and $\mathbf{S}^{n^2\times n^2}_+$ are evidently closed, by a similar argument as in [36] it is also easy to see that the cone of quartic SOS forms $\mathbf{\Sigma}^2_{n,4} := \operatorname{sym}\operatorname{cone}\left\{A\otimes A \,|\, A\in \mathbf{S}^{n^2}\right\}$ is closed. It only remains to show that the cone of quartic SOQ forms $\mathbf{\Sigma}^4_{n,4}$ is closed. In fact, we have a slightly stronger result below, which ensures the closedness of $\mathbf{\Sigma}^4_{n,4}$:

Lemma 3.2 *If* $\mathbf{D}\subseteq \mathbf{R}^n$*, then*

$$\operatorname{cl}\operatorname{cone}\{a\otimes a\otimes a\otimes a \,|\, a\in\mathbf{D}\} = \operatorname{cone}\{a\otimes a\otimes a\otimes a \,|\, a\in\operatorname{cl}\mathbf{D}\}.$$

Proof. Suppose that $\mathcal{F}\in\operatorname{cl}\operatorname{cone}\{a\otimes a\otimes a\otimes a \,|\, a\in\mathbf{D}\}$, then there is a sequence of quartic forms $\mathcal{F}^k\in\operatorname{cone}\{a\otimes a\otimes a\otimes a \,|\, a\in\mathbf{D}\}$ $(k=1,2,\ldots)$, such that $\mathcal{F}=\lim_{k\to\infty}\mathcal{F}^k$. Since the dimension of $\mathbf{S}^{n^4}$ is $N := \binom{n+3}{4}$, it follows from Carathéodory's theorem that for any given $\mathcal{F}^k$, there exists an $n\times(N+1)$ matrix Z^k, such that

$$\mathcal{F}^k = \sum_{i=1}^{N+1} z^k(i)\otimes z^k(i)\otimes z^k(i)\otimes z^k(i),$$

where $z^k(i)$ is the i-th column vector of Z^k, and is a positive multiple of a vector in $\mathbf{D}$. Now define $\operatorname{tr}\mathcal{F}^k = \sum_{j=1}^n \mathcal{F}^k_{jjjj}$, then

$$\sum_{i=1}^{N+1}\sum_{j=1}^{n}(Z^k_{ji})^4 = \operatorname{tr}\mathcal{F}^k \to \operatorname{tr}\mathcal{F}.$$

Thus, the sequence $\{Z^k\}$ is bounded, and have a cluster point Z^*, satisfying $\mathcal{F} = \sum_{i=1}^{N+1} z^*(i) \otimes z^*(i) \otimes z^*(i) \otimes z^*(i)$. Note that each column of Z^* is also a positive multiple of a vector in $\mathrm{cl}\,\mathbf{D}$, it follows that $\mathcal{F} \in \mathrm{cone}\,\{a \otimes a \otimes a \otimes a \,|\, a \in \mathrm{cl}\,\mathbf{D}\}$. The converse containing relationship is trivial. $\square$

The cone of quartic SOQ forms is closely related to the fourth moment of a multi-dimensional random variable. Given an n-dimensional random variable $\xi = (\xi_1, \cdots, \xi_n)^{\mathrm{T}}$ on the support set $\mathbf{D} \subseteq \mathbf{R}^n$ with density function p, its fourth moment is a super-symmetric fourth order tensor $\mathcal{M} \in \mathbf{S}^{n^4}$, whose (i,j,k,ℓ)-th entry is

$$\mathcal{M}_{ijk\ell} = \mathsf{E}\,[\xi_i\xi_j\xi_k\xi_\ell] = \int_{\mathbf{D}} x_i x_j x_k x_\ell \, p(x) dx.$$

Suppose the fourth moment of ξ is finite, then by the closedness of $\mathbf{\Sigma}_{n,4}^4$, we have

$$\mathcal{M} = \mathsf{E}\,[\xi \otimes \xi \otimes \xi \otimes \xi] = \int_{\mathbf{D}} (x \otimes x \otimes x \otimes x)\, p(x) dx \in \mathrm{cone}\,\{a \otimes a \otimes a \otimes a \,|\, a \in \mathbf{R}^n\} = \mathbf{\Sigma}_{n,4}^4.$$

Conversely, for any $\mathcal{M} \in \mathbf{\Sigma}_{n,4}^4$, it is easy to verify that there exists an n-dimensional random variable whose fourth moment is just $\mathcal{M}$. Thus, the set of all the finite fourth moments of n-dimensional random variables is exactly $\mathbf{\Sigma}_{n,4}^4$, similar to the fact that all possible covariance matrices form the cone of positive semidefinite matrices.

3.2 Alternative Representations

In this subsection we present some alternative forms of the same cones that we have discussed. Some of these alternative representations are more convenient to use in various applications.

Theorem 3.3 *For the quartic polynomials cones introduced, we have the following equivalent representations:*

1. *For the cone of quartic SOS forms*

$$\mathbf{\Sigma}_{n,4}^2 := \mathrm{sym\,cone}\left\{A \otimes A \,|\, A \in \mathbf{S}^{n^2}\right\} = \mathrm{sym}\left\{\mathcal{F} \in \overrightarrow{\mathbf{S}}^{n^4} \,|\, \mathcal{F}(X,X) \geq 0,\, \forall\, X \in \mathbf{S}^{n^2}\right\};$$

2. *For the cone of quartic matrix PSD forms*

$$\begin{aligned} \mathbf{S}_+^{n^2 \times n^2} &:= \left\{\mathcal{F} \in \mathbf{S}^{n^4} \,|\, \mathcal{F}(X,X) \geq 0,\, \forall\, X \in \mathbf{R}^{n^2}\right\} \\ &= \left\{\mathcal{F} \in \mathbf{S}^{n^4} \,|\, \mathcal{F}(X,X) \geq 0,\, \forall\, X \in \mathbf{S}^{n^2}\right\} \\ &= \mathbf{S}^{n^4} \bigcap \mathrm{cone}\left\{A \otimes A \,|\, A \in \mathbf{S}^{n^2}\right\}; \end{aligned}$$

3. For the cone of quartic SOQ forms

$$\mathbf{\Sigma}^4_{n,4} := \text{cone}\,\{a\otimes a\otimes a\otimes a \,|\, a\in \mathbf{R}^n\} = \text{sym}\,\text{cone}\left\{A\otimes A \,|\, A\in \mathbf{S}^{n^2}_+\right\}.$$

Recall that $\overrightarrow{\mathbf{S}}^{n^4}$ is the set of partial-symmetric fourth order tensors in $\mathbf{R}^{n^4}$, defined in Section 2.1. The remaining of this subsection is devoted to the proof of Theorem 3.3.

Let us first study the equivalent representations for $\mathbf{\Sigma}^2_{n,4}$ and $\mathbf{S}^{n^2\times n^2}_+$. To verify a quartic matrix PSD form, we should check the operations of quartic forms on matrices. In fact, the quartic matrix PSD forms can be extended to the space of partial-symmetric tensors $\overrightarrow{\mathbf{S}}^{n^4}$. It is not hard to verify that for any $\mathcal{F}\in\overrightarrow{\mathbf{S}}^{n^4}$, it holds that

$$\mathcal{F}(X,Y)=\mathcal{F}(X^{\mathrm{T}},Y)=\mathcal{F}(X,Y^{\mathrm{T}})=\mathcal{F}(Y,X)\quad \forall\, X,Y\in\mathbf{R}^{n^2}, \tag{6}$$

which implies that $\mathcal{F}(X,Y)$ is invariant under the transpose operation as well as the operation to swap the X and Y matrices. Indeed, it is easy to see that the partial-symmetry of $\mathcal{F}$ is a necessary and sufficient condition for (6) to hold. We have the following property for quartic matrix PSD forms in $\overrightarrow{\mathbf{S}}^{n^4}$, similar to that for $\mathbf{S}^{n^2\times n^2}_+$ in Theorem 3.3.

Lemma 3.4 *For partial-symmetric four order tensors, it holds that*

$$\begin{aligned}
\overrightarrow{\mathbf{S}}^{n^2\times n^2}_+ &:= \left\{\mathcal{F}\in\overrightarrow{\mathbf{S}}^{n^4} \,|\, \mathcal{F}(X,X)\ge 0,\,\forall\, X\in\mathbf{R}^{n^2}\right\}\\
&= \left\{\mathcal{F}\in\overrightarrow{\mathbf{S}}^{n^4} \,|\, \mathcal{F}(X,X)\ge 0,\,\forall\, X\in\mathbf{S}^{n^2}\right\} &(7)\\
&= \text{cone}\left\{A\otimes A \,|\, A\in\mathbf{S}^{n^2}\right\}. &(8)
\end{aligned}$$

Proof. Observe that for any skew-symmetric $Y\in\mathbf{R}^{n^2}$, i.e., $Y^{\mathrm{T}}=-Y$, we have

$$\mathcal{F}(X,Y)=-\mathcal{F}(X,-Y)=-\mathcal{F}(X,Y^{\mathrm{T}})=-\mathcal{F}(X,Y)\quad\forall\, X\in\mathbf{R}^{n^2},$$

which implies that $\mathcal{F}(X,Y)=0$. As any square matrix can be written as the sum of a symmetric matrix and a skew-symmetric matrix, say for $Z\in\mathbf{R}^{n^2}$, by letting $X=(Z+Z^{\mathrm{T}})/2$ which is symmetric, and $Y=(Z-Z^{\mathrm{T}})/2$ which is skew-symmetric, we have $Z=X+Y$. Therefore,

$$\mathcal{F}(Z,Z)=\mathcal{F}(X+Y,X+Y)=\mathcal{F}(X,X)+2\mathcal{F}(X,Y)+\mathcal{F}(Y,Y)=\mathcal{F}(X,X).$$

This implies the equivalence between $\mathcal{F}(X,X)\ge 0,\,\forall\, X\in\mathbf{R}^{n^2}$ and $\mathcal{F}(X,X)\ge 0,\,\forall\, X\in\mathbf{S}^{n^2}$, which proves (7).

To prove (8), first note that cone $\left\{A\otimes A \,|\, A\in\mathbf{S}^{n^2}\right\}\subseteq\{\mathcal{F}\in\overrightarrow{\mathbf{S}}^{n^4} \,|\, \mathcal{F}(X,X)\ge 0,\,\forall\, X\in\mathbf{R}^{n^2}\}$. Conversely, given any $\mathcal{G}\in\overrightarrow{\mathbf{S}}^{n^4}$ with $\mathcal{G}(X,X)\ge 0,\,\forall\, X\in\mathbf{R}^{n^2}$, we may rewrite $\mathcal{G}$ as an $n^2\times n^2$ symmetric matrix $M_{\mathcal{G}}$. Therefore

$$(\text{vec}\,(X))^{\mathrm{T}} M_{\mathcal{G}}\,\text{vec}\,(X)=\mathcal{G}(X,X)\ge 0\quad\forall\, X\in\mathbf{R}^{n^2},$$

which implies that $M_{\mathcal{G}}$ is positive semidefinite. Let $M_{\mathcal{G}} = \sum_{i=1}^m z^i (z^i)^{\mathrm{T}}$, where

$$z^i = \left(z^i_{11}, \cdots, z^i_{1n}, \cdots, z^i_{n1}, \cdots, z^i_{nn}\right)^{\mathrm{T}} \quad \forall\, 1 \le i \le m.$$

Note that for any $1 \le k, \ell \le n$, $\mathcal{G}_{k\ell\ell k} = \sum_{i=1}^m z^i_{k\ell} z^i_{\ell k}$, $\mathcal{G}_{k\ell k\ell} = \sum_{i=1}^m (z^i_{k\ell})^2$ and $\mathcal{G}_{\ell k\ell k} = \sum_{i=1}^m (z^i_{\ell k})^2$, as well as $\mathcal{G}_{k\ell\ell k} = \mathcal{G}_{k\ell k\ell} = \mathcal{G}_{\ell k\ell k}$ by partial-symmetry of $\mathcal{G}$. We have

$$\sum_{i=1}^m (z^i_{k\ell} - z^i_{\ell k})^2 = \sum_{i=1}^m (z^i_{k\ell})^2 + \sum_{i=1}^m (z^i_{\ell k})^2 - 2\sum_{i=1}^m z^i_{k\ell} z^i_{\ell k} = \mathcal{G}_{k\ell k\ell} + \mathcal{G}_{\ell k\ell k} - 2\mathcal{G}_{k\ell\ell k} = 0,$$

which implies that $z^i_{k\ell} = z^i_{\ell k}$ for any $1 \le k, \ell \le n$. Therefore, we may construct a symmetric matrix $Z^i \in \mathbf{S}^{n^2}$, such that $\mathrm{vec}\,(Z^i) = z^i$ for all $1 \le i \le m$. We have $\mathcal{G} = \sum_{i=1}^m Z^i \otimes Z^i$, and so (8) is proven. □

The first part of Theorem 3.3 follows from (8) by applying the symmetrization operation on both sides, while the second part of Theorem 3.3 follows from (7) and (8) by restricting to $\mathbf{S}^{n^4}$.

Let us now turn to proving the last part of Theorem 3.3, which is an alternative representation of the quartic SOQ forms. Obviously we need only to show that

$$\mathrm{sym\,cone}\left\{A \otimes A \,|\, A \in \mathbf{S}^{n^2}_+\right\} \subseteq \mathrm{cone}\,\{a \otimes a \otimes a \otimes a \,|\, a \in \mathbf{R}^n\}\,.$$

Since there is a one-to-one mapping from quartic forms to fourth order super-symmetric tensors, it suffices to show that for any $A \in \mathbf{S}^{n^2}_+$, the function $(x^{\mathrm{T}}Ax)^2$ can be written as a form of $\sum_{i=1}^m (x^{\mathrm{T}}a^i)^4$ for some $a^1, \cdots, a^m \in \mathbf{R}^n$. Note that the so-called Hilbert's identity (see e.g., Barvinok [5]) asserts the following:

> For any fixed positive integers d and n, there always exist m real vectors $a^1, \cdots, a^m \in \mathbf{R}^n$ such that $(x^{\mathrm{T}}x)^d = \sum_{i=1}^m (x^{\mathrm{T}}a^i)^{2d}$.

In fact, when $d = 2$, He *et al.* [17] proposed a polynomial-time algorithm to find the afore-mentioned representations where the number m is bounded by a polynomial of n, although in the original version of Hilbert's identity m is exponential in n. Since we have $A \in \mathbf{S}^{n^2}_+$, replacing x by $A^{1/2}y$ in Hilbert's identity when $d = 2$, one gets $(y^{\mathrm{T}}Ay)^2 = \sum_{i=1}^m (y^{\mathrm{T}}A^{1/2}a^i)^4$. The desired decomposition follows, and this proves the last part of Theorem 3.3.

3.3 Duality

In this subsection, we shall discuss the duality relationships among these four cones of quartic forms. Note that $\mathbf{S}^{n^4}$ is the ground vector space within which the duality is defined, unless otherwise specified.

Theorem 3.5 *The cone of quartic PSD forms and the cone of quartic SOQ forms are primal-dual pair, i.e.,* $\mathbf{\Sigma}^4_{n,4} = \left(\mathbf{S}^{n^4}_+\right)^*$ *and* $\mathbf{S}^{n^4}_+ = \left(\mathbf{\Sigma}^4_{n,4}\right)^*$*. The cone of quartic SOS forms and the cone of quartic matrix PSD forms are primal-dual pair, i.e.,* $\mathbf{S}^{n^2\times n^2}_+ = \left(\mathbf{\Sigma}^2_{n,4}\right)^*$ *and* $\mathbf{\Sigma}^2_{n,4} = \left(\mathbf{S}^{n^2\times n^2}_+\right)^*$*.*

Remark that the primal-dual relationship between $\mathbf{\Sigma}^4_{n,4}$ and $\mathbf{S}^{n^4}_+$ was also studied in [34]. Let us indeed start by discussing the primal-dual pair $\mathbf{\Sigma}^4_{n,4}$ and $\mathbf{S}^{n^4}_+$. In Proposition 1 of [36], Sturm and Zhang proved that for the quadratic forms, $\{A \in \mathbf{S}^{n^2} \,|\, x^{\mathrm{T}} A x \ge 0,\, \forall\, x \in \mathbf{D}\}$ and $\mathrm{cone}\,\{aa^{\mathrm{T}} \,|\, a \in \mathbf{D}\}$ are a primal-dual pair for any closed $\mathbf{D} \subseteq \mathbf{R}^n$. We observe that a similar structure holds for the quartic forms as well. The first part of Theorem 3.5 then follows from next lemma.

Lemma 3.6 *If* $\mathbf{D} \subseteq \mathbf{R}^n$ *is closed, then* $\mathbf{S}^{n^4}_+(\mathbf{D}) := \{\mathcal{F} \in \mathbf{S}^{n^4} \,|\, \mathcal{F}(x,x,x,x) \ge 0,\, \forall\, x \in \mathbf{D}\}$ *and* $\mathrm{cone}\,\{a\otimes a\otimes a\otimes a \,|\, a \in \mathbf{D}\}$ *are a primal-dual pair, i.e.,*

$$\mathbf{S}^{n^4}_+(\mathbf{D}) = \left(\mathrm{cone}\,\{a\otimes a\otimes a\otimes a \,|\, a \in \mathbf{D}\}\right)^* \tag{9}$$

and

$$\left(\mathbf{S}^{n^4}_+(\mathbf{D})\right)^* = \mathrm{cone}\,\{a\otimes a\otimes a\otimes a \,|\, a \in \mathbf{D}\}.$$

Proof. Since $\mathrm{cone}\,\{a\otimes a\otimes a\otimes a \,|\, a \in \mathbf{D}\}$ is closed by Lemma 3.2, we only need to show (9). In fact, if $\mathcal{F} \in \mathbf{S}^{n^4}_+(\mathbf{D})$, then $\mathcal{F} \bullet (a\otimes a\otimes a\otimes a) = \mathcal{F}(a,a,a,a) \ge 0$ for all $a \in \mathbf{D}$. Thus $\mathcal{F} \bullet \mathcal{G} \ge 0$ for all $\mathcal{G} \in \mathrm{cone}\,\{a\otimes a\otimes a\otimes a \,|\, a \in \mathbf{D}\}$, which implies that $\mathcal{F} \in \left(\mathrm{cone}\,\{a\otimes a\otimes a\otimes a \,|\, a \in \mathbf{D}\}\right)^*$. Conversely, if $\mathcal{F} \in \left(\mathrm{cone}\,\{a\otimes a\otimes a\otimes a \,|\, a \in \mathbf{D}\}\right)^*$, then $\mathcal{F} \bullet \mathcal{G} \ge 0$ for all $\mathcal{G} \in \mathrm{cone}\,\{a\otimes a\otimes a\otimes a \,|\, a \in \mathbf{D}\}$. In particular, by letting $\mathcal{G} = x\otimes x\otimes x\otimes x$, we have $\mathcal{F}(x,x,x,x) = \mathcal{F} \bullet (x\otimes x\otimes x\otimes x) \ge 0$ for all $x \in \mathbf{D}$, which implies that $\mathcal{F} \in \mathbf{S}^{n^4}_+(\mathbf{D})$. ☐

Let us turn to the primal-dual pair of $\mathbf{S}^{n^2\times n^2}_+$ and $\mathbf{\Sigma}^2_{n,4}$. For technical reasons, we shall momentarily lift the ground space from $\mathbf{S}^{n^4}$ to the space of partial-symmetric tensors $\overrightarrow{\mathbf{S}}^{n^4}$. This enlarges all the dual objects. To distinguish these two dual objects, let us use the notation '$\mathbf{K}^{\overrightarrow{*}}$' to indicate the dual of convex cone $\mathbf{K} \in \mathbf{S}^{n^4} \subseteq \overrightarrow{\mathbf{S}}^{n^4}$ generated in the space $\overrightarrow{\mathbf{S}}^{n^4}$, while '$\mathbf{K}^*$' is the dual of $\mathbf{K}$ generated in the space $\mathbf{S}^{n^4}$.

Lemma 3.7 *For partial-symmetric tensors, the cone* $\overrightarrow{\mathbf{S}}^{n^2\times n^2}_+$ *is self-dual with respect to the space* $\overrightarrow{\mathbf{S}}^{n^4}$*, i.e.,* $\overrightarrow{\mathbf{S}}^{n^2\times n^2}_+ = \left(\overrightarrow{\mathbf{S}}^{n^2\times n^2}_+\right)^{\overrightarrow{*}}$*.*

Proof. According to Proposition 1 of [36] and the partial-symmetry of $\overrightarrow{\mathbf{S}}^{n^4}$, we have

$$\left(\mathrm{cone}\,\left\{A\otimes A \,|\, A \in \mathbf{S}^{n^2}\right\}\right)^{\overrightarrow{*}} = \left\{\mathcal{F} \in \overrightarrow{\mathbf{S}}^{n^4} \,|\, \mathcal{F}(X,X) \ge 0,\, \forall\, X \in \mathbf{S}^{n^2}\right\}.$$

By Lemma 3.4, we have

$$\overrightarrow{\mathbf{S}}_{+}^{n^2\times n^2} = \left\{\mathcal{F}\in\overrightarrow{\mathbf{S}}^{n^4}\,|\,\mathcal{F}(X,X)\ge 0,\,\forall\, X\in\mathbf{S}^{n^2}\right\} = \mathrm{cone}\left\{A\otimes A\,|\,A\in\mathbf{S}^{n^2}\right\}.$$

Thus $\overrightarrow{\mathbf{S}}_{+}^{n^2\times n^2}$ is self-dual with respect to the space $\overrightarrow{\mathbf{S}}^{n^4}$. □

Notice that by definition and Lemma 3.7, we have

$$\mathbf{\Sigma}_{n,4}^2 = \mathrm{sym}\,\mathrm{cone}\left\{A\otimes A\,|\,A\in\mathbf{S}^{n^2}\right\} = \mathrm{sym}\,\overrightarrow{\mathbf{S}}_{+}^{n^2\times n^2} = \mathrm{sym}\left(\overrightarrow{\mathbf{S}}_{+}^{n^2\times n^2}\right)^{\overrightarrow{*}},$$

and by the alternative representation in Theorem 3.3 we have

$$\mathbf{S}_{+}^{n^2\times n^2} = \mathbf{S}^{n^4}\bigcap\mathrm{cone}\left\{A\otimes A\,|\,A\in\mathbf{S}^{n^2}\right\} = \mathbf{S}^{n^4}\bigcap\overrightarrow{\mathbf{S}}_{+}^{n^2\times n^2}.$$

Therefore the duality between $\mathbf{S}_{+}^{n^2\times n^2}$ and $\mathbf{\Sigma}_{n,4}^2$ follows immediately from the following lemma.

Lemma 3.8 *If $\mathbf{K}\subseteq\overrightarrow{\mathbf{S}}^{n^4}$ is a closed convex cone and $\mathbf{K}^{\overrightarrow{*}}$ is its dual with respect to the space $\overrightarrow{\mathbf{S}}^{n^4}$, then $\mathbf{K}\bigcap\mathbf{S}^{n^4}$ and $\mathrm{sym}\,\mathbf{K}^{\overrightarrow{*}}$ are a primal-dual pair with respect to the space $\mathbf{S}^{n^4}$, i.e., $\left(\mathbf{K}\bigcap\mathbf{S}^{n^4}\right)^* = \mathrm{sym}\,\mathbf{K}^{\overrightarrow{*}}$ and $\mathbf{K}\bigcap\mathbf{S}^{n^4} = \left(\mathrm{sym}\,\mathbf{K}^{\overrightarrow{*}}\right)^*$.*

Proof. For any $\mathcal{G}\in\mathrm{sym}\,\mathbf{K}^{\overrightarrow{*}}\subseteq\mathbf{S}^{n^4}$, there is a $\mathcal{G}'\in\mathbf{K}^{\overrightarrow{*}}\subseteq\overrightarrow{\mathbf{S}}^{n^4}$, such that $\mathcal{G}=\mathrm{sym}\,\mathcal{G}'\in\mathbf{S}^{n^4}$. We then have $\mathcal{G}_{ijk\ell}=\frac{1}{3}(\mathcal{G}'_{ijk\ell}+\mathcal{G}'_{ikj\ell}+\mathcal{G}'_{i\ell jk})$. Thus for any $\mathcal{F}\in\mathbf{K}\bigcap\mathbf{S}^{n^4}\subseteq\mathbf{S}^{n^4}$, it follows that

$$\begin{aligned}\mathcal{F}\bullet\mathcal{G} &= \sum_{1\le i,j,k,\ell\le n}\frac{\mathcal{F}_{ijk\ell}(\mathcal{G}'_{ijk\ell}+\mathcal{G}'_{ikj\ell}+\mathcal{G}'_{i\ell jk})}{3}\\ &= \sum_{1\le i,j,k,\ell\le n}\frac{\mathcal{F}_{ijk\ell}\mathcal{G}'_{ijk\ell}+\mathcal{F}_{ikj\ell}\mathcal{G}'_{ikj\ell}+\mathcal{F}_{i\ell jk}\mathcal{G}'_{i\ell jk}}{3}\\ &= \mathcal{F}\bullet\mathcal{G}'\ge 0.\end{aligned}$$

Therefore $\mathcal{G}\in\left(\mathbf{K}\bigcap\mathbf{S}^{n^4}\right)^*$, implying that $\mathrm{sym}\,\mathbf{K}^{\overrightarrow{*}}\subseteq\left(\mathbf{K}\bigcap\mathbf{S}^{n^4}\right)^*$.

Moreover, if $\mathcal{F}\in\left(\mathrm{sym}\,\mathbf{K}^{\overrightarrow{*}}\right)^*\subseteq\mathbf{S}^{n^4}$, then for any $\mathcal{G}'\in\mathbf{K}^{\overrightarrow{*}}\subseteq\overrightarrow{\mathbf{S}}^{n^4}$, we have $\mathcal{G}=\mathrm{sym}\,\mathcal{G}'\in\mathrm{sym}\,\mathbf{K}^{\overrightarrow{*}}$, and $\mathcal{G}'\bullet\mathcal{F}=\mathcal{G}\bullet\mathcal{F}\ge 0$. Therefore $\mathcal{F}\in\left(\mathbf{K}^{\overrightarrow{*}}\right)^{\overrightarrow{*}}=\mathrm{cl}\,\mathbf{K}=\mathbf{K}$, which implies that $\left(\mathrm{sym}\,\mathbf{K}^{\overrightarrow{*}}\right)^*\subseteq\left(\mathbf{K}\bigcap\mathbf{S}^{n^4}\right)$. Finally, the duality relationship holds by the bipolar theorem and the closedness of these cones. □

3.4 The Hierarchical Structure

The last part of this section is to present a hierarchy among these four cones of quartic forms. The main result is summarized in the theorem below.

Theorem 3.9 *If $n \geq 4$, then*

$$\mathbf{\Sigma}^4_{n,4} \subsetneq \mathbf{S}_+^{n^2\times n^2} \subsetneq \mathbf{\Sigma}^2_{n,4} \subsetneq \mathbf{S}_+^{n^4}.$$

For the low dimension cases ($n \leq 3$), we shall present it in Section 5.2. Evidently a quartic SOS form is quartic PSD, implying $\mathbf{\Sigma}^2_{n,4} \subseteq \mathbf{S}_+^{n^4}$. By invoking the duality operation and using Theorem 3.5 we have $\mathbf{\Sigma}^4_{n,4} \subseteq \mathbf{S}_+^{n^2\times n^2}$, while by the alternative representation in Theorem 3.3 we have $\mathbf{S}_+^{n^2\times n^2} = \mathbf{S}^{n^4} \bigcap \operatorname{cone}\left\{A\otimes A \,|\, A\in \mathbf{S}^{n^2}\right\}$, and by the very definition we have $\mathbf{\Sigma}^2_{n,4} = \operatorname{sym}\operatorname{cone}\left\{A\otimes A \,|\, A\in \mathbf{S}^{n^2}\right\}$. Therefore $\mathbf{S}_+^{n^2\times n^2} \subseteq \mathbf{\Sigma}^2_{n,4}$. Finally, the strict containing relationship is a result of the following examples.

Example 3.1 *(Quartic forms in $\mathbf{S}_+^{n^4} \setminus \mathbf{\Sigma}^2_{n,4}$ when $n=4$). Let $g_1(x) = {x_1}^2(x_1-x_4)^2 + {x_2}^2(x_2-x_4)^2 + {x_3}^2(x_3-x_4)^2 + 2x_1x_2x_3(x_1+x_2+x_3-2x_4)$ and $g_2(x) = {x_1}^2{x_2}^2 + {x_2}^2{x_3}^2 + {x_3}^2{x_1}^2 + {x_4}^4 - 4x_1x_2x_3x_4$, then both $g_1(x)$ and $g_2(x)$ are in $\mathbf{S}_+^{4^4} \setminus \mathbf{\Sigma}^2_{4,4}$. We refer the interested readers to [33] for more information on these two cones.*

Example 3.2 *(A quartic form in $\mathbf{S}_+^{n^2\times n^2} \setminus \mathbf{\Sigma}^4_{n,4}$ when $n=4$). Construct $\mathcal{F} \in \mathbf{S}^{4^4}$, whose only nonzero entries (taking into account the super-symmetry) are $\mathcal{F}_{1122}=4$, $\mathcal{F}_{1133}=4$, $\mathcal{F}_{2233}=4$, $\mathcal{F}_{1144}=9$, $\mathcal{F}_{2244}=9$, $\mathcal{F}_{3344}=9$, $\mathcal{F}_{1234}=6$, $\mathcal{F}_{1111}=29$, $\mathcal{F}_{2222}=29$, $\mathcal{F}_{3333}=29$, and $\mathcal{F}_{4444} = 3+\frac{25}{7}$. One may verify straightforwardly that $\mathcal{F}$ can be decomposed as $\sum_{i=1}^7 A^i\otimes A^i$, with $A^1 = \begin{bmatrix} \sqrt{7} & 0 & 0 & 0 \\ 0 & \sqrt{7} & 0 & 0 \\ 0 & 0 & \sqrt{7} & 0 \\ 0 & 0 & 0 & \frac{5}{\sqrt{7}} \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{bmatrix}$, $A^4 = \begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix}$, $A^5 = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $A^6 = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, and $A^7 = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. According to Theorem 3.3, we have $\mathcal{F} \in \mathbf{S}_+^{4^2\times 4^2}$. Recall $g_2(x)$ in Example 3.1, which is quartic PSD. Denote $\mathcal{G}$ to be the super-symmetric tensor associated with $g_2(x)$, thus $\mathcal{G} \in \mathbf{S}_+^{4^4}$. One computes that $\mathcal{G} \bullet \mathcal{F} = 4+4+4+3+\frac{25}{7}-24 < 0$. By the duality result as stipulated in Theorem 3.5, we conclude that $\mathcal{F} \notin \mathbf{\Sigma}^4_{4,4}$.*

Example 3.3 *(A quartic form in $\mathbf{\Sigma}^2_{n,4} \setminus \mathbf{S}_+^{n^2\times n^2}$ when $n=3$). Let $g_3(x) = 2{x_1}^4 + 2{x_2}^4 + \frac{1}{2}{x_3}^4 + 6{x_1}^2{x_3}^2 + 6{x_2}^2{x_3}^2 + 6{x_1}^2{x_2}^2$, which is obviously quartic SOS. Now recycle the notation and denote $\mathcal{G} \in \mathbf{\Sigma}^2_{3,4}$ to be the super-symmetric tensor associated with $g_3(x)$, and we have $\mathcal{G}_{1111}=2$, $\mathcal{G}_{2222}=2$,*

$\mathcal{G}_{3333} = \frac{1}{2}$, $\mathcal{G}_{1122} = 1$, $\mathcal{G}_{1133} = 1$, *and* $\mathcal{G}_{2233} = 1$. *If we let* $X = \mathrm{Diag}\,(1, 1, -4) \in \mathbf{S}^{3^2}$, *then*

$$\mathcal{G}(X,X) = \sum_{1\le i,j,k,\ell\le 3} \mathcal{G}_{ijk\ell} X_{ij} X_{k\ell} = \sum_{1\le i,k\le 3} \mathcal{G}_{iikk} X_{ii} X_{kk} = \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & \frac{1}{2} \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix} = -2,$$

implying that $\mathcal{G} \notin \mathbf{S}_+^{3^2\times 3^2}$.

4 Cones Related to Convex Quartic Forms

In this section we shall study the cone of quartic sos-convex forms $\mathbf{\Sigma}^2_{\nabla^2_{n,4}}$, and the cone of quartic forms which are both SOS and convex $\mathbf{\Sigma}^2_{n,4} \bigcap \mathbf{S}^{n^4}_{cvx}$. The aim is to incorporate these two new cones into the hierarchical structure as depicted in Theorem 3.9.

Theorem 4.1 *If* $n \ge 4$, *then*

$$\mathbf{\Sigma}^4_{n,4} \subsetneq \mathbf{S}_+^{n^2\times n^2} \subsetneq \mathbf{\Sigma}^2_{\nabla^2_{n,4}} \subseteq \left(\mathbf{\Sigma}^2_{n,4} \bigcap \mathbf{S}^{n^4}_{cvx}\right) \subsetneq \mathbf{\Sigma}^2_{n,4} \subsetneq \mathbf{S}^{n^4}_+.$$

As mentioned in Section 2.2, an sos-convex homogeneous quartic polynomial function is both SOS and convex (see also [3]), which implies that $\mathbf{\Sigma}^2_{\nabla^2_{n,4}} \subseteq \left(\mathbf{\Sigma}^2_{n,4} \bigcap \mathbf{S}^{n^4}_{cvx}\right)$. Moreover, the following example shows that a quartic SOS form is not necessarily convex.

Example 4.1 *(A quartic form in* $\mathbf{\Sigma}^2_{n,4} \setminus \mathbf{S}^{n^4}_{cvx}$ *when* $n = 2$*). Let* $g_4(x) = (x^{\mathrm{T}}Ax)^2$ *with* $A \in \mathbf{S}^{n^2}$, *and its Hessian matrix is* $\nabla^2 g_4(x) = 8Axx^{\mathrm{T}}A + 4x^{\mathrm{T}}AxA$. *In particular, by letting* $A = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}$ *and* $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, *we have* $\nabla^2 f(x) = \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix} + \begin{bmatrix} -12 & 0 \\ 0 & 4 \end{bmatrix} \not\succeq 0$, *implying that* $g_4(x)$ *is not convex.*

The above example suggests that $\left(\mathbf{\Sigma}^2_{n,4} \bigcap \mathbf{S}^{n^4}_{cvx}\right) \subsetneq \mathbf{\Sigma}^2_{n,4}$ when $n \ge 2$. Next we shall prove the assertion that $\mathbf{S}_+^{n^2\times n^2} \subsetneq \mathbf{\Sigma}^2_{\nabla^2_{n,4}}$ when $n \ge 3$. To this end, let us first quote a result on the sos-convex functions due to Ahmadi and Parrilo [3]:

> If $f(x)$ is a polynomial with its Hessian matrix being $\nabla^2 f(x)$, then $f(x)$ is sos-convex if and only if $y^{\mathrm{T}}\nabla^2 f(x) y$ is a sum of squares in (x, y).

For a quartic form $\mathcal{F}(x,x,x,x)$, its Hessian matrix is $12\mathcal{F}(x,x,\cdot,\cdot)$. Therefore, $\mathcal{F}$ is quartic sos-convex if and only if $\mathcal{F}(x,x,y,y)$ is is a sum of squares in (x,y). Now if $\mathcal{F} \in \mathbf{S}_+^{n^2\times n^2}$, then by

Theorem 3.3 we may find matrices $A^1, \cdots, A^m \in \mathbf{S}^{n^2}$ such that $\mathcal{F} = \sum_{t=1}^m A^t \otimes A^t$. We have

$$\mathcal{F}(x,x,y,y) = \mathcal{F}(x,y,x,y) = \sum_{t=1}^m \sum_{1\le i,j,k,\ell\le n} x_i y_j x_k y_\ell A^t_{ij} A^t_{k\ell}$$
$$= \sum_{t=1}^m \left(\sum_{1\le i,j\le n} x_i y_j A^t_{ij} \right) \left(\sum_{1\le k,\ell\le n} x_k y_\ell A^t_{k\ell} \right) = \sum_{t=1}^m \left(x^{\mathrm{T}} A^t y \right)^2,$$

which is a sum of squares in (x,y), hence sos-convex. This proves $\mathbf{S}_+^{n^2\times n^2} \subseteq \mathbf{\Sigma}^2_{\nabla^2_{n,4}}$, and the example below rules out the equality when $n \ge 3$.

Example 4.2 *(A quartic form in $\mathbf{\Sigma}^2_{\nabla^2_{n,4}} \setminus \mathbf{S}_+^{n^2\times n^2}$ when $n=3$). Recall $g_3(x) = 2{x_1}^4 + 2{x_2}^4 + \frac{1}{2}{x_3}^4 + 6{x_1}^2{x_3}^2 + 6{x_2}^2{x_3}^2 + 6{x_1}^2{x_2}^2$ in Example 3.3, which is shown not to be quartic matrix PSD. Moreover, it is straightforward to compute that*

$$\nabla^2 g_3(x) = 24 \begin{pmatrix} x_1 \\ x_2 \\ \frac{x_3}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \frac{x_3}{2} \end{pmatrix}^{\mathrm{T}} + 12 \begin{pmatrix} 0 \\ x_3 \\ x_2 \end{pmatrix} \begin{pmatrix} 0 \\ x_3 \\ x_2 \end{pmatrix}^{\mathrm{T}} + 12 \begin{pmatrix} x_3 \\ 0 \\ x_1 \end{pmatrix} \begin{pmatrix} x_3 \\ 0 \\ x_1 \end{pmatrix}^{\mathrm{T}} + 12 \begin{bmatrix} {x_2}^2 & 0 & 0 \\ 0 & {x_1}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \succeq 0,$$

which implies that $g_3(x)$ is quartic sos-convex.

A natural question regarding to the hierarchical structure in Theorem 4.1 is whether $\mathbf{\Sigma}^2_{n,4} \bigcap \mathbf{S}^{n^4}_{cvx} = \mathbf{\Sigma}^2_{\nabla^2_{n,4}}$ or not. In fact, the relationship between convex, sos-convex, and SOS is a highly interesting subject which attracted many speculations in the literature. Ahmadi and Parrilo [3] gave an explicit example with a three dimensional homogeneous form of degree eight, which they showed to be both convex and SOS but not sos-convex. However, for quartic polynomials (degree four), such an explicit instant in $\left(\mathbf{\Sigma}^2_{n,4} \bigcap \mathbf{S}^{n^4}_{cvx}\right) \setminus \mathbf{\Sigma}^2_{\nabla^2_{n,4}}$ is not in sight. Notwithstanding the difficulty in constructing an explicit quartic example, on the premise that $P \neq NP$ in Section 5 we will show that $\left(\mathbf{\Sigma}^2_{n,4} \bigcap \mathbf{S}^{n^4}_{cvx}\right) \neq \mathbf{\Sigma}^2_{\nabla^2_{n,4}}$. With that piece of information, the chain of containing relations manifested in Theorem 4.1 is complete, under the assumption that $P \neq NP$. However, the following open question remains:

Question 4.1 *Find an explicit instant in $\left(\mathbf{\Sigma}^2_{n,4} \bigcap \mathbf{S}^{n^4}_{cvx}\right) \setminus \mathbf{\Sigma}^2_{\nabla^2_{n,4}}$: a quartic form that is both SOS and convex, but not sos-convex.*

The two newly introduced cones in this section are related to the convexity properties. Some more words on convex quartic forms are in order here. As mentioned in Section 2.2, for a quartic form $\mathcal{F} \in \mathbf{S}^{n^4}_{cvx}$, its Hessian matrix is $12\mathcal{F}(x,x,\cdot,\cdot)$. Therefore, $\mathcal{F}$ is convex if and only if $\mathcal{F}(x,x,\cdot,\cdot) \succeq 0$ for all $x \in \mathbf{R}^n$, which is equivalent to $\mathcal{F}(x,x,y,y) \ge 0$ for all $x,y \in \mathbf{R}^n$. In fact, it is also equivalent to $\mathcal{F}(X,Y) \geq 0$ for all $X, Y \in \mathbf{S}_+^{n^2}$. To see why, we first decompose the positive semidefinite matrices X and Y, and let $X = \sum_{i=1}^n x^i (x^i)^{\mathrm{T}}$ and $Y = \sum_{j=1}^n y^j (y^j)^{\mathrm{T}}$ (see e.g., Sturm and Zhang [36]). Then

$$\begin{aligned} \mathcal{F}(X,Y) &= \mathcal{F}\left(\sum_{i=1}^n x^i (x^i)^{\mathrm{T}}, \sum_{j=1}^n y^j (y^j)^{\mathrm{T}}\right) \\ &= \sum_{1 \leq i,j \leq n} \mathcal{F}\left(x^i (x^i)^{\mathrm{T}}, y^j (y^j)^{\mathrm{T}}\right) \\ &= \sum_{1 \leq i,j \leq n} \mathcal{F}\left(x^i, x^i, y^j, y^j\right) \geq 0, \end{aligned}$$

if $\mathcal{F}(x,x,y,y) \geq 0$ for all $x, y \in \mathbf{R}^n$. Note that the converse is trivial, as it reduces to let X and Y be rank-one positive semidefinite matrices. Thus we have the following equivalence for the quartic convex forms.

Proposition 4.2 *For a given quartic form $\mathcal{F} \in \mathbf{S}^{n^4}$, the following statements are equivalent:*

- $\mathcal{F}(x,x,x,x)$ *is convex;*
- $\mathcal{F}(x,x,\cdot,\cdot)$ *is positive semidefinite for all* $x \in \mathbf{R}^n$*;*
- $\mathcal{F}(x,x,y,y) \geq 0$ *for all* $x, y \in \mathbf{R}^n$*;*
- $\mathcal{F}(X,Y) \geq 0$ *for all* $X, Y \in \mathbf{S}_+^{n^2}$.

For the relationship between the cone of quartic convex forms and the cone of quartic SOS forms, Example 4.1 has ruled out the possibility that $\mathbf{\Sigma}_{n,4}^2 \subseteq \mathbf{S}_{cvx}^{n^4}$, while Blekherman [6] proved that $\mathbf{\Sigma}_{n,4}^2$ is not contained in $\mathbf{S}_{cvx}^{n^4}$. Therefore these two cones are indeed distinctive. According to Blekherman [6], the cone of quartic convex forms is actually much bigger than the cone of quartic SOS forms when n is sufficiently large. However, at this point we are not aware of any explicit instant belonging to $\mathbf{S}_{cvx}^{n^4} \setminus \mathbf{\Sigma}_{n,4}^2$. In fact, according to a recent working paper of Ahmadi *et al.* [1], this kind of instants exist only when $n \geq 4$. Anyway, the following challenge remains:

Question 4.2 *Find an explicit instant in $\mathbf{S}_{cvx}^{n^4} \setminus \mathbf{\Sigma}_{n,4}^2$: a quartic convex form that is not quartic SOS.*

5 Complexity and Low Dimensions of the Quartic Cones

In this section, we shall study the computational complexity issues for the membership queries regarding these cones of quartic forms. Unlike their quadratic counterparts where the positive semidefniteness can be checked in polynomial-time, the case for the quartic cones are substantially subtler. We also study the low dimension cases of these cones, as a complement to the result of hirarchary relationship in Theorem 4.1.

5.1 Complexity

Let us start with easy cases. It is well known that deciding whether a general polynomial function is SOS can be done in polynomial-time, by resorting to checking the feasibility of an SDP. Therefore, the membership query for $\mathbf{\Sigma}^2_{n,4}$ can be done in polynomial-time. By the duality relationship claimed in Theorem 3.5, the membership query for $\mathbf{S}_+^{n^2\times n^2}$ can also be done in polynomial-time. In fact, for any quartic form $\mathcal{F}\in\mathbf{S}^{n^4}$, we may rewrite $\mathcal{F}$ as an $n^2\times n^2$ matrix, to be denoted by $M_{\mathcal{F}}$, and then Theorem 3.3 assures that $\mathcal{F}\in\mathbf{S}_+^{n^2\times n^2}$ if and only if $M_{\mathcal{F}}$ is positive semedefinite, which can be checked in polynomial-time. Moreover, as discussed in Section 4, a quartic form $\mathcal{F}$ is sos-convex if and only if $y^{\mathrm{T}}\left(\nabla^2\mathcal{F}(x,x,x,x)\right)y = 12\mathcal{F}(x,x,y,y)$ is a sum of squares in (x,y), which can also be checked in polynomial-time. Therefore, the membership checking problem for $\mathbf{\Sigma}^2_{\nabla^2_{n,4}}$ can be done in polynomial-time as well. Summarizing, we have:

Proposition 5.1 *Whether a quartic form belongs to $\mathbf{\Sigma}^2_{n,4}$, $\mathbf{S}_+^{n^2\times n^2}$, or $\mathbf{\Sigma}^2_{\nabla^2_{n,4}}$, can be verified in polynomial-time.*

Unfortunately, the membership checking problems for all the other cones that we have discussed so far are difficult. To see why, let us introduce a famous cone of quadratic functions: the co-positive cone

$$\mathbf{C} := \left\{A\in\mathbf{S}^{n^2}\,|\,x^{\mathrm{T}}Ax\ge 0,\,\forall\, x\in\mathbf{R}^n_+\right\},$$

whose membership query is known to be co-NP-complete. The dual of the co-positive cone is the cone of completely positive matrices, defined as

$$\mathbf{C}^* := \mathrm{cone}\left\{xx^{\mathrm{T}}\,|\,x\in\mathbf{R}^n_+\right\}.$$

Recently, Dickinson and Gijben [14] gave a formal proof for the NP-completeness of the membership problem for $\mathbf{C}^*$.

Proposition 5.2 *It is co-NP-complete to check if a quartic form belongs to $\mathbf{S}_+^{n^4}$ (the cone of quartic PSD forms).*

Proof. We shall reduce the problem to checking the membership of the co-positive cone $\mathbf{C}$. In particular, given any matrix $A\in\mathbf{S}^{n^2}$, construct an $\mathcal{F}\in\mathbf{S}^{n^4}$, whose only nonzero entries are

$$\mathcal{F}_{iikk}=\mathcal{F}_{ikik}=\mathcal{F}_{ikki}=\mathcal{F}_{kiik}=\mathcal{F}_{kiki}=\mathcal{F}_{kkii}=\begin{cases}\frac{A_{ik}}{3} & i\ne k,\\ A_{ik} & i=k.\end{cases}\tag{10}$$

For any $x \in \mathbf{R}^n$,

$$\begin{aligned}\mathcal{F}(x,x,x,x) &= \sum_{1\le i<k\le n} \left(\mathcal{F}_{iikk}+\mathcal{F}_{ikik}+\mathcal{F}_{ikki}+\mathcal{F}_{kiik}+\mathcal{F}_{kiki}+\mathcal{F}_{kkii}\right) {x_i}^2 {x_k}^2 + \sum_{i=1}^n \mathcal{F}_{iiii} {x_i}^4 \\ &= \sum_{1\le i,k\le n} A_{ik} {x_i}^2 {x_k}^2 = (x\circ x)^{\mathrm{T}} A (x\circ x), \qquad (11)\end{aligned}$$

where the symbol '$\circ$' represents the Hadamard product. Denote $y = x\circ x \ge 0$, and then $\mathcal{F}(x,x,x,x) \ge 0$ if and only if $y^{\mathrm{T}} A y \ge 0$. Therefore $A \in \mathbf{C}$ if and only if $\mathcal{F} \in \mathbf{S}_+^{n^4}$ and the reduction is complete. □

Proposition 5.3 *It is NP-hard to check if a quartic form belongs to* $\mathbf{\Sigma}_{n,4}^4$ *(the cone of quartic SOQ forms).*

Proof. Similarly, the problem can be reduced to checking the membership of the completely positive cone $\mathbf{C}^*$. In particular, given any matrix $A \in \mathbf{S}^{n^2}$, construct an $\mathcal{F} \in \mathbf{S}^{n^4}$, whose only nonzero entries are defined exactly as in (10). If $A \in \mathbf{C}^*$, then $A = \sum_{t=1}^m a^t (a^t)^{\mathrm{T}}$ for some $a^1, \cdots, a^m \in \mathbf{R}_+^n$. By the construction of $\mathcal{F}$, we have

$$\mathcal{F}_{iikk} = \mathcal{F}_{ikik} = \mathcal{F}_{ikki} = \mathcal{F}_{kiik} = \mathcal{F}_{kiki} = \mathcal{F}_{kkii} = \begin{cases} \sum_{t=1}^m \frac{a_i^t a_k^t}{3} & i \ne k, \\ \sum_{t=1}^m \left(a_i^t\right)^2 & i = k. \end{cases}$$

Denote $A^t = \mathrm{Diag}\,(a^t) \in \mathbf{S}_+^{n^2}$ for all $1 \le t \le m$. It is straightforward to verify that

$$\mathcal{F} = \sum_{t=1}^m \mathrm{sym}\,\left(A^t \otimes A^t\right) = \mathrm{sym}\,\left(\sum_{t=1}^m A^t \otimes A^t\right).$$

Therefore by Theorem 3.3 we have $\mathcal{F} \in \mathbf{\Sigma}_{n,4}^4$.

Conversely, if $A \notin \mathbf{C}^*$, then there exits a vector $y \in \mathbf{R}_+^n$, such that $y^{\mathrm{T}} A y < 0$. Define a vector $x \in \mathbf{R}_+^n$ with $x_i = \sqrt{y_i}$ for all $1 \le i \le n$. By (11), we have

$$\mathcal{F} \bullet (x\otimes x\otimes x\otimes x) = \mathcal{F}(x,x,x,x) = (x\circ x)^{\mathrm{T}} A (x\circ x) = y^{\mathrm{T}} A y < 0.$$

Therefore, by the duality relationship in Theorem 3.5, we have $\mathcal{F} \notin \mathbf{\Sigma}_{n,4}^4$. Since $A \in \mathbf{C}^*$ if and only if $\mathcal{F} \in \mathbf{\Sigma}_{n,4}^4$ and so it follows that $\mathbf{\Sigma}_{n,4}^4$ is a hard cone. □

Recently, Burer [8] showed that a large class of mixed-binary quadratic programs can be formulated as co-positive programs where a linear function is minimized over a linearly constrained subset of the cone of completely positive matrices. Later, Burer and Dong [9] extended this equivalence to general nonconvex quadratically constrained quadratic program whose feasible region is nonempty and bounded. From the proof of Proposition 5.3, the cone of completely positive matrices can be imbedded into the cone of quartic SOQ forms. Evidently, these mixed-binary quadratic programs can also be formulated as linear conic program with the cone $\mathbf{\Sigma}^4_{n,4}$. In fact, the modeling power of $\mathbf{\Sigma}^4_{n,4}$ is much greater, which we shall discuss in Section 6 for further illustration.

Before concluding this subsection, a final remark on the cone $\mathbf{\Sigma}^2_{n,4} \bigcap \mathbf{S}^{n^4}_{cvx}$ is in order. Recall the recent breakthrough [2] mentioned in Section 1, that checking the convexity of a quartic form is strongly NP-hard. However, if we are given more information, that the quartic form to be considered is a sum of squares, will this make the membership easier? The answer is still no, as the following theorem asserts.

Theorem 5.4 *Deciding the convexity of a quartic SOS form is strongly NP-hard. In particular, it is strongly NP-hard to check if a quartic form belongs to $\mathbf{\Sigma}^2_{n,4} \bigcap \mathbf{S}^{n^4}_{cvx}$.*

Proof. Let $G = (V, E)$ be a graph with V being the set of n vertices and E being the set of edges. Define the following bi-quadratic form associated with graph G as follows:

$$b_G(x, y) := 2 \sum_{(i,j)\in E} x_i x_j y_i y_j.$$

Ling *et al.* [24] showed that the problem $\max_{\|x\|_2=\|y\|_2=1} b_G(x, y)$ is strongly NP-hard. Define

$$b_{G,\lambda}(x, y) := \lambda(x^{\mathrm{T}}x)(y^{\mathrm{T}}y) - b_G(x, y) = \lambda(x^{\mathrm{T}}x)(y^{\mathrm{T}}y) - 2 \sum_{(i,j)\in E} x_i x_j y_i y_j.$$

Then determining the nonnegativity of $b_{G,\lambda}(x, y)$ in (x, y) is also strongly NP-hard, due to the fact that the problem $\max_{\|x\|_2=\|y\|_2=1} b_G(x, y)$ can be polynomially reduced to it. Let us now construct a quartic form in (x, y) as

$$f_{G,\lambda}(x, y) := b_{G,\lambda}(x, y) + n^2 \left(\sum_{i=1}^n {x_i}^4 + \sum_{i=1}^n {y_i}^4 + \sum_{1\le i<j\le n} {x_i}^2 {x_j}^2 + \sum_{1\le i<j\le n} {y_i}^2 {y_j}^2 \right).$$

Observe that

$$f_{G,\lambda}(x, y) = g_{G,\lambda}(x, y) + \sum_{(i,j)\in E} (x_i x_j - y_i y_j)^2 + (n^2 - 1) \sum_{(i,j)\in E} ({x_i}^2 {x_j}^2 + {y_i}^2 {y_j}^2),$$

where $g_{G,\lambda}(x, y) := \lambda(x^{\mathrm{T}}x)(y^{\mathrm{T}}y) + n^2 \left(\sum_{i=1}^n ({x_i}^4 + {y_i}^4) + \sum_{(i,j)\notin E} ({x_i}^2 {x_j}^2 + {y_i}^2 {y_j}^2) \right)$. Therefore $f_{G,\lambda}(x, y)$ is quartic SOS in (x, y). Moreover, according to Theorem 2.3 of [2] with $\gamma = 2$, we know that $f_{G,\lambda}(x, y)$ is convex if and only if $b_{G,\lambda}(x, y)$ is nonnegative. The latter being strongly NP-hard, therefore checking the convexity of the quartic SOS form $f_{G,\lambda}(x, y)$ is also strongly NP-hard. □

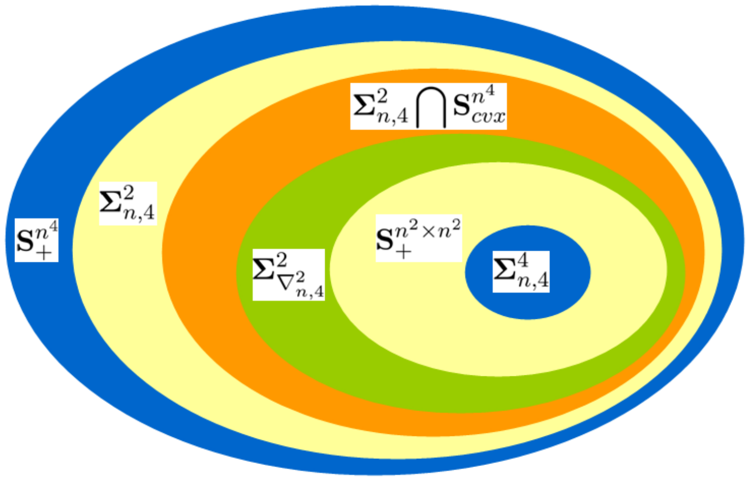

Figure 1: Hierarchy for the cones of quartic forms

With the help of Theorem 5.4 and Proposition 5.1, which claim that $\mathbf{\Sigma}^2_{n,4} \bigcap \mathbf{S}^{n^4}_{cvx}$ is a hard cone while $\mathbf{\Sigma}^2_{\nabla^2_{n,4}}$ is easy, we conclude the following complete hierarchical structure to clarify one last containing relationship in Theorem 4.1. (Note that Theorem 4.1 only concludes $\mathbf{\Sigma}^2_{\nabla^2_{n,4}} \subseteq \left(\mathbf{\Sigma}^2_{n,4} \bigcap \mathbf{S}^{n^4}_{cvx}\right)$.)

Corollary 5.5 *Assuming $P \neq NP$, for general n we have*

$$\mathbf{\Sigma}^4_{n,4} \subsetneq \mathbf{S}^{n^2\times n^2}_+ \subsetneq \mathbf{\Sigma}^2_{\nabla^2_{n,4}} \subsetneq \left(\mathbf{\Sigma}^2_{n,4} \bigcap \mathbf{S}^{n^4}_{cvx}\right) \subsetneq \mathbf{\Sigma}^2_{n,4} \subsetneq \mathbf{S}^{n^4}_+. \tag{12}$$

The relationship among these six cones of quartic forms is depicted in Figure 1, where a primal-dual pair is painted by the same color. The chain of containing relationship is useful especially when some of the cones are hard while others are easy. One obvious possible application is to use an 'easy' cone either as restriction or as relaxation of a hard one. Such scheme is likely to be useful in the design of approximation algorithms.

5.2 The Low Dimensional Cases

The chain of containing relations (12) holds for general dimension n. Essentially the strict containing relations are true for $n \geq 4$, except that we do not know if $\mathbf{\Sigma}^2_{\nabla^2_{n,4}} \subsetneq \left(\mathbf{\Sigma}^2_{n,4} \bigcap \mathbf{S}^{n^4}_{cvx}\right)$ holds true or not. To complete the picture, in this subsection we discuss quartic forms in low dimensional cases: $n = 2$ and $n = 3$. Specifically, when $n = 2$, the six cones of quartic forms reduce to two distinctive ones; while $n = 3$, they reduce to three distinctive cones.

Proposition 5.6 *For the cone of bi-variate quartic forms, it holds that*

$$\mathbf{\Sigma}^4_{2,4} = \mathbf{S}_+^{2^2\times 2^2} = \mathbf{\Sigma}^2_{\nabla^2_{2,4}} = \left(\mathbf{\Sigma}^2_{2,4} \bigcap \mathbf{S}^{2^4}_{cvx}\right) \subsetneq \mathbf{\Sigma}^2_{2,4} = \mathbf{S}^{2^4}_+.$$

Proof. By a well known equivalence between nonnegative polynomial and sum of squares due to Hilbert [19] (for bi-variate quartic polynomials), we have $\mathbf{\Sigma}^2_{n,4} = \mathbf{S}^{n^4}_+$ for $n \le 3$, by noticing that Hilbert's result is true for *inhomogeneous* polynomials and our cones are for homogeneous forms. Now, the duality relationship in Theorem 3.5 leads to $\mathbf{\Sigma}^4_{2,4} = \mathbf{S}_+^{2^2\times 2^2}$. Next let us focus on the relationship between $\mathbf{S}_+^{2^2\times 2^2}$ and $\mathbf{\Sigma}^2_{2,4} \bigcap \mathbf{S}^{2^4}_{cvx}$. In fact we shall prove below that $\mathbf{S}^{2^4}_{cvx} \subseteq \mathbf{S}_+^{2^2\times 2^2}$, i.e., any bi-variate convex quartic form is quartic matrix PSD.

For bi-variate convex quartic form $\mathcal{F}$ with

$$\mathcal{F}_{1111} = a_1, \mathcal{F}_{1112} = a_2, \mathcal{F}_{1122} = a_3, \mathcal{F}_{1222} = a_4, \mathcal{F}_{2222} = a_5,$$

we have $f(x) = \mathcal{F}(x,x,x,x) = a_1{x_1}^4 + 4a_2{x_1}^3x_2 + 6a_3{x_1}^2{x_2}^2 + 4a_4x_1{x_2}^3 + a_5{x_2}^4$, and

$$\nabla^2 f(x) = 12 \begin{bmatrix} a_1{x_1}^2 + 2a_2x_1x_2 + a_3{x_2}^2 & a_2{x_1}^2 + 2a_3x_1x_2 + a_4{x_2}^2 \\ a_2{x_1}^2 + 2a_3x_1x_2 + a_4{x_2}^2 & a_3{x_1}^2 + 2a_4x_1x_2 + a_5{x_2}^2 \end{bmatrix} \succeq 0 \quad \forall\, x_1, x_2 \in \mathbf{R}. \tag{13}$$

Denote $A^1 = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}$, $A^2 = \begin{bmatrix} a_2 & a_3 \\ a_3 & a_4 \end{bmatrix}$ and $A^3 = \begin{bmatrix} a_3 & a_4 \\ a_4 & a_5 \end{bmatrix}$, and (13) is equivalent to

$$\begin{bmatrix} x^{\mathrm{T}}A^1x & x^{\mathrm{T}}A^2x \\ x^{\mathrm{T}}A^2x & x^{\mathrm{T}}A^3x \end{bmatrix} \succeq 0 \quad \forall\, x \in \mathbf{R}^2. \tag{14}$$

According to Theorem 4.8 and the subsequent discussions in [26], it follows that (14) is equivalent to $\begin{bmatrix} A^1 & A^2 \\ A^2 & A^3 \end{bmatrix} \succeq 0$. Therefore,

$$\mathcal{F}(X,X) = (\mathrm{vec}\,(X))^{\mathrm{T}} \begin{bmatrix} A^1 & A^2 \\ A^2 & A^3 \end{bmatrix} \mathrm{vec}\,(X) \ge 0 \quad \forall\, X \in \mathbf{R}^{2^2},$$

implying that $\mathcal{F}$ is quartic matrix PSD. This proves $\mathbf{S}_+^{2^2\times 2^2} = \mathbf{\Sigma}^2_{2,4} \bigcap \mathbf{S}^{2^4}_{cvx}$. Finally, Example 4.1 for $\mathbf{\Sigma}^2_{2,4} \setminus \mathbf{S}^{2^4}_{cvx}$ leads to $\mathbf{\Sigma}^2_{2,4} \bigcap \mathbf{S}^{2^4}_{cvx} \ne \mathbf{\Sigma}^2_{2,4}$. □

It remains to consider the case $n = 3$. Our previous discussion concluded that $\mathbf{\Sigma}^2_{3,4} = \mathbf{S}^{3^4}_+$, and so by duality $\mathbf{\Sigma}^4_{3,4} = \mathbf{S}_+^{3^2\times 3^2}$. Moreover, in a recent working paper Ahmadi *et al.* [1] showed that every tri-variate convex quartic polynomial is sos-convex, implying $\mathbf{\Sigma}^2_{\nabla^2_{3,4}} = \left(\mathbf{\Sigma}^2_{3,4} \bigcap \mathbf{S}^{3^4}_{cvx}\right)$. So we have at most three distinctive cones of quartic forms. Example 4.1 in $\mathbf{\Sigma}^2_{2,4} \setminus \mathbf{S}^{2^4}_{cvx}$ and Example 4.2 in $\mathbf{\Sigma}^2_{\nabla^2_{3,4}} \setminus \mathbf{S}_+^{3^2\times 3^2}$ show that there are in fact three distinctive cones.

Proposition 5.7 *For the cone of tri-variate quartic forms, it holds that*

$$\mathbf{\Sigma}^4_{3,4} = \mathbf{S}_+^{3^2\times 3^2} \subsetneq \mathbf{\Sigma}^2_{\nabla^2_{3,4}} = \left(\mathbf{\Sigma}^2_{3,4} \bigcap \mathbf{S}^{3^4}_{cvx}\right) \subsetneq \mathbf{\Sigma}^2_{3,4} = \mathbf{S}^{3^4}_+.$$

6 Quartic Conic Programming

The study of quartic forms in the previous sections gives rise some new modeling opportunities. In this section we shall discuss quartic conic programming, i.e., optimizing a linear function over the intersection of an affine subspace and a cone of quartic forms. In particular, we shall investigate the following quartic conic programming model:

$$\begin{array}{lll} (QCP) & \max & \mathcal{C} \bullet \mathcal{X} \\ & \text{s.t.} & \mathcal{A}^i \bullet \mathcal{X} = b_i,\ i = 1,\ldots,m \\ & & \mathcal{X} \in \mathbf{\Sigma}^4_{n,4}, \end{array}$$

where $\mathcal{C}, \mathcal{A}^i \in \mathbf{S}^{n^4}$ and $b_i \in \mathbf{R}$ for $i = 1,\ldots,m$. As we will see later, a large class of non-convex quartic polynomial optimization models can be formulated as a special class of (QCP). In fact we will study a few concrete examples to show the modeling power of the quartic forms that we introduced.

6.1 Quartic Polynomial Optimization

Quartic polynomial optimization received much attention in the recent years; see e.g., [25, 24, 15, 16, 35, 23]. Essentially, all the models studied involve optimization of a quartic polynomial function subject to some linear and/or homogenous quadratic constraints, including spherical constraints, binary constraints, the intersection of co-centered ellipsoids, and so on. Below we consider a very general quartic polynomial optimization model:

$$\begin{array}{lll} (P) & \max & p(x) \\ & \text{s.t.} & (a^i)^{\mathrm{T}} x = b_i,\ i = 1,\ldots,m \\ & & x^{\mathrm{T}} A^j x = c_j,\ j = 1,\ldots,l \\ & & x \in \mathbf{R}^n, \end{array}$$

where $p(x)$ is a general inhomogeneous quartic polynomial function.

We first homogenize $p(x)$ by introducing a new homogenizing variable, say x_{n+1}, which is set to one, and get a homogeneous quartic form

$$p(x) = \mathcal{F}(\bar{x},\bar{x},\bar{x},\bar{x}) = \mathcal{F} \bullet (\bar{x}\otimes\bar{x}\otimes\bar{x}\otimes\bar{x}),$$

where $\mathcal{F} \in \mathbf{S}^{(n+1)^4}$, $\bar{x} = \begin{pmatrix} x \\ x_{n+1} \end{pmatrix}$ and $x_{n+1} = 1$. By adding some redundant constraints, we have an equivalent formulation of (P):

$$
\begin{array}{ll}
\max & \mathcal{F}(\bar{x}, \bar{x}, \bar{x}, \bar{x}) \\
\text{s.t.} & (a^i)^{\mathrm{T}} x = b_i,\ \left((a^i)^{\mathrm{T}} x\right)^2 = {b_i}^2,\ \left((a^i)^{\mathrm{T}} x\right)^4 = {b_i}^4,\ i = 1, \dots, m \\
& x^{\mathrm{T}} A^j x = c_j,\ \left(x^{\mathrm{T}} A^j x\right)^2 = {c_j}^2,\ j = 1, \dots, l \\
& \bar{x} = \begin{pmatrix} x \\ 1 \end{pmatrix} \in \mathbf{R}^{n+1}.
\end{array}
$$

The objective function of the above problem can be taken as a linear function of $\bar{x} \otimes \bar{x} \otimes \bar{x} \otimes \bar{x}$, and we introduce new variables of a super-symmetric fourth order tensor $\bar{\mathcal{X}} \in \mathbf{S}^{(n+1)^4}$. The notations x, X, and $\mathcal{X}$ extract part of the entries of $\bar{\mathcal{X}}$, which are defined as:

$$
\begin{array}{lll}
x \in \mathbf{R}^n & x_i = \bar{\mathcal{X}}_{i,n+1,n+1,n+1} & \forall\, 1 \le i \le n, \\
X \in \mathbf{S}^{n^2} & X_{i,j} = \bar{\mathcal{X}}_{i,j,n+1,n+1} & \forall\, 1 \le i, j \le n, \\
\mathcal{X} \in \mathbf{S}^{n^4} & \mathcal{X}_{i,j,k,\ell} = \bar{\mathcal{X}}_{i,j,k,\ell} & \forall\, 1 \le i, j, k, \ell \le n.
\end{array}
$$

Essentially they can be treated as linear constraints on $\bar{\mathcal{X}}$. Now by taking $\bar{\mathcal{X}} = \bar{x} \otimes \bar{x} \otimes \bar{x} \otimes \bar{x}$, $\mathcal{X} = x \otimes x \otimes x \otimes x$, and $X = x \otimes x$, we may equivalently represent the above problem as a quartic conic programming model with a rank-one constraint:

$$
\begin{array}{lll}
(Q) & \max & \mathcal{F} \bullet \bar{\mathcal{X}} \\
& \text{s.t.} & (a^i)^{\mathrm{T}} x = b_i,\ (a^i \otimes a^i) \bullet X = {b_i}^2,\ (a^i \otimes a^i \otimes a^i \otimes a^i) \bullet \mathcal{X} = {b_i}^4,\ i = 1, \dots, m \\
& & A^j \bullet X = c_j,\ (A^j \otimes A^j) \bullet \mathcal{X} = {c_j}^2,\ j = 1, \dots, l \\
& & \bar{\mathcal{X}}_{n+1,n+1,n+1,n+1} = 1,\ \bar{\mathcal{X}} \in \mathbf{\Sigma}^4_{n+1,4},\ \operatorname{rank}(\bar{\mathcal{X}}) = 1.
\end{array}
$$

Dropping the rank-one constraint, we obtain a relaxation problem, which is exactly in the form of quartic conic program (QCP):

$$
\begin{array}{lll}
(RQ) & \max & \mathcal{F} \bullet \bar{\mathcal{X}} \\
& \text{s.t.} & (a^i)^{\mathrm{T}} x = b_i,\ (a^i \otimes a^i) \bullet X = {b_i}^2,\ (a^i \otimes a^i \otimes a^i \otimes a^i) \bullet \mathcal{X} = {b_i}^4,\ i = 1, \dots, m \\
& & A^j \bullet X = c_j,\ (A^j \otimes A^j) \bullet \mathcal{X} = {c_j}^2,\ j = 1, \dots, l \\
& & \bar{\mathcal{X}}_{n+1,n+1,n+1,n+1} = 1,\ \bar{\mathcal{X}} \in \mathbf{\Sigma}^4_{n+1,4}.
\end{array}
$$

Interestingly, the relaxation from (Q) to (RQ) is not lossy; or, to put it differently, (RQ) is a tight relaxation of (Q), under some mild conditions.

Theorem 6.1 *If $A^j \in \mathbf{S}^{n^2}_+$ for all $1 \le j \le l$ in the model (P), then (RQ) is equivalent to (P) in the sense that: (i) they have the same optimal value; (ii) if $\bar{\mathcal{X}}$ is optimal to (RQ), then x is in the convex hull of the optimal solution of (P). Moreover, the minimization counterpart of (P) is also equivalent to the minimization counterpart of (RQ).*

This result shows that (P) is in fact a conic quartic program (QCP) when the matrices A^j's in (P) are positive semidefinite. Notice that the model (P) actually includes quadratic inequality constraints $x^{\mathrm{T}}A^jx \le c_j$ as its subclasses, for one can always add a slack variable $y_j \in \mathbf{R}$ with $x^{\mathrm{T}}A^jx + {y_j}^2 = c_j$, while reserving the new data matrix $\begin{bmatrix} A^j & 0 \\ 0 & 1 \end{bmatrix}$ in the quadratic term still being positive semidefinite. The proof of Theorem 6.1 is dedicated to Appendix A.

As mentioned before, Burer [8] established the equivalence between a large class of mixed-binary quadratic programs and co-positive programs. Theorem 6.1 may be regarded as a quartic extension of Burer's result. The virtue of this equivalence is to alleviate the highly non-convex objective and/or constraints of (QCP) and retain the problem in convex form, although the difficulty is all absorbed into the dealing of the quartic cone, which is nonetheless a convex one. Note that this is characteristically a property for polynomial of degree higher than 2: the SDP relaxation for similar quadratic models can never be tight.

6.2 Biquadratic Assignment Problems

The biquadratic assignment problem $(BQAP)$ is a generalization of the quadratic assignment problem (QAP), which is to minimize a quartic polynomial of an assignment matrix:

$$
\begin{array}{lll}
(BQAP) & \min & \sum_{1\le i,j,k,\ell,s,t,u,v\le n} \mathcal{A}_{ijk\ell}\mathcal{B}_{stuv}X_{is}X_{jt}X_{ku}X_{\ell v} \\
 & \text{s.t.} & \sum_{i=1}^n X_{ij} = 1,\ j = 1,\dots,n \\
 & & \sum_{j=1}^n X_{ij} = 1,\ i = 1,\dots,n \\
 & & X_{ij} \in \{0,1\},\ i,j = 1,\dots,n \\
 & & X \in \mathbf{R}^{n^2},
\end{array}
$$

where $\mathcal{A}, \mathcal{B} \in \mathbf{R}^{n^4}$. This problem was first considered by Burkard *et al.* [10] and was shown to have applications in the VLSI synthesis problem. After that, several heuristics for $(BQAP)$ were developed by Burkard and Cela [11], and Mavridou *et al.* [27].

In this subsection we shall show that $(BQAP)$ can be formulated as a quartic conic program (QCP). First notice that the objective function of $(BQAP)$ is a fourth order polynomial function with respect to the variables X_{ij}'s, where X is taken as an n^2-dimensional vector. The assignment constraints $\sum_{i=1}^n X_{ij} = 1$ and $\sum_{j=1}^n X_{ij} = 1$ are clearly linear equality constraints. Finally by imposing a new variable $x_0 \in \mathbf{R}$, and the binary constraints $X_{ij} \in \{0,1\}$ is equivalent to

$$
\begin{pmatrix} X_{ij} \\ x_0 \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} X_{ij} \\ x_0 \end{pmatrix} = \frac{1}{4} \quad \text{and} \quad x_0 = \frac{1}{2},
$$

where the coefficient matrix in the quadratic term is indeed positive semidefinite. Applying Theorem 6.1 we have the following result:

Corollary 6.2 *The biquadratic assignment problem* $(BQAP)$ *can be formulated as a quartic conic program* (QCP)*.*

6.3 Eigenvalues of Fourth Order Super-Symmetric Tensor

The notion of eigenvalue for a matrix has been extended to tensors; see e.g., [30, 31, 32]. Versatile extensions turned out to be possible, among which Qi [30] proposed the following notion of the *Z-eigenvalues*: Restricting to the space of fourth order super-symmetric tensors $\mathbf{S}^{n^4}$, $\lambda \in \mathbf{R}$ is called Z-eigenvalue of the super-symmetric tensor $\mathcal{F} \in \mathbf{S}^{n^4}$, if the following system holds

$$\begin{cases} \mathcal{F}(x,x,x,\cdot) = \lambda\, x, \\ x^{\mathrm{T}} x = 1, \end{cases}$$

where $x \in \mathbf{R}^n$ is the corresponding eigenvector with respect to λ. Notice that the Z-eigenvalues are the usual eigenvalues for a symmetric matrix, when restricting to the space of symmetric matrices $\mathbf{S}^{n^2}$. We refer interested readers to [31] for various properties of Z-eigenvalue and [32] for its applications in polynomial optimizations.

Observe that x is a Z-eigenvector of the fourth order tenor $\mathcal{F}$ if and only if x is a KKT point to following polynomial optimization problem:

$$\begin{array}{lll} (E) & \max & \mathcal{F}(x,x,x,x) \\ & \text{s.t.} & x^{\mathrm{T}} x = 1. \end{array}$$

Furthermore, x is the Z-eigenvector with respect to the largest (resp. smallest) Z-eigenvalue of $\mathcal{F}$ if and only if x is optimal to (E) (resp. the minimization counterpart of (E)). As the quadratic constraint $x^{\mathrm{T}}x = 1$ satisfies the condition in Theorem 6.1, we reach the following conclusion:

Corollary 6.3 *The problem of finding a Z-eigenvector with respect to the largest or smallest Z-eigenvalue of a fourth order super-symmetric tensor* $\mathcal{F}$ *can be formulated as a quartic conic program* (QCP)*.*

To conclude this section, as well as the whole paper, we remark here that quartic conic problems have many potential application, alongside their many intriguing theoretical properties. The hierarchical structure of the quartic cones that we proved in the previous sections paves a way for possible relaxation methods to be viable. For instance, according to the hierarchy relationship (12), by relaxing the cone $\mathbf{\Sigma}^4_{n,4}$ to an easy cone $\mathbf{S}_+^{n^2 \times n^2}$ lends a hand to solve the quartic conic problem approximately. The quality of such solution methods and possible enhancements remain our future research topic.

References

[1] A.A. Ahmadi, G. Blekherman, and P.A. Parrilo, *Every Convex Ternary Quartic is SOS-Convex*, Working Paper, Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, 2011. 18, 23

[2] A.A. Ahmadi, A. Olshevsky, P.A. Parrilo, and J.N. Tsitsiklis, *NP-hardness of Deciding Convexity of Quartic Polynomials and Related Problems*, Preprint, December 2010. http://arxiv.org/abs/1012.1908v1 1, 2, 3, 21

[3] A.A. Ahmadi and P.A. Parrilo, *A Convex Polynomial that Is Not SOS-Convex*, Mathematical Programming, Series A, to appear. 3, 7, 16, 17

[4] A.A. Ahmadi and P.A. Parrilo, *On the Equivalence of Algebraic Conditions for Convexity and Quasiconvexity of Polynomials*, Proceedings of the 49th IEEE Conference on Decision and Control, 2010. 7

[5] A. Barvinok, *A Course in Convexity*, Graduate Studies in Mathematics Volume 54, American Mathematical Society, 2002. 12

[6] G. Blekherman, *Convex Forms That Are Not Sums of Squares*, Preprint, October 2009. http://arxiv.org/abs/0910.0656 3, 18

[7] P. Biswas, T.-C. Liang, T.-C. Wang, and Y. Ye, *Semidefinite Programming Based Algorithms for Sensor Network Localization*, ACM Transactions on Sensor Networks, 2, 188–220, 2006. 2, 4

[8] S. Burer, *On the Copositive Representation of Binary and Continuous Nonconvex Quadratic Programs*, Mathematical Programming, Series A, 120, 479–495, 2009. 20, 26, 33

[9] S. Burer and H. Dong, *Representing Quadratically Constrained Quadratic Programs as Generalized Copositive Programs*, Technical Report. http://www.optimization-online.org/DB_FILE/2011/07/3095.pdf 20

[10] R.E. Burkard, E. Cela and B. Klinz, *On the biquadratic assignment problem*. In: Pardalos, P.M., Wolkowicz, H. (Eds.), Quadratic Assignment and Related Problems, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Vol. 16. AMS, Rhode Island, 117–146, 1994. 26

[11] R.E. Burkard and E. Cela, *Heuristis for Biqudratic Assignment Problems and Their Computational Comparison*, European Journal of Operational Research 83, 283–300, 1995. 26

[12] C. Chen and P.P. Vaidyanathan, *MIMO Radar Waveform Optimization With Prior Information of the Extended Target and Clutter*, IEEE Transactions on Signal Processing, 57, 3533–3544, 2009. 2

[13] G. Dahl, J.M. Leinaas, J. Myrheim, and E. Ovrum, *A Tensor Product Matrix Approximation Problem in Quantum Physics*, Linear Algebra and Applications, 420, 711–725, 2007. 2

[14] P.J.C. Dickinson and L. Gijben, *On the Computational Complexity of Membership Problems for the Completely Positive Cone and Its Dual*, Technical Report. http://www.optimization-online.org/DB_HTML/2011/05/3041.html 19

[15] S. He, Z. Li, and S. Zhang, *Approximation Algorithms for Homogeneous Polynomial Optimization with Quadratic Constraints*, Mathematical Programming, Series B, 125, 353–383, 2010. 2, 24

[16] S. He, Z. Li, and S. Zhang, *General Constrained Polynomial Optimization: An Approximation Approach*, Technical Report SEEM2009-06, Department of Systems Engineering & Engineering Management, The Chinese University of Hong Kong, 2009. 2, 24

[17] S. He, B. Jiang, Z. Li, and S. Zhang, *A Note on Hilbert's Identity*, Technical Report SEEM2011-01, Department of Systems Engineering & Engineering Management, The Chinese University of Hong Kong, 2011. 12

[18] J.W. Helton and J. Nie, *Semidefinite Representation of Convex Sets*, Mathematical Programming, Series A, 122, 21–64, 2010. 3, 7, 8

[19] D. Hilbert, *Über die Darstellung Definiter Formen als Summe von Formenquadraten*, Mathematische Annalen, 32, 342–350, 1888. 3, 23

[20] P.M. Kleniati, P. Parpas, and B. Rustem, *Partitioning Procedure for Polynomial Optimization: Application to Portfolio Decisions with Higher Order Moments*, COMISEF Working Papers Series, WPS-023, 2009. 2, 3

[21] J.B. Lasserre, *Global Optimization with Polynomials and the Problem of Moments*, SIAM Journal on Optimization, 11, 769–817, 2001. 2

[22] M. Laurent, *Sums of Squares, Moment Matrices and Optimization over Polynomials*, in: Putinar, M., Sullivant, S. (eds.) Emerging Applications of Algebraic Geometry, Series: The IMA Volumes in Mathematics and its Applications, vol. 149, Springer, Berlin, 2009. 2

[23] Z. Li, *Polynomial Optimization Problems—Approximation Algorithms and Applications*, Ph.D. Thesis, The Chinese Univesrity of Hong Kong, June 2011. 2, 24

[24] C. Ling, J. Nie, L. Qi, and Y. Ye, *Biquadratic Optimization Over Unit Spheres and Semidefinite Programming Relaxations*, SIAM Journal on Optimization, 20, 1286–1310, 2009. 2, 3, 21, 24

[25] Z.-Q. Luo and S. Zhang, *A Semidefinite Relaxation Scheme for Multivariate Quartic Polynomial Optimization With Quadratic Constraints*, SIAM Journal on Optimization, 20, 1716–1736, 2010. 2, 24

[26] Z.-Q. Luo, J.F. Sturm, and S. Zhang, *Multivariate Nonnegative Quadratic Mappings*, SIAM Journal on Optimization, 14, 1140–1162, 2004. 2, 23

[27] T. Mavridou, P.M. Pardalos, L.S. Pitsoulis and M.G.C. Resende, *A GRASP for the biquadratic assignment problem*, European Journal of Operational Research 105, 613–621, 1998. 26

[28] P.M. Pardalos and S.A. Vavasis, *Open Questions in Complexity Theory for Numerical Optimization*, Mathematical Programming, 57, 337–339, 1992. 2, 3

[29] P.A. Parrilo, *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*, PhD Dissertation, California Institute of Technology, 2000. 2

[30] L. Qi, *Extrema of a Real Polynomial*, Journal of Global Optimization, 30, 405–433, 2004. 27

[31] L. Qi, *Eigenvalues of a Real Supersymmetric Tensor*, Journal of Symbolic Computation, 40, 1302–1324, 2005. 27

[32] L. Qi, F. Wang and Y. Wang, *Z-eigenvalue Methods for a Global Polynomial Optimization Problem*, Mathematical Programming, Series A, 118, 301–316, 2009. 27

[33] B. Reznick, *Some Concrete Aspects of Hilbert's 17th Problem*, Real Algebraic Geometry and Ordered Structures, Contemporary Mathematics, 253, American Mathematical Society, 2000. 15

[34] B. Reznick, *Some of Even Powers of Real Linear Forms*, Memoirs of the American Mathematical Society, Volume 96, Number 463, March 1992. 3, 13

[35] A. M.-C. So, *Deterministic Approximation Algorithms for Sphere Constrained Homogeneous Polynomial Optimization Problems*, Mathematical Programming, Series B, to appear. 2, 24

[36] J.F. Sturm and S. Zhang, *On Cones of Nonnegative Quadratic Functions*, Mathematics of Operations Research, 28, 246–267, 2003. 2, 3, 9, 13, 18

A Proof of Theorem 6.1

Here we only prove the equivalence relation for the maximization problems, since the proof for their minimization counterparts is exactly the same. That is, we shall prove the equivalence between (Q) and (RQ).

To start with, let us first investigate the feasible regions of these two problems, to be denoted by $\text{feas}\,(Q)$ and $\text{feas}\,(RQ)$ respectively. The relationship between $\text{feas}\,(Q)$ and $\text{feas}\,(RQ)$ is revealed by the following lemma.

Lemma A.1 *It holds that* $\text{conv}\,(\text{feas}\,(Q)) \subseteq \text{feas}\,(RQ) = \text{conv}\,(\text{feas}\,(Q)) + \mathbf{P}$*, where*

$$\mathbf{P} := \text{cone}\left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \otimes \begin{pmatrix} x \\ 0 \end{pmatrix} \otimes \begin{pmatrix} x \\ 0 \end{pmatrix} \otimes \begin{pmatrix} x \\ 0 \end{pmatrix} \middle| \begin{array}{ll} (a^i)^{\mathrm{T}} x = 0 & \forall\, 1 \le i \le m, \\ x^{\mathrm{T}} A^j x = 0 & \forall\, 1 \le j \le l \end{array} \right\} \subset \mathbf{\Sigma}^4_{n+1,4}.$$

Proof. First, it is obvious that $\text{conv}\,(\text{feas}\,(Q)) \subseteq \text{feas}\,(RQ)$, since (RQ) is a relaxation of (Q) and $\text{feas}\,(RQ)$ is convex. Next we notice that the recession cone of $\text{feas}\,(RQ)$ is equal to

$$\left\{ \bar{\mathcal{X}} \in \mathbf{\Sigma}^4_{n+1,4} \middle| \begin{array}{ll} \mathcal{X}_{n+1,n+1,n+1,n+1} = 0, & \\ (a^i)^{\mathrm{T}} x = 0,\ (a^i \otimes a^i) \bullet X = 0,\ (a^i \otimes a^i \otimes a^i \otimes a^i) \bullet \mathcal{X} = 0 & \forall\, 1 \le i \le m, \\ A^j \bullet X = 0,\ (A^j \otimes A^j) \bullet \mathcal{X} = 0 & \forall\, 1 \le j \le l \end{array} \right\}.$$

Observing that $\bar{\mathcal{X}} \in \mathbf{\Sigma}^4_{n+1,4}$ and $\mathcal{X}_{n+1,n+1,n+1,n+1} = 0$, it is easy to see that $x = 0$ and $X = 0$. Thus the recession cone of $\text{feas}\,(RQ)$ is reduced to

$$\left\{ \bar{\mathcal{X}} \in \mathbf{\Sigma}^4_{n+1,4} \middle| \begin{array}{ll} \mathcal{X}_{n+1,n+1,n+1,n+1} = 0,\ x = 0,\ X = 0, & \\ (a^i \otimes a^i \otimes a^i \otimes a^i) \bullet \mathcal{X} = 0 & \forall\, 1 \le i \le m, \\ (A^j \otimes A^j) \bullet \mathcal{X} = 0 & \forall\, 1 \le j \le l \end{array} \right\} \supseteq \mathbf{P},$$

which proves $\text{feas}\,(RQ) \supseteq \text{conv}\,(\text{feas}\,(Q)) + \mathbf{P}$.

Finally, we shall show the inverse inclusion, i.e., $\text{feas}\,(RQ) \subseteq \text{conv}\,(\text{feas}\,(Q)) + \mathbf{P}$. Suppose $\bar{\mathcal{X}} \in \text{feas}\,(RQ)$, then it can be decomposed as

$$\bar{\mathcal{X}} = \sum_{k \in K} \begin{pmatrix} y^k \\ \alpha_k \end{pmatrix} \otimes \begin{pmatrix} y^k \\ \alpha_k \end{pmatrix} \otimes \begin{pmatrix} y^k \\ \alpha_k \end{pmatrix} \otimes \begin{pmatrix} y^k \\ \alpha_k \end{pmatrix}, \tag{15}$$

where $\alpha_k \in \mathbf{R}$, $y^k \in \mathbf{R}^n$ for all $k \in K$. Immediately we have

$$\sum_{k \in K} {\alpha_k}^4 = \mathcal{X}_{n+1,n+1,n+1,n+1} = 1. \tag{16}$$

Now divide the index set K into two parts, with $K_0 := \{k \in K \mid \alpha_k = 0\}$ and $K_1 := \{k \in K \mid \alpha_k \neq 0\}$, and let $z^k = y^k/\alpha_k$ for all $k \in K_1$. The decomposition (15) is then equivalent to

$$\bar{\mathcal{X}} = \sum_{k\in K_1} {\alpha_k}^4 \binom{z^k}{1} \otimes \binom{z^k}{1} \otimes \binom{z^k}{1} \otimes \binom{z^k}{1} + \sum_{k\in K_0} \binom{y^k}{0} \otimes \binom{y^k}{0} \otimes \binom{y^k}{0} \otimes \binom{y^k}{0}.$$

If we can prove that

$$\binom{z^k}{1} \otimes \binom{z^k}{1} \otimes \binom{z^k}{1} \otimes \binom{z^k}{1} \in \text{feas}\,(Q) \quad \forall\, k \in K_1 \tag{17}$$

$$\binom{y^k}{0} \otimes \binom{y^k}{0} \otimes \binom{y^k}{0} \otimes \binom{y^k}{0} \in \mathbf{P} \qquad \forall\, k \in K_0 \tag{18}$$

then by (16), we shall have $\bar{\mathcal{X}} \in \text{conv}\,(\text{feas}\,(Q)) + \mathbf{P}$, proving the inverse inclusion.

In the following we shall prove (17) and (18). Since $\bar{\mathcal{X}} \in \text{feas}\,(RQ)$, together with $x = \sum_{k\in K} {\alpha_k}^3 y^k$, $X = \sum_{k\in K} {\alpha_k}^2 y^k \otimes y^k$, and $\mathcal{X} = \sum_{k\in K} y^k \otimes y^k \otimes y^k \otimes y^k$, we obtain the following equalities:

$$\begin{aligned}
&\sum_{k\in K} {\alpha_k}^3 (a^i)^{\mathrm{T}} y^k = b_i,\ \sum_{k\in K} {\alpha_k}^2 \left((a^i)^{\mathrm{T}} y^k\right)^2 = {b_i}^2,\ \sum_{k\in K} \left((a^i)^{\mathrm{T}} y^k\right)^4 = {b_i}^4, \quad \forall\, 1 \le i \le m\\
&\sum_{k\in K} {\alpha_k}^2 (y^k)^{\mathrm{T}} A^j y^k = c_j,\ \sum_{k\in K} \left((y^k)^{\mathrm{T}} A^j y^k\right)^2 = {c_j}^2, \qquad \forall\, 1 \le j \le l.
\end{aligned}$$

As a direct consequence of the above equalities and (16), we have

$$\begin{aligned}
\left(\sum_{k\in K} {\alpha_k}^2 \cdot \alpha_k (a^i)^{\mathrm{T}} y^k\right)^2 &= {b_i}^2 = \left(\sum_{k\in K} {\alpha_k}^4\right)\left(\sum_{k\in K} {\alpha_k}^2 \left((a^i)^{\mathrm{T}} y^k\right)^2\right), \quad \forall\, 1 \le i \le m\\
\left(\sum_{k\in K} {\alpha_k}^2 \left((a^i)^{\mathrm{T}} y^k\right)^2\right)^2 &= {b_i}^4 = \left(\sum_{k\in K} {\alpha_k}^4\right)\left(\sum_{k\in K} \left((a^i)^{\mathrm{T}} y^k\right)^4\right), \quad \forall\, 1 \le i \le m\\
\left(\sum_{k\in K} {\alpha_k}^2 (y^k)^{\mathrm{T}} A^j y^k\right)^2 &= {c_j}^2 = \left(\sum_{k\in K} {\alpha_k}^4\right)\left(\sum_{k\in K} \left((y^k)^{\mathrm{T}} A^j y^k\right)^2\right), \quad \forall\, 1 \le j \le l.
\end{aligned}$$

Noticing that the equalities hold for the above Cauchy-Schwarz inequalities, it follows that for every $1 \le i \le m$ and every $1 \le j \le l$, there exist $\delta_i, \epsilon_i, \theta_j \in \mathbf{R}$, such that

$$\delta_i {\alpha_k}^2 = \alpha_k (a^i)^{\mathrm{T}} y^k,\ \epsilon_i {\alpha_k}^2 = \left((a^i)^{\mathrm{T}} y^k\right)^2 \text{ and } \theta_j {\alpha_k}^2 = (y^k)^{\mathrm{T}} A^j y^k \quad \forall\, k \in K. \tag{19}$$

If $\alpha_k = 0$, then $(a^i)^{\mathrm{T}} y^k = 0$ and $(y^k)^{\mathrm{T}} A^j y^k = 0$, which implies (18). Moreover, due to (19) and (16),

$$\delta_i = \delta_i \left(\sum_{k\in K} {\alpha_k}^4\right) = \sum_{k\in K} \delta_i {\alpha_k}^2 \cdot {\alpha_k}^2 = \sum_{k\in K} \alpha_k (a^i)^{\mathrm{T}} y^k \cdot {\alpha_k}^2 = b_i \quad \forall\, 1 \le i \le m.$$

Similarly, we have $\theta_j = c_j$ for all $1 \le j \le l$. If $\alpha_k \neq 0$, noticing $z^k = y^k/\alpha_k$, it follows from (19) that

$$\begin{aligned} (a^i)^{\mathrm{T}} z^k &= (a^i)^{\mathrm{T}} y^k/\alpha_k = \delta_i = b_i & \forall\, 1 \le i \le m \\ (z^k)^{\mathrm{T}} A^j z^k &= (y^k)^{\mathrm{T}} A^j y^k/{\alpha_k}^2 = \theta_j = c_j & \forall\, 1 \le j \le l, \end{aligned}$$

which implies (17). □

To prove the original Theorem 6.1, we notice that if A^j is positive semidefinite, then $x^{\mathrm{T}} A^j x = 0 \Longleftrightarrow A^j x = 0$. Therefore, $\begin{pmatrix} x \\ 0 \end{pmatrix} \otimes \begin{pmatrix} x \\ 0 \end{pmatrix} \otimes \begin{pmatrix} x \\ 0 \end{pmatrix} \otimes \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathbf{P}$ implies that x is a recession direction of the feasible region for (P). With this property and using a similar argument of Theorem 2.6 in [8], Theorem 6.1 follows immediately.