Approximation Methods for Complex Polynomial Optimization

Bo JIANG [*] Zhening LI [†] Shuzhong ZHANG [‡]

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Abstract

Complex polynomial optimization problems arise from real-life applications including radar code design, MIMO beamforming, and quantum mechanics. In this paper, we study complex polynomial optimization models whereby the objective function takes one of the following three forms: (1) multilinear; (2) homogeneous polynomial; (3) a conjugate symmetric form. On the constraint side, the decision variables belong to one of the following three sets: (1) the m-th roots of complex unity; (2) the complex unity; (3) the Euclidean sphere. We first discuss the multilinear objective function. Polynomial-time approximation algorithms are proposed for such problems with assured worst-case performance ratios, which depend only on the dimensions of the model. Then we introduce complex homogenous polynomial functions and establish key linkages between complex multilinear form and the complex polynomial functions. Approximation algorithms for the above-mentioned complex polynomial optimization models with worst-case performance ratios are presented.

Keywords: polynomial optimization, complex programming, complex tensor, approximation algorithm, tensor relaxation.

Mathematics Subject Classification: 90C59, 90C10, 15A69, 90C26.

This paper is dedicated to Masao Fukushima in celebration of his 65th birthday.

[*]Department of Industrial and Systems Engineering, University of Minnesota, Minneapolis, MN 55455, USA. Email: jiang373@umn.edu

[†]Department of Mathematics, Shanghai University, Shanghai 200444, China. Email: zheningli@gmail.com. The research of this author was supported in part by Natural Science Foundation of Shanghai, Grant 12ZR1410100.

[‡]Department of Industrial and Systems Engineering, University of Minnesota, Minneapolis, MN 55455, USA. Email: zhangs@umn.edu. The research of this author was supported in part by National Science Foundation, Grant CMMI-1161242.

1 Introduction

Polynomial optimization has received much attention in the recent years. The reason for this surge of interests is twofold. On the one hand, there is an emerging wide range of applications for polynomial optimization, for instance from biomedical engineering, control theory, graph theory, investment science, material science, quantum mechanics, signal processing, speech recognition; for specific references, see e.g., [17]. On the other hand, polynomial optimization has been found to be deeply rooted in a theoretical sense. Following the seminal work of Lasserre [16] and Parrilo [21], sum of squares (SOS) methods have become a cornerstone for general polynomial optimization problems. Recent developments can be found in the handbook by Anjos and Lasserre [3]. Since most of polynomial optimization problems are NP-hard, on the front of approximate solutions, various approximation algorithms have been proposed for solving certain types of high degree polynomial optimization models; we refer interested readers to the recent monograph of Li et al. [17].

Hitherto, polynomial optimization models under investigation are mostly in the domain of real numbers. Motivated by applications from signal processing, in this paper we set out to study several new classes of discrete and continuous polynomial optimization models in the complex domain. The detailed descriptions of these models can be found in Section 2. As a matter of fact, there are scattered results on complex polynomial optimization in the literature. When the objective function is quadratic, the MAX-3-CUT problem is a typical instance for the 3rd roots of unity constraint. Unity circle constrained complex optimization arises from the study of robust optimization as well as control theory [25, 5]. In particular, complex quadratic form optimization over unity constraints studied by Toker and Ozbay [25] are called complex programming. If the degree of complex polynomial is beyond quadratic, say quartic, several applications in signal processing can be found in the literature. Maricic et al. [20] proposed a quartic polynomial model for blind channel equalization in digital communication. Aittomäki and Koivunen [1] discussed the problem of beam-pattern synthesis in array signal processing problem and formulated it to be a complex quartic minimization problem. Chen and Vaidyanathan [7] studied MIMO radar waveform optimization with prior information of the extended target and clutter, by relaxing a quartic complex model. Most recently, Aubry et al. [4] managed to design a radar waveform sharing an ambiguity function behavior by resorting to a complex optimization problem. In quantum entanglement, Hilling and Sudbery [12] formulated a typical problem as a complex form optimization problem under spherical constraint, which is one of the three classes of models studied in this paper. Inspired by their work, Zhang and Qi [27] discussed the quantum eigenvalue problem, which arises from the geometric measure of entanglement of a multipartite symmetric pure state, in the complex tensor space. In fact, complex polynomial and complex tensor are interesting on their own. Eigenvalue and eigenvectors in the complex domain were already proposed and studied by Qi [22], whereas the name E-eigenvalue was coined. In a very recent working paper of Jiang et al. [15], necessary and sufficient conditions are discovered for general complex polynomial function to always take real values, based on which they extended the definitions of eigenvalues for conjugate type tensors.

Like its real-case counterpart, complex polynomial optimization is also NP-hard in general.

Therefore, approximation algorithms for complex models are on high demand. However, in the literature approximation algorithms are mostly considered for quadratic models only. Ben-Tal et al. [5] first studied complex quadratic optimization whose objective function is restricted nonnegative by using complex matrix cube theorem. Zhang and Huang [26], So et al. [24] considered complex quadratic form maximization under the m-th roots of unity constraints and unity constraints. Later, Huang and Zhang [14] also considered bilinear form complex optimization models under similar constraints. For real valued polynomial optimization problems, Luo and Zhang [19] first considered approximation algorithms for quartic optimization. At the same time, Ling et al. [18] considered a special quartic optimization model. Basically, the problem is to maximize a biquadratic form over two spherical constraints. Significant progresses have recently been made by He et al. [9, 10, 11], where the authors derived a series of approximation methods for optimization of any fixed degree polynomial function under various constrains. So [23] further considered spherically constrained homogeneous polynomial optimization and proposed a deterministic algorithm with an improved approximation ratio. For most recent development on approximation algorithms for homogeneous polynomial optimization, we refer the interested readers to [8, 13].

To the best of our knowledge, there is no result on approximation methods for general degree complex polynomial optimization as such, except for the practice of transforming a general high degree complex polynomial to the real case by doubling the problem dimension, and then resorting to the existing approximation algorithms for the real-valued polynomials [9, 10, 11, 23, 8, 13]. The latter approach, however, may lose the handle on the structure of the problem, hence misses nice properties of the complex polynomial functions. As a result, the computational costs may increase while the solution qualities may deteriorate. Exploiting the special structure of the complex model, it is often possible to get better approximation bounds, e.g, [26]. With this in mind, in this paper we shall study the complex polynomial optimization in its direct form. Let us start with some preparations next.

2 Models, notations, and organization

Throughout this paper, for any complex number $z = a + \boldsymbol{i}b \in \mathbb{C}$ with $a, b \in \mathbb{R}$, its real part is denoted by $\operatorname{Re} z = a$, and its modulus by $|z| = \sqrt{z^{\mathrm{H}} z} = \sqrt{a^2 + b^2}$. For $x \in \mathbb{C}^n$, its norm is denoted by $\|x\| := \left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}}$.

Given a d-th order complex tensor $\mathcal{F} = (\mathcal{F}_{i_1 i_2 \cdots i_d}) \in \mathbb{C}^{n_1 \times n_2 \times \cdots \times n_d}$, its associated multilinear form is defined as

$$L(x^1, x^2, \cdots, x^d) := \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_d=1}^{n_d} \mathcal{F}_{i_1 i_2 \cdots i_d}\, x^1_{i_1} x^2_{i_2} \cdots x^d_{i_d},$$

where the variables $x^k \in \mathbb{C}^{n_k}$ for $k = 1, 2, \ldots, d$, with 'L' standing for 'multilinearity'.

Closely related to multilinear form is homogeneous polynomial function, or, more explicitly

$$H(x) := \sum_{1 \le i_1 \le i_2 \le \cdots \le i_d \le n} a_{i_1 i_2 \cdots i_d} x_{i_1} x_{i_2} \cdots x_{i_d},$$

where the variable $x \in \mathbb{C}^n$, with 'H' standing for 'homogeneous polynomial'. Associated with any homogeneous polynomial is a super-symmetric complex tensor $\mathcal{F} \in \mathbb{C}^{n^d}$; i.e., its entries $\mathcal{F}_{i_1 i_2 \cdots i_d}$'s are invariant under permutations of its indices $\{i_1, i_2, \cdots, i_d\}$. In this sense,

$$\mathcal{F}_{i_1 i_2 \cdots i_d} = \frac{a_{i_1 i_2 \cdots i_d}}{|\Pi(i_1 i_2 \cdots i_d)|} \quad \forall\, 1 \le i_1 \le i_2 \le \cdots \le i_d \le n,$$

where $\Pi(i_1 i_2 \cdots i_d)$ is the set of all distinct permutations of the indices $\{i_1, i_2, \cdots, i_d\}$.

In light of multilinear form L associated with a super-symmetric tensor, homogeneous polynomial H is obtained by letting $x^1 = x^2 = \cdots = x^d$; i.e., $H(x) = L(\underbrace{x, x, \cdots, x}_{d})$. Furthermore, He et al. [9] established an essential linkage between multilinear forms and homogeneous polynomials in the real domain.

Lemma 2.1 *Suppose $x^1, x^2, \cdots, x^d \in \mathbb{R}^n$, and $\xi_1, \xi_2, \cdots, \xi_d$ are i.i.d. symmetric Bernoulli random variables (taking 1 and -1 with equal probability). For any super-symmetric tensor $\mathcal{F} \in \mathbb{R}^{n^d}$ with its associated multilinear form L and homogeneous polynomial H, it holds that*

$$\mathsf{E}\left[\prod_{i=1}^{d} \xi_i H\left(\sum_{k=1}^{d} \xi_k x^k\right)\right] = d!L(x^1, x^2, \cdots, x^d).$$

With Lemma 2.1 in place, tensor relaxation [9] is proposed to solve homogeneous polynomial optimization problems, by relaxing the objective function to a multilinear form.

In terms of the optimization, the real part of the above functions (multilinear form and homogeneous polynomial) is usually considered. Jiang et al. [15] introduced conjugate partial-symmetric complex tensors, which are extended from Hermitian matrices.

Definition 2.2 *An even order complex tensor $\mathcal{F} = (\mathcal{F}_{i_1 i_2 \cdots i_{2d}}) \in \mathbb{C}^{n^{2d}}$ is called conjugate partial-symmetric if*
(1) $\mathcal{F}_{i_1 \cdots i_d\, i_{d+1} \cdots i_{2d}} = \overline{\mathcal{F}_{i_{d+1} \cdots i_{2d}\, i_1 \cdots i_d}}$ *and*
(2) $\mathcal{F}_{i_1 \cdots i_d\, i_{d+1} \cdots i_{2d}} = \mathcal{F}_{j_1 \cdots j_d\, j_{d+1} \cdots j_{2d}} \quad \forall\, (j_1 \cdots j_d) \in \Pi(i_1 \cdots i_d),\ (j_{d+1} \cdots j_{2d}) \in \Pi(i_{d+1} \cdots i_{2d})$.

Associated with any conjugate partial-symmetric tensor, the following conjugate form

$$C(\overline{x}, x) := L(\underbrace{\overline{x}, \cdots, \overline{x}}_{d}, \underbrace{x, \cdots, x}_{d}) = \sum_{1 \le i_1, \cdots, i_d, j_1, \cdots, j_d \le n} \mathcal{F}_{i_1 \cdots i_d j_1 \cdots j_d}\, \overline{x_{i_1} \cdots x_{i_d}} x_{j_1} \cdots x_{j_d}$$

always takes real value for any $x \in \mathbb{C}^n$. Besides, any conjugate form C uniquely determines a conjugate partial-symmetric complex tensor. For details, one is refereed to [15]. In the above expression, 'C' signifies 'conjugate'.

The following commonly encountered constraint sets for complex polynomial optimization are considered in this paper:

- The m-th roots of unity constraint: $\mathbf{\Omega}_m = \{1, \omega_m, \cdots, \omega_m^{m-1}\}$, where $\omega_m = e^{\boldsymbol{i}\frac{2\pi}{m}} = \cos\frac{2\pi}{m} + \boldsymbol{i}\sin\frac{2\pi}{m}$. Denote $\mathbf{\Omega}_m^n = \{x \in \mathbb{C}^n \mid x_i \in \mathbf{\Omega}_m,\ i = 1, 2, \cdots, n\}$.

- The unity constraint: $\mathbf{\Omega}_\infty = \{z \in \mathbb{C} \,|\, |z| = 1\}$. Denote $\mathbf{\Omega}_\infty^n = \{x \in \mathbb{C}^n \,|\, x_i \in \mathbf{\Omega}_\infty,\, i = 1, 2, \cdots, n\}$.
- The complex spherical constraint: $\mathbf{S}^n = \{x \in \mathbb{C}^n \,|\, \|x\| = 1\}$.

Throughout this paper we assume $m \geq 3$, to ensure that the decision variables being considered are essentially complex.

In this paper, we shall discuss various complex polynomial optimization models. The objective function will be one of the three afore-mentioned complex polynomial functions (L, H, and C), or their real parts whenever is applicable; the constraint set is one of the three kinds as discussed above. The organization of the paper is as follows. Maximizing multilinear form over three types of constraint sets will be discussed in Section 3, i.e., models (L_m), (L_∞) and (L_S), with the subscription indicating the constraint for: the m-th roots of unity, the unity, and the complex sphere, respectively. Section 4 deals with maximization of homogeneous polynomial over three types of constraints, i.e., models $(H_m), (H_\infty)$ and (H_S). Finally, Section 5 discusses maximization of conjugate form over three types of constraints, i.e., models $(C_m), (C_\infty)$ and (C_S).

As a matter of notation, for any maximization problem $(P) : \max_{x \in X} p(x)$, we denote $v(P)$ to be the optimal value, and $\underline{v}(P)$ to be the optimal value of its minimization counterpart $(\min_{x \in X} p(x))$.

Definition 2.3 *(1) A maximization problem* $(P) : \max_{x \in X} p(x)$ *admits a polynomial-time approximation algorithm with approximation ratio* $\tau \in (0, 1]$*, if* $v(P) \geq 0$ *and a feasible solution* $\hat{x} \in X$ *can be found in polynomial-time, such that* $p(\hat{x}) \geq \tau v(P)$*.*
(2) A maximization problem $(P) : \max_{x \in X} p(x)$ *admits a polynomial-time approximation algorithm with* relative *approximation ratio* $\tau \in (0, 1]$*, if a feasible solution* $\hat{x} \in X$ *can be found in polynomial-time, such that* $p(\hat{x}) - \underline{v}(P) \geq \tau\,(v(P) - \underline{v}(P))$*.*

In this paper, we reserve τ to denote the approximation ratio. All the optimization models considered in this paper are NP-hard in general, even restricting the domain to be real. We shall propose polynomial-time approximation algorithms with worse-case performance ratios for the models concerned, when the degree of these polynomial functions, d or $2d$, is fixed. These approximation ratios are depended only on the dimensions of the problems, or data-independent. We shall start off by presenting Table 1 which summarizes the approximation results and the organization of the paper.

Table 1: Organization of the paper and the approximation results

Section	Model	Theorem	Approximation performance ratio
3.1	(L_m)	3.4	$\tau_m^{d-2}\,(2\tau_m-1)\left(\prod_{k=1}^{d-2} n_k\right)^{-\frac{1}{2}}$ where $\tau_m = \frac{m^2}{4\pi}\sin^2\frac{\pi}{m}$
3.2	(L_∞)	3.6	$0.7118\left(\frac{\pi}{4}\right)^{d-2}\left(\prod_{k=1}^{d-2} n_k\right)^{-\frac{1}{2}}$
3.3	(L_S)	3.7	$\left(\prod_{k=1}^{d-2} n_k\right)^{-\frac{1}{2}}$
4.1	(H_m)	4.3, 4.4	$\tau_m^{d-2}\,(2\tau_m-1)\,d!d^{-d}n^{-\frac{d-2}{2}}$
4.2	(H_∞)	4.5	$0.7118(\frac{\pi}{4})^{d-2}d!d^{-d}n^{-\frac{d-2}{2}}$
4.3	(H_S)	4.6	$d!d^{-d}n^{-\frac{d-2}{2}}$
5.1	(C_m)	5.3, 5.4	$\tau_m^{2d-2}(2\tau_m-1)(d!)^2(2d)^{-2d}n^{-(d-1)}$
5.2	(C_∞)	5.5	$0.7118\left(\frac{\pi}{4}\right)^{2d-2}(d!)^2(2d)^{-2d}n^{-(d-1)}$
5.2	(C_S)	5.6	$(d!)^2(2d)^{-2d}n^{-(d-1)}$

3 Complex multilinear form optimization

Let us consider optimization of complex multilinear forms, under three types of constraints described in Section 2. Specifically, the models under consideration are:

$$\begin{array}{lll}(L_m) & \max & \operatorname{Re} L(x^1, x^2, \cdots, x^d)\\ & \text{s.t.} & x^k \in \mathbf{\Omega}_m^{n_k},\ k = 1, 2, \ldots, d;\\ (L_\infty) & \max & \operatorname{Re} L(x^1, x^2, \cdots, x^d)\\ & \text{s.t.} & x^k \in \mathbf{\Omega}_\infty^{n_k},\ k = 1, 2, \ldots, d;\\ (L_S) & \max & \operatorname{Re} L(x^1, x^2, \cdots, x^d)\\ & \text{s.t.} & x^k \in \mathbf{S}^{n_k},\ k = 1, 2, \ldots, d.\end{array}$$

Associated with multilinear form objective is a d-th order complex tensor $\mathcal{F} \in \mathbb{C}^{n_1\times n_2\times\cdots\times n_d}$. Without loss of generality, we assume that $n_1 \le n_2 \le \cdots \le n_d$ and $\mathcal{F} \neq 0$. The multilinear form optimization models are interesting on their own. For example, typical optimization problem in quantum entanglement problem [12] is in the formulation of (L_S).

3.1 Multilinear form in the m-th roots of unity

When $d = 2$, (L_m) is already NP-hard, even for $m = 2$. In this case, (L_m) is to compute $\infty \mapsto 1$-norm of a matrix, and the best approximation bound is $\frac{2\ln(1+\sqrt{2})}{\pi} \approx 0.56$ due to Alon and Naor [2]. Huang and Zhang [14] studied general m when $d = 2$, and proposed polynomial-time randomized approximation algorithm with constant worst-case performance ratio. Specifically the ratio is $\frac{m^2}{4\pi}(1-\cos\frac{2\pi}{m})-1 = 2\tau_m - 1$ for $m \ge 3$, where $\tau_m := \frac{m^2}{8\pi}(1-\cos\frac{2\pi}{m}) = \frac{m^2}{4\pi}\sin^2\frac{\pi}{m}$ throughout this paper.

To proceed to the general degree d, let us start with the case $d = 3$.

$$
\begin{array}{lll}
(L_m^3) & \max & \operatorname{Re} L(x, y, z) \\
 & \text{s.t.} & x \in \mathbf{\Omega}_m^{n_1},\ y \in \mathbf{\Omega}_m^{n_2},\ z \in \mathbf{\Omega}_m^{n_3}.
\end{array}
$$

Denote $W = xy^{\mathrm{T}}$. It is easy to observe that $W_{ij} = x_i y_j \in \mathbf{\Omega}_m$ for all (i, j), implying $W \in \mathbf{\Omega}_m^{n_1 \times n_2}$. The above problem can be relaxed to

$$
\begin{array}{lll}
(L_m^2) & \max & \operatorname{Re} \hat{L}(W, z) := \operatorname{Re} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \sum_{k=1}^{n_3} \mathcal{F}_{ijk}\, W_{ij} z_k \\
 & \text{s.t.} & W \in \mathbf{\Omega}_m^{n_1 \times n_2},\ z \in \mathbf{\Omega}_m^{n_3}.
\end{array}
$$

This is exactly (L_m) with $d = 2$, which admits a polynomial-time approximation algorithm with approximation ratio $2\tau_m - 1$ in [14]. Denote the approximate solution of (L_m^2) to be $(\hat{W}, \hat{z})$, i.e.,

$$
\operatorname{Re} \hat{L}(\hat{W}, \hat{z}) \geq (2\tau_m - 1) v(L_m^2) \geq (2\tau_m - 1) v(L_m^3). \tag{1}
$$

The key step is to recover (x, y) from $\hat{W}$. For this purpose, we introduce the following decomposition routine (DR).

DR (Decomposition Routine) 3.1

- *Input:* $\hat{W} \in \mathbf{\Omega}_m^{n_1 \times n_2}$.
- *Construct*

$$
\tilde{W} = \begin{bmatrix} I & \hat{W}/\sqrt{n_1} \\ \hat{W}^{\mathrm{H}}/\sqrt{n_1} & \hat{W}^{\mathrm{H}}\hat{W}/n_1 \end{bmatrix} \succeq 0 \text{ (Hermitian positive semidefinite).}
$$

- *Randomly generate*

$$
\begin{pmatrix} \xi \\ \eta \end{pmatrix} \sim \mathcal{N}(0, \tilde{W}).
$$

- *For* $i = 1, 2, \ldots, n_1$, *let*

$$
\hat{x}_i := \omega_m^{\ell} \text{ if } \arg \xi_i \in \left[\frac{\ell}{m} 2\pi, \frac{\ell + 1}{m} 2\pi\right) \text{ for some } \ell \in \mathbb{Z};
$$

  and for $j = 1, 2, \ldots, n_2$, *let*

$$
\hat{y}_j := \omega_m^{-\ell} \text{ if } \arg \eta_j \in \left[\frac{\ell}{m} 2\pi, \frac{\ell + 1}{m} 2\pi\right) \text{ for some } \ell \in \mathbb{Z}.
$$

- *Output:* $(\hat{x}, \hat{y}) \in \mathbf{\Omega}_m^{n_1 + n_2}$.

It was shown in [26] that

$$\mathsf{E}[\hat{x}_i\hat{y}_j] = \frac{m(2-\omega_m-\omega_m^{-1})}{8\pi^2}\sum_{\ell=0}^{m-1}\omega_m^{\ell}\left(\arccos\left(-\mathrm{Re}\,\omega_m^{-\ell}\tilde{W}_{i,n_1+j}\right)\right)^2. \tag{2}$$

There are some useful properties regarding (2) as shown below; the proofs can be found in the appendix.

Lemma 3.2 *Define* $F_m:\mathbb{C}\mapsto\mathbb{C}$ *with* $F_m(x) := \frac{m(2-\omega_m-\omega_m^{-1})}{8\pi^2}\sum_{\ell=0}^{m-1}\omega_m^{\ell}\left(\arccos\left(-\mathrm{Re}\,\omega_m^{-\ell}x\right)\right)^2$.
(1) If $a\in\mathbb{C}$ *and* $b\in\mathbf{\Omega}_m$*, then* $F_m(ab)=bF_m(a)$.
(2) If $a\in\mathbb{R}$*, then* $F_m(a)\in\mathbb{R}$.

As $(\hat{W},\hat{z})$ is a feasible solution of (L_m^2), $\hat{W}_{ij}\in\mathbf{\Omega}_m$. By Lemma 3.2, we have for all (i,j)

$$\mathsf{E}[\hat{x}_i\hat{y}_j] = F_m(\tilde{W}_{i,n_1+j}) = F_m(\hat{W}_{ij}/\sqrt{n_1}) = \hat{W}_{ij}F_m(1/\sqrt{n_1}) \text{ and } F_m(1/\sqrt{n_1})\in\mathbb{R}. \tag{3}$$

We are now able to evaluate the objective value of $(\hat{x},\hat{y},\hat{z})$:

$$\begin{aligned}
\mathsf{E}\left[\mathrm{Re}\,L(\hat{x},\hat{y},\hat{z})\right] &= \mathsf{E}\left[\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{k=1}^{n_3}\mathrm{Re}\,\mathcal{F}_{ijk}\,\hat{x}_i\hat{y}_j\hat{z}_k\right]\\
&= \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{k=1}^{n_3}\mathrm{Re}\,\mathcal{F}_{ijk}\,\mathsf{E}\left[\hat{x}_i\hat{y}_j\right]\hat{z}_k\\
&= \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{k=1}^{n_3}\mathrm{Re}\,\mathcal{F}_{ijk}\,\hat{W}_{ij}F_m(1/\sqrt{n_1})\hat{z}_k\\
&= F_m\left(1/\sqrt{n_1}\right)\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{k=1}^{n_3}\mathrm{Re}\,\mathcal{F}_{ijk}\,\hat{W}_{ij}\hat{z}_k\\
&= F_m\left(1/\sqrt{n_1}\right)\mathrm{Re}\,\hat{L}(\hat{W},\hat{z}).
\end{aligned}$$

Furthermore, according to the appendix of [26], we have

$$F_m\left(1/\sqrt{n_1}\right) \geq \frac{m^2(1-\cos\frac{2\pi}{m})}{8\pi\sqrt{n_1}} = \frac{\tau_m}{\sqrt{n_1}}. \tag{4}$$

Combined with (1), we finally get

$$\mathsf{E}\left[\mathrm{Re}\,L(\hat{x},\hat{y},\hat{z})\right] = F_m\left(1/\sqrt{n_1}\right)\mathrm{Re}\,\hat{L}(\hat{W},\hat{z}) \geq \frac{\tau_m}{\sqrt{n_1}}(2\tau_m-1)v(L_m^3).$$

Theorem 3.3 *When* $d=3$*,* (L_m) *admits a polynomial-time randomized approximation algorithm with approximation ratio* $\frac{\tau_m(2\tau_m-1)}{\sqrt{n_1}}$.

By a similar method and using induction, the above discussion is readily extended to any fixed degree d.

Theorem 3.4 *(L_m) admits a polynomial-time randomized approximation algorithm with approximation ratio $\tau(L_m) := \tau_m^{d-2}\,(2\tau_m - 1)\left(\prod_{k=1}^{d-2} n_k\right)^{-\frac{1}{2}}$, i.e., a feasible solution $(\hat{x}^1, \hat{x}^2, \cdots, \hat{x}^d)$ can be found in polynomial-time, such that*

$$\mathsf{E}\left[\operatorname{Re} L(\hat{x}^1, \hat{x}^2, \cdots, \hat{x}^{d-1})\right] \geq \tau(L_m) v(L_m).$$

Proof. The proof is based on induction on the degree d. The case for $d = 2$ or $d = 3$ is known to be true. The inductive step can be similarly derived from Theorem 3.3.

For general d, denote $W = x^1 (x^d)^{\mathrm{T}}$ and (L_m) is then relaxed to

$$\begin{array}{lll} (L_m^{d-1}) & \max & \operatorname{Re} \hat{L}(W, x^2, \cdots, x^{d-1}) := \operatorname{Re} \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_d=1}^{n_d} \mathcal{F}_{i_1 i_2 \cdots i_d} W_{i_1 i_d} x_{i_2}^2 \cdots x_{i_{d-1}}^{d-1} \\ & \text{s.t.} & W \in \mathbf{\Omega}_m^{n_1 \times n_d},\ x^k \in \mathbf{\Omega}_m^{n_k},\ k = 2, 3, \cdots, d-1. \end{array}$$

By induction we are able to find $(\hat{W}, \hat{x}^2, \cdots, \hat{x}^{d-1})$, such that

$$\begin{aligned} \mathsf{E}\left[\operatorname{Re} \hat{L}(\hat{W}, \hat{x}^2, \cdots, \hat{x}^{d-1})\right] &\geq \tau_m^{d-3}\,(2\tau_m - 1)\left(\prod_{k=2}^{d-2} n_k\right)^{-\frac{1}{2}} v(L_m^{d-1}) \\ &\geq \tau_m^{d-3}\,(2\tau_m - 1)\left(\prod_{k=2}^{d-2} n_k\right)^{-\frac{1}{2}} v(L_m). \end{aligned}$$

Applying DR 3.1 with input $\hat{W}$ and output $(\hat{x}^1, \hat{x}^d)$, and using (3) and (4), we conclude that

$$\begin{aligned} \mathsf{E}\left[\operatorname{Re} L(\hat{x}^1, \hat{x}^2, \cdots, \hat{x}^d)\right] &= \mathsf{E}\left[\operatorname{Re} \hat{L}\left(\hat{x}^1 (\hat{x}^d)^{\mathrm{T}}, \hat{x}^2, \cdots, \hat{x}^{d-1}\right)\right] \\ &= \mathsf{E}\left[\operatorname{Re} \hat{L}\left(\mathsf{E}\left[\hat{x}^1 (\hat{x}^d)^{\mathrm{T}} \middle| \hat{W}\right], \hat{x}^2, \cdots, \hat{x}^{d-1}\right)\right] \\ &= \mathsf{E}\left[\operatorname{Re} \hat{L}\left(\hat{W} F_m\left(1/\sqrt{n_1}\right), \hat{x}^2, \cdots, \hat{x}^{d-1}\right)\right] \\ &= F_m\left(1/\sqrt{n_1}\right) \mathsf{E}\left[\operatorname{Re} \hat{L}(\hat{W}, \hat{x}^2, \cdots, \hat{x}^{d-1})\right] \\ &\geq \frac{\tau_m}{\sqrt{n_1}} \cdot \tau_m^{d-3}\,(2\tau_m - 1)\left(\prod_{k=2}^{d-2} n_k\right)^{-\frac{1}{2}} v(L_m) \\ &= \tau(L_m) v(L_m). \end{aligned}$$

□

3.2 Multilinear form with unity constraints

Let us now turn to the optimization model with unity constraint (L_∞), which can be taken as the model (L_m) when $m \to \infty$:

$$\begin{array}{lll} (L_\infty) & \max & \operatorname{Re} L(x^1, x^2, \cdots, x^d) \\ & \text{s.t.} & x^k \in \mathbf{\Omega}_\infty^{n_k},\ k = 1, 2, \ldots, d. \end{array}$$

When $d = 2$, (L_∞) was studied in [14] and a polynomial-time approximation algorithm with approximation ratio 0.7118 was presented. To treat the high degree case, one may again apply induction in the proof of Theorem 3.4. However, DR 3.1 should be slightly modified in order to apply the decomposition procedure for $\mathbf{\Omega}_\infty$.

DR (Decomposition Routine) 3.5

- *Input:* $\hat{W} \in \mathbf{\Omega}_\infty^{n_1 \times n_2}$.
- *Construct* $\tilde{W} = \begin{bmatrix} I & \hat{W}/\sqrt{n_1} \\ \hat{W}^{\mathrm{H}}/\sqrt{n_1} & \hat{W}^{\mathrm{H}}\hat{W}/n_1 \end{bmatrix} \succeq 0$.
- *Randomly generate* $\binom{\xi}{\eta} \sim \mathcal{N}(0, \tilde{W})$.
- *Let* $\hat{x}_i = e^{\boldsymbol{i} \arg \xi_i}$ *for* $i = 1, 2, \ldots, n_1$, *and let* $\hat{y}_j = e^{-\boldsymbol{i} \arg \eta_j}$ *for* $j = 1, 2, \ldots, n_2$.
- *Output:* $(\hat{x}, \hat{y}) \in \mathbf{\Omega}_\infty^{n_1+n_2}$.

The estimation of $(\hat{x}, \hat{y})$ is then

$$\mathsf{E}[\hat{x}_i \hat{y}_j] = F_\infty(\tilde{W}_{i,n_1+j}) = F_\infty(\hat{W}_{ij}/\sqrt{n_1}) \quad \forall\,(i,j).$$

It was calculated in [26] that

$$F_\infty(a) := \lim_{m \to \infty} F_m(a) = \frac{\pi}{4} a + \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{((2k)!)^2}{2^{4k+1}(k!)^4(k+1)} |a|^{2k} a.$$

Similar as in Lemma 3.2:

$$\begin{aligned} & F_\infty(ab) = bF_\infty(a) \quad \forall\, a \in \mathbb{C},\, b \in \mathbf{\Omega}_\infty, \\ & F_\infty(a) \in \mathbb{R} \quad \forall\, a \in \mathbb{R}, \\ & F_\infty(a) \geq \frac{\pi}{4} a \quad \forall\, a > 0. \end{aligned}$$

By applying the result in [14] for case $d = 2$ and using a similar argument as Theorem 3.4, we have the following main result of this subsection.

Theorem 3.6 (L_∞) *admits a polynomial-time randomized approximation algorithm with approximation ratio* $\tau(L_\infty) := 0.7118 \left(\frac{\pi}{4}\right)^{d-2} \left(\prod_{k=1}^{d-2} n_k\right)^{-\frac{1}{2}}$.

3.3 Multilinear form with spherical constraints

Let us turn to our last model for multilinear form optimization:

$$\begin{array}{lll} (L_S) & \max & \operatorname{Re} L(x^1, x^2, \cdots, x^d) \\ & \text{s.t.} & x^k \in \mathbf{S}^{n_k},\ k = 1, 2, \ldots, d. \end{array}$$

Model (L_S) is also known as computing the largest singular value (the real part) of a d-th order complex tensor $\mathcal{F}$. The case when $\mathcal{F}$ is real was widely studied [9, 23, 6, 17]. In particular, He et al. [9] introduced the recursive procedure and eigen-decomposition based approximation algorithm with approximation ratio $\left(\prod_{k=1}^{d-2} n_k\right)^{-\frac{1}{2}}$. Using a similar argument, we have the following result.

Theorem 3.7 (L_S) *admits a deterministic polynomial-time approximation algorithm with approximation ratio* $\tau(L_S) := \left(\prod_{k=1}^{d-2} n_k\right)^{-\frac{1}{2}}$.

When $d = 2$, (L_S) is to compute the largest singular value of a complex matrix, and is therefore solvable in polynomial-time, which also follows as a consequence of Theorem 3.7. The proof of Theorem 3.7 is similar to that of [9] for the real case. The main ingredients include establishing the initial step for the case $d = 2$, and then establishing a decomposition routine, which is shown as follows, to enable the induction.

DR (Decomposition Routine) 3.8

- *Input:* $\hat{W} \in \mathbb{C}^{n_1 \times n_2}$.
- *Find the left singular vector* $\hat{x} \in \mathbf{S}^{n_1}$ *and the right singular vector* $\hat{y} \in \mathbf{S}^{n_2}$ *corresponding to the largest singular value of* $\hat{W}$.
- *Output:* $\hat{x} \in \mathbf{S}^{n_1}, \hat{y} \in \mathbf{S}^{n_2}$.

Remark that if we directly apply the result for the real case in [9] by treating tensor $\mathcal{F} \in \mathbb{R}^{2n_1 \times 2n_2 \times \cdots \times 2n_d}$, then the approximation ratio will be $\left(\prod_{k=1}^{d-2} 2n_k\right)^{-\frac{1}{2}}$, which is certainly worse than $\tau(L_S)$.

4 Complex homogeneous polynomial optimization

This section is concerned with the optimization of complex homogeneous polynomial $H(x)$, associated with super-symmetric complex tensor $\mathcal{F} \in \mathbb{C}^{n^d}$. Specifically, the models under considerations are:

$$\begin{array}{lll} (H_m) & \max & \operatorname{Re} H(x) \\ & \text{s.t.} & x \in \mathbf{\Omega}_m^n; \\ (H_\infty) & \max & \operatorname{Re} H(x) \\ & \text{s.t.} & x \in \mathbf{\Omega}_\infty^n; \\ (H_S) & \max & \operatorname{Re} H(x) \\ & \text{s.t.} & x \in \mathbf{S}^n. \end{array}$$

Denote L to be multilinear form associated with $\mathcal{F}$, and then $H(x) = L(\underbrace{x, x, \cdots, x}_{d})$. By applying the tensor relaxation method established in [9], the above models are then relaxed to the following multilinear form optimization models discussed in Section 3:

$$
\begin{array}{lll}
(LH_m) & \max & \operatorname{Re} L(x^1, x^2, \cdots, x^d) \\
 & \text{s.t.} & x^k \in \mathbf{\Omega}_m^n,\ k = 1, 2, \ldots, d; \\
(LH_\infty) & \max & \operatorname{Re} L(x^1, x^2, \cdots, x^d) \\
 & \text{s.t.} & x^k \in \mathbf{\Omega}_\infty^n,\ k = 1, 2, \ldots, d; \\
(LH_S) & \max & \operatorname{Re} L(x^1, x^2, \cdots, x^d) \\
 & \text{s.t.} & x^k \in \mathbf{S}^n,\ k = 1, 2, \ldots, d.
\end{array}
$$

The approximation results in Section 3 can return good approximation solutions for these relaxed models. The key next step is to obtain good solutions for the original homogeneous polynomial optimizations. Similar to Lemma 2.1, we establish a linkage between functions L and H in the complex domain. The proof of Lemma 4.1 can be found in the appendix.

Lemma 4.1 *Let $m \in \{3, 4, \cdots, \infty\}$. Suppose $x^1, x^2, \cdots, x^d \in \mathbb{C}^n$, and $\mathcal{F} \in \mathbb{C}^{n^d}$ is a super-symmetric complex tensor with its associated multilinear form L and homogeneous polynomial H. If $\xi_1, \xi_2, \cdots, \xi_d$ are i.i.d. uniform distribution on $\mathbf{\Omega}_m$, then*

$$
\mathsf{E}\left[\prod_{i=1}^d \overline{\xi_i} H\left(\sum_{k=1}^d \xi_k x^k\right)\right] = d! L(x^1, x^2 \cdots, x^d) \text{ and } \mathsf{E}\left[\prod_{i=1}^d \xi_i H\left(\sum_{k=1}^d \xi_k x^k\right)\right] = 0.
$$

4.1 Homogeneous polynomial in the m-th roots of unity

Let us now focus on the model $(H_m) : \max_{x \in \mathbf{\Omega}_m^n} \operatorname{Re} H(x)$. By Lemma 4.1, for any fixed $\hat{x}^1, \hat{x}^2, \cdots, \hat{x}^d \in \mathbb{C}^n$, we can find $\beta_1, \beta_2, \cdots, \beta_d \in \mathbf{\Omega}_m$ in polynomial-time, such that

$$
\operatorname{Re} \prod_{i=1}^d \overline{\beta_i} H\left(\frac{1}{d} \sum_{k=1}^d \beta_k \hat{x}^k\right) \geq \operatorname{Re} d^{-d} d! L(\hat{x}^1, \hat{x}^2, \cdots, \hat{x}^d). \tag{5}
$$

For any $1 \leq i \leq n$, if $\hat{x}_i^k \in \mathbf{\Omega}_m$ for all $1 \leq k \leq d$, then $\frac{1}{d} \sum_{k=1}^d \beta_k \hat{x}_i^k \in \operatorname{conv}(\mathbf{\Omega}_m)$. As shown below, we are able to get a solution from $\operatorname{conv}(\mathbf{\Omega}_m)$ to one of its vertices $(\mathbf{\Omega}_m)$.

Lemma 4.2 *Suppose $m \in \{3, 4, \cdots, \infty\}$, and $x \in \mathbb{C}^n$ with $x_i \in \operatorname{conv}(\mathbf{\Omega}_m)$ for all $1 \leq i \leq n$.*
(1) If $H(x)$ is a complex homogeneous polynomial associated with square-free (meaning that its entry is zero whenever two of its indices are identical) super-symmetric tensor $\mathcal{F} \in \mathbb{C}^{n^d}$, then $y, z \in \mathbf{\Omega}_m^n$ can be found in polynomial-time, such that $\operatorname{Re} H(y) \leq \operatorname{Re} H(x) \leq \operatorname{Re} H(z)$.
(2) If $\operatorname{Re} H(x)$ is convex, then $z \in \mathbf{\Omega}_m^n$ can be found in polynomial-time, such that $\operatorname{Re} H(x) \leq \operatorname{Re} H(z)$.

Proof. If $H(x)$ is square-free, by fixing $x_2, x_3, \cdots, x_n$ as constants and taking x_1 as the only decision variable, we may write

$$
\operatorname{Re} H(x) = \operatorname{Re} h_1(x_2, x_3, \cdots, x_n) + \operatorname{Re} x_1 h_2(x_2, x_3, \cdots, x_n) =: \operatorname{Re} h(x_1).
$$

Since $\mathrm{Re}\, h(x_1)$ is a linear function of x_1, its optimal value over $\mathrm{conv}\,(\mathbf{\Omega}_m)$ is attained at one of its vertices. For instance, $z_1 \in \mathbf{\Omega}_m$ can be found easily such that $\mathrm{Re}\, h(z_1) \geq \mathrm{Re}\, h(x_1)$. Now, repeat the same procedures for $x_2, x_3, \cdots, x_n$, and let them be replaced by $z_2, z_3, \cdots, z_n$ respectively. Then $z \in \mathbf{\Omega}_m^n$ satisfies $\mathrm{Re}\, H(z) \geq \mathrm{Re}\, H(x)$. Using the same argument, we may find $y \in \mathbf{\Omega}_m^n$, such that $\mathrm{Re}\, H(y) \leq \mathrm{Re}\, H(x)$. The case that $\mathrm{Re}\, H(x)$ is convex can be proven similarly. □

Now we are ready to prove the main results in this subsection.

Theorem 4.3 *Suppose $H(x)$ is square-free or $\mathrm{Re}\, H(x)$ is convex.*
(1) If $m \mid (d-1)$, then (H_m) admits a polynomial-time randomized approximation algorithm with approximation ratio $\tau(H_m) := \tau_m^{d-2}(2\tau_m - 1)\, d! d^{-d} n^{-\frac{d-2}{2}}$.
(2) If $m \nmid 2d$, then (H_m) admits a polynomial-time randomized approximation algorithm with approximation ratio $\frac{1}{2}\tau(H_m)$.

Proof. Relaxing (H_m) to (LH_m), we find a feasible solution $(\hat{x}^1, \hat{x}^2, \cdots, \hat{x}^d)$ of (LH_m) in polynomial-time with approximation ratio $\tau_m^{d-2}(2\tau_m - 1)\, n^{-\frac{d-2}{2}}$ by Theorem 3.4. Then by (5), we further find $\beta \in \mathbf{\Omega}_m^d$, such that

$$\mathrm{Re} \prod_{i=1}^d \overline{\beta_i} H\left(\frac{1}{d}\sum_{k=1}^d \beta_k \hat{x}^k\right) \geq \mathrm{Re}\, d! d^{-d} L(\hat{x}^1, \hat{x}^2, \cdots, \hat{x}^d) \geq \tau(H_m) v(LH_m) \geq \tau(H_m) v(H_m).$$

Let us denote $\hat{x} := \frac{1}{d}\sum_{k=1}^d \beta_k \hat{x}^k$. Clearly we have $\hat{x}_i \in \mathrm{conv}\,(\mathbf{\Omega}_m)$ for $i = 1, 2, \ldots, n$.

(1) If $m \mid (d-1)$, then $d = 1 + mp$ for some $p \in \mathbb{Z}$. As $\beta_i \in \mathbf{\Omega}_m$, we have

$$H\left(\hat{x} \prod_{i=1}^d \overline{\beta_i}\right) = \left(\prod_{i=1}^d \overline{\beta_i}\right)^d H(\hat{x}) = \prod_{i=1}^d \overline{\beta_i}^{1+mp} H(\hat{x}) = \prod_{i=1}^d \overline{\beta_i} H(\hat{x}).$$

Since $\hat{x}_j \prod_{i=1}^d \overline{\beta_i} \in \mathrm{conv}\,(\mathbf{\Omega}_m)$ for $j = 1, 2, \ldots, n$, noticing $H(x)$ is square-free or $\mathrm{Re}\, H(x)$ is convex, and applying Lemma 4.2, we are able to find $y \in \mathbf{\Omega}_m^n$ in polynomial-time, such that

$$\mathrm{Re}\, H(y) \geq \mathrm{Re}\, H\left(\hat{x} \prod_{i=1}^d \overline{\beta_i}\right) = \mathrm{Re} \prod_{i=1}^d \overline{\beta_i} H(\hat{x}) \geq \tau(H_m) v(H_m).$$

(2) Let $\Phi = \left\{H(\omega_m^\ell \hat{x}) \,|\, \ell = 0, 1, \ldots, m-1\right\}$. As $H(\omega_m^\ell \hat{x}) = \omega_m^{d\ell} H(\hat{x})$ for $\ell = 0, 1, \ldots, m-1$, the elements of Φ is evenly distributed on the unity circle with radius $|H(\hat{x})|$ in the complex plane. Since $\omega_m^{d\ell} = e^{\boldsymbol{i}\frac{2d\ell\pi}{m}}$ and $m \nmid 2d$, it is easy to verify that $|\Phi| \geq 3$. Let ϕ be the minimum angle between Φ and the real axis, or equivalently $|H(\hat{x})| \cos\phi = \max_{x \in \Phi} \mathrm{Re}\, x$. Clearly $0 \leq \phi \leq \frac{\pi}{3}$ by $|\Phi| \geq 3$. Let $H(\omega_m^t \hat{x}) = \arg\max_{x \in \Phi} \mathrm{Re}\, x$. As $\omega_m^t \hat{x}_j \in \mathrm{conv}\,(\mathbf{\Omega}_m)$ for $j = 1, 2, \ldots, n$, again by Lemma 4.2, we are able to find $y \in \mathbf{\Omega}_m^n$ in polynomial-time, such that

$$\mathrm{Re}\, H(y) \geq \mathrm{Re}\, H(\omega_m^t \hat{x}) = |H(\hat{x})| \cos\phi \geq \frac{1}{2}|H(\hat{x})| \geq \frac{1}{2}\mathrm{Re} \prod_{i=1}^d \overline{\beta_i} H(\hat{x}) \geq \frac{1}{2}\tau(H_m) v(H_m).$$

□

Remark that condition (1) in Theorem 4.3 is a special case of (2); however in that special case a better approximation ratio than (2) is obtained. When $d \geq 4$ is even, almost all of the optimization models of homogeneous polynomials in the real domain (e.g., [9, 11, 23, 17]) only admit *relative* approximation ratios. Interestingly, in the complex domain, as Theorem 4.3 suggests, absolute approximation ratios are possible for some m when d is even.

When $m \mid 2d$, the approach in (2) of Theorem 4.3 may not work, since $|\Phi| \leq 2$. The worst case performance of the approximate solution cannot be guaranteed any more. However a relative approximation bound is possible for any m, as long as $H(x)$ is square-free.

Theorem 4.4 *If $H(x)$ is square-free, then (H_m) admits a polynomial-time randomized approximation algorithm with relative approximation ratio $\frac{1}{4}\tau(H_m)$.*

Proof. Relaxing (H_m) to (LH_m), we may find a feasible solution $(\hat{x}^1, \hat{x}^2, \cdots, \hat{x}^d)$ of (LH_m) in polynomial-time with approximation ratio $\tau_m^{d-2}(2\tau_m - 1)\, n^{-\frac{d-2}{2}}$ by Theorem 3.4, such that

$$d!d^{-d}\mathrm{Re}\, L(\hat{x}^1, \hat{x}^2, \cdots, \hat{x}^d) \geq d!d^{-d}\tau_m^{d-2}(2\tau_m - 1)\, n^{-\frac{d-2}{2}} v(LH_m) = \tau(H_m)v(LH_m) \geq \tau(H_m)v(H_m).$$

Let $\xi_1, \xi_2, \cdots, \xi_d$ be i.i.d. uniform distribution on $\mathbf{\Omega}_m$, and we have $\frac{1}{d}\sum_{k=1}^d \xi_k \hat{x}_i^k \in \mathrm{conv}(\mathbf{\Omega}_m)$ for $i = 1, 2, \ldots, n$. As $H(x)$ is square-free, by Lemma 4.2, there exists $y \in \mathbf{\Omega}_m^n$, such that

$$\mathrm{Re}\, H\left(\frac{1}{d}\sum_{k=1}^d \xi_k \hat{x}^k\right) \geq \mathrm{Re}\, H(y) \geq \underline{v}(H_m). \tag{6}$$

According to Lemma 4.1, it follows that

$$\mathsf{E}\left[\mathrm{Re}\prod_{i=1}^d \overline{\xi_i} H\left(\sum_{k=1}^d \xi_k \hat{x}^k\right)\right] = \mathrm{Re}\, d!L(\hat{x}^1, \hat{x}^2, \cdots, \hat{x}^d) \text{ and } \mathsf{E}\left[\mathrm{Re}\prod_{i=1}^d \xi_i H\left(\sum_{k=1}^d \xi_k \hat{x}^k\right)\right] = 0.$$

Combining the above two identities leads to

$$\begin{aligned}
\mathrm{Re}\, d!d^{-d}L(\hat{x}^1, \hat{x}^2, \cdots, \hat{x}^d) &= \mathsf{E}\left[\mathrm{Re}\prod_{i=1}^d \overline{\xi_i} H\left(\frac{1}{d}\sum_{k=1}^m \xi_k \hat{x}^k\right)\right] + \mathsf{E}\left[\mathrm{Re}\prod_{i=1}^d \xi_i H\left(\frac{1}{d}\sum_{k=1}^m \xi_k \hat{x}^k\right)\right] \\
&= \mathsf{E}\left[\mathrm{Re}\left(\prod_{i=1}^d \overline{\xi_i} + \prod_{i=1}^d \xi_i\right) H\left(\frac{1}{d}\sum_{k=1}^d \xi_k \hat{x}^k\right)\right] \\
&= \mathsf{E}\left[\left(\prod_{i=1}^d \overline{\xi_i} + \prod_{i=1}^d \xi_i\right) \mathrm{Re}\, H\left(\frac{1}{d}\sum_{k=1}^d \xi_k \hat{x}^k\right)\right] \\
&= \mathsf{E}\left[\left(\prod_{i=1}^d \overline{\xi_i} + \prod_{i=1}^d \xi_i\right) \left(\mathrm{Re}\, H\left(\frac{1}{d}\sum_{k=1}^d \xi_k \hat{x}^k\right) - \underline{v}(H_m)\right)\right] \\
&\leq \mathsf{E}\left[\left|\prod_{i=1}^d \overline{\xi_i} + \prod_{i=1}^d \xi_i\right| \cdot \left|\mathrm{Re}\, H\left(\frac{1}{d}\sum_{k=1}^d \xi_k \hat{x}^k\right) - \underline{v}(H_m)\right|\right] \\
&\leq 2\,\mathsf{E}\left[\mathrm{Re}\, H\left(\frac{1}{d}\sum_{k=1}^d \xi_k \hat{x}^k\right) - \underline{v}(H_m)\right],
\end{aligned}$$

where the last step is due to (6). By randomizing, we are able to find $\beta \in \mathbf{\Omega}_m^d$, such that

$$\operatorname{Re} H\left(\frac{1}{d}\sum_{k=1}^{d}\beta_k \hat{x}^k\right) - \underline{v}(H_m) \geq \frac{1}{2}\operatorname{Re} d! d^{-d} L(\hat{x}^1, \hat{x}^2, \cdots, \hat{x}^d) \geq \frac{1}{2}\tau(H_m)v(H_m).$$

Let us now separately discuss two cases. In the first case, if $v(H_m) \geq \frac{1}{2}\left(v(H_m) - \underline{v}(H_m)\right)$, then the above further leads to

$$\operatorname{Re} H\left(\frac{1}{d}\sum_{k=1}^{d}\beta_k \hat{x}^k\right) - \underline{v}(H_m) \geq \frac{1}{2}\tau(H_m)v(H_m) \geq \frac{1}{4}\tau(H_m)\left(v(H_m) - \underline{v}(H_m)\right).$$

Otherwise, we have $v(H_m) \leq \frac{1}{2}\left(v(H_m) - \underline{v}(H_m)\right)$, which implies $-\underline{v}(H_m) \geq \frac{1}{2}\left(v(H_m) - \underline{v}(H_m)\right)$, and this leads to

$$\operatorname{Re} H(0) - \underline{v}(H_m) = 0 - \underline{v}(H_m) \geq \frac{1}{2}\left(v(H_m) - \underline{v}(H_m)\right) \geq \frac{1}{4}\tau(H_m)\left(v(H_m) - \underline{v}(H_m)\right).$$

Combing these two cases, we shall uniformly get $\hat{x} = \arg\max\left\{\operatorname{Re} H\left(\frac{1}{d}\sum_{k=1}^{d}\beta_k \hat{x}^k\right), \operatorname{Re} H(0)\right\}$ satisfying $\operatorname{Re} H(\hat{x}) - \underline{v}(H_m) \geq \frac{1}{4}\tau(H_m)\left(v(H_m) - \underline{v}(H_m)\right)$. Finally, by noticing $\hat{x}_i \in \operatorname{conv}(\mathbf{\Omega}_m)$ for $i = 1, 2, \ldots, n$ and $H(x)$ is square-free, and applying Lemma 4.2, we are able to find $z \in \mathbf{\Omega}_m^n$ in polynomial-time, such that

$$\operatorname{Re} H(z) - \underline{v}(H_m) \geq \operatorname{Re} H(\hat{x}) - \underline{v}(H_m) \geq \frac{1}{4}\tau(H_m)\left(v(H_m) - \underline{v}(H_m)\right).$$

□

Before concluding this subsection, we remark that (H_m) can be equivalently transferred to polynomial optimization over discrete variables in the real case, which was discussed in [11]. Essentially, by letting $x = y + \boldsymbol{i}z$ with $y, z \in \mathbb{R}^n$, $\operatorname{Re} H(x)$ can be rewritten as a homogeneous polynomial of (y, z), where for each $i = 1, 2, \ldots, n$, $(y_i, z_i) = \left(\cos\frac{2k\pi}{m}, \sin\frac{2k\pi}{m}\right)$ for some $k \in \mathbb{Z}$. By applying the Lagrange polynomial interpolation technique, the problem can then be transferred to an inhomogeneous polynomial optimization with binary constraints, which will yield a worst case relative approximation ratio as well. However, comparing to the bounds obtained in Theorem 4.4, the direct transformation to the real case is much worse and more costly to implement.

4.2 Homogeneous polynomial with unity constraints

Let us now turn to the case $m \to \infty$. In that case, (H_m) becomes

$$\begin{array}{lll} (H_\infty) & \max & \operatorname{Re} H(x) \\ & \text{s.t.} & x \in \mathbf{\Omega}_\infty^n. \end{array}$$

It is not hard to verity (see the proof of Theorem 4.5) that (H_∞) is actually equivalent to

$$\begin{array}{ll} \max & |H(x)| \\ \text{s.t.} & x \in \mathbf{\Omega}_\infty^n. \end{array}$$

For the case $d = 2$, the above problem was studied by Toker and Ozbay [25], and was termed complex programming. Unlike the case of the m-th roots of unity, where certain conditions on m and d are required to secure approximation ratios, model (H_∞) actually always admits a polynomial-time approximation ratio for any fixed d.

Theorem 4.5 *If $H(x)$ is square-free or $\operatorname{Re} H(x)$ is convex, then (H_∞) admits a polynomial-time randomized approximation algorithm with approximation ratio $\tau(H_\infty) := 0.7118(\frac{\pi}{4})^{d-2} d! d^{-d} n^{-\frac{d-2}{2}}$.*

Proof. Relaxing (H_∞) to (LH_∞), we may find a feasible solution $(\hat{x}^1, \hat{x}^2, \cdots, \hat{x}^d)$ of (LH_∞) in polynomial-time with approximation ratio $0.7118\left(\frac{\pi}{4}\right)^{d-2} n^{-\frac{d-2}{2}}$ by Theorem 3.6, i.e.,

$$\operatorname{Re} L(\hat{x}^1, \hat{x}^2, \cdots, \hat{x}^d) \geq 0.7118 \left(\frac{\pi}{4}\right)^{d-2} n^{-\frac{d-2}{2}} v(LH_\infty).$$

Then by Lemma 4.1, we further find $\beta \in \mathbf{\Omega}_\infty^d$ by randomization, such that

$$\operatorname{Re} \prod_{i=1}^d \overline{\beta_i} H\left(\frac{1}{d}\sum_{k=1}^d \beta_k \hat{x}^k\right) \geq \operatorname{Re} d^{-d} d! L(\hat{x}^1, \hat{x}^2, \cdots, \hat{x}^d) \geq \tau(H_\infty) v(LH_\infty) \geq \tau(H_\infty) v(H_\infty).$$

Let $\phi = \arg H\left(\frac{1}{d}\sum_{k=1}^d \beta_k \hat{x}^k\right)$, and we get

$$H\left(\frac{e^{-\boldsymbol{i}\phi/d}}{d}\sum_{k=1}^d \beta_k \hat{x}^k\right) = e^{-\boldsymbol{i}\phi} H\left(\frac{1}{d}\sum_{k=1}^d \beta_k \hat{x}^k\right) = \left|H\left(\frac{1}{d}\sum_{k=1}^d \beta_k \hat{x}^k\right)\right| \geq \operatorname{Re} \prod_{i=1}^d \overline{\beta_i} H\left(\frac{1}{d}\sum_{k=1}^d \beta_k \hat{x}^k\right).$$

Finally, by noticing that each component of $\frac{e^{-\boldsymbol{i}\phi/d}}{d}\sum_{k=1}^d \beta_k \hat{x}^k$ is in $\operatorname{conv}(\mathbf{\Omega}_\infty)$, and applying Lemma 4.2, we are able to find $y \in \mathbf{\Omega}_\infty^n$ in polynomial-time, such that

$$\operatorname{Re} H(y) \geq \operatorname{Re} H\left(\frac{e^{-\boldsymbol{i}\phi/d}}{d}\sum_{k=1}^d \beta_k \hat{x}^k\right) \geq \operatorname{Re} \prod_{i=1}^d \overline{\beta_i} H\left(\frac{1}{d}\sum_{k=1}^d \beta_k \hat{x}^k\right) \geq \tau(H_\infty) v(H_\infty).$$

□

4.3 Homogeneous polynomial with spherical constraint

Our last model in this section is spherically constrained homogeneous polynomial optimization in the complex domain

$$\begin{array}{lll} (H_S) & \max & \operatorname{Re} H(x) \\ & \text{s.t.} & x \in \mathbf{S}^n. \end{array}$$

The model is equivalent to $\max_{x \in \mathbf{S}^n} |H(x)|$, which is also equivalent to computing the largest eigenvalue of a super-symmetric complex tensor $\mathcal{F} \in \mathbb{C}^{n^d}$.

The real counterpart of (H_S) is studied in the literature; see [9, 23, 17]. The problem is related to computing the largest Z-eigenvalue of a super-symmetric tensor, or equivalently, finding the best rank-one approximation of a super-symmetric tensor [6, 27]. Again, in principle, the complex case can be transformed to the real case by letting $x = y + \boldsymbol{i}z$ with $y, z \in \mathbb{R}^n$, which however increases the number of the variables as well as the dimension of the data tensor $\mathcal{F}$. As a result, this will cause a deterioration in the approximation quality. Moreover, in the real case, (H_S) only admits a *relative* approximation ratio when d is even. Interestingly, for any fixed d, an absolute approximation ratio is possible for the complex case.

Theorem 4.6 (H_S) *admits admits a deterministic polynomial-time approximation algorithm with approximation ratio* $\tau(H_S) := d! d^{-d} n^{-\frac{d-2}{2}}$.

Proof. Like in the proof of Theorem 4.5, by relaxing (H_S) to (LH_S), we first find a feasible solution $(\hat{x}^1, \hat{x}^2, \cdots, \hat{x}^d)$ of (LH_S) with approximation ratio $n^{-\frac{d-2}{2}}$ (Theorem 3.7). Then by Lemma 4.1, we further find $\beta \in \mathbf{\Omega}_\infty^d$, such that

$$\operatorname{Re} \prod_{i=1}^d \overline{\beta_i} H\left(\frac{1}{d}\sum_{k=1}^d \beta_k \hat{x}^k\right) \geq \operatorname{Re} d^{-d} d! L(\hat{x}^1, \hat{x}^2, \cdots, \hat{x}^d) \geq \tau(H_S) v(LH_S) \geq \tau(H_S) v(H_S).$$

Let $\hat{x} = \frac{1}{d}\sum_{k=1}^d \beta_k \hat{x}^k$ and $\phi = \arg H(\hat{x})$. By triangle inequality we have $\|\hat{x}\| \leq \frac{1}{d}\sum_{k=1}^d \|\beta_k \hat{x}^k\| = 1$. Finally, $e^{-\boldsymbol{i}\phi/d}\hat{x}/\|\hat{x}\|$ is a feasible solution of (H_S), satisfying

$$H\left(e^{-\boldsymbol{i}\phi/d}\frac{\hat{x}}{\|\hat{x}\|}\right) = e^{-\boldsymbol{i}\phi}\|\hat{x}\|^{-d} H(\hat{x}) = \|\hat{x}\|^{-d}|H(\hat{x})| \geq |H(\hat{x})| \geq \operatorname{Re}\prod_{i=1}^d \overline{\beta_i} H(\hat{x}) \geq \tau(H_S) v(H_S).$$

□

We remark that the above result does not require $H(x)$ to be square-free or $\operatorname{Re} H(x)$ to be convex, which is a condition for Theorems 4.3 and 4.5.

5 Conjugate form optimization

Our last set of optimization models involve the so-called conjugate forms:

$$\begin{array}{lll} (C_m) & \max & C(\overline{x}, x) \\ & \text{s.t.} & x \in \mathbf{\Omega}_m^n; \\ (C_\infty) & \max & C(\overline{x}, x) \\ & \text{s.t.} & x \in \mathbf{\Omega}_\infty^n; \\ (C_S) & \max & C(\overline{x}, x) \\ & \text{s.t.} & x \in \mathbf{S}^n. \end{array}$$

Recall that the conjugate form $C(\overline{x}, x) = L(\underbrace{\overline{x}, \cdots, \overline{x}}_d, \underbrace{x, \cdots, x}_d)$ is associated with a conjugate partial-symmetric tensor $\mathcal{F} \in \mathbb{C}^{n^{2d}}$ (cf. Section 2 for details).

These models are known to have wide applications as well. For instance, (C_m) and (C_∞) with degree 4 are used in the design of radar waveforms [4] sharing an ambiguity function. (C_∞) includes (H_∞) as its special case, since (H_∞) is equivalent to $\max_{x \in \mathbf{\Omega}_\infty^n} |H(x)|$, where $|H(x)|^2$ is a special class for $C(\overline{x},x)$. Therefore, complex programming ((H_∞) with $d=2$) studied by Toker and Ozbay [25] also belongs to (C_∞). Similarly, (C_S) also includes (H_S) as its special case.

Let us now focus on approximation algorithms. Observe that for any conjugate partial-symmetric tensor $\mathcal{F}$ with its associated conjugate form $C(\overline{x},x)$:

$$C(\overline{x},x) = \mathrm{Re}\, L(x^1,\cdots,x^d,x^{d+1},\cdots,x^{2d}) \text{ when } x^1=\cdots=x^d=\overline{x} \text{ and } x^{d+1}=\cdots=x^{2d}=x.$$

Therefore, (C_m), (C_∞) and (C_S) can be relaxed to the following multilinear optimization models:

$$\begin{array}{lll}(LC_m) & \max & \mathrm{Re}\, L(x^1,\cdots,x^d,x^{d+1},\cdots,x^{2d}) \\ & \text{s.t.} & x^k \in \boldsymbol{\Omega}_m^n,\ k=1,2,\ldots,2d; \\ (LC_\infty) & \max & \mathrm{Re}\, L(x^1,\cdots,x^d,x^{d+1},\cdots,x^{2d}) \\ & \text{s.t.} & x^k \in \boldsymbol{\Omega}_\infty^n,\ k=1,2,\ldots,2d; \\ (LC_S) & \max & \mathrm{Re}\, L(x^1,\cdots,x^d,x^{d+1},\cdots,x^{2d}) \\ & \text{s.t.} & x^k \in \mathbf{S}^n,\ k=1,2,\ldots,2d.\end{array}$$

By the approximation results established in Section 3, we are able to find good approximate solutions for these multilinear form optimization models. In order to generate good approximate solutions for the original conjugate form optimizations, we need the following new linkage between the conjugate form and the multilinear form.

Lemma 5.1 *Let $m \in \{3,4,\cdots,\infty\}$. Suppose $x^1,x^2,\cdots,x^{2d} \in \mathbb{C}^n$, and $\mathcal{F} \in \mathbb{C}^{n^{2d}}$ is a conjugate partial-symmetric tensor with its associated multilinear form L and conjugate form C. If $\xi_1,\xi_2,\cdots,\xi_{2d}$ are i.i.d. uniform distribution on $\boldsymbol{\Omega}_m$, then*

$$\mathsf{E}\left[\left(\prod_{i=1}^d \xi_i\right)\left(\prod_{i=d+1}^{2d}\overline{\xi_i}\right) C\left(\sum_{k=1}^d \overline{\xi_k} x^k + \sum_{k=d+1}^{2d}\overline{\xi_k x^k}, \sum_{k=1}^d \xi_k \overline{x^k} + \sum_{k=d+1}^{2d}\xi_k x^k\right)\right] = (d!)^2 L(x^1,x^2,\cdots,x^{2d}).$$

The proof of Lemma 5.1 can be found in the appendix. By randomization we find $\beta \in \boldsymbol{\Omega}_m^{2d}$ in polynomial-time, such that

$$\mathrm{Re}\left(\prod_{i=1}^d \beta_i\right)\left(\prod_{i=d+1}^{2d}\overline{\beta_i}\right) C\left(\overline{x_\beta},x_\beta\right) \geq (d!)^2 2d^{-2d}\mathrm{Re}\, L(x^1,x^2,\cdots,x^{2d}), \tag{7}$$

where

$$x_\beta := \frac{1}{2d}\sum_{k=1}^d \beta_k \overline{x^k} + \frac{1}{2d}\sum_{k=d+1}^{2d}\beta_k x^k. \tag{8}$$

5.1 Conjugate form in the m-th roots of unity

For (C_m), by relaxing to (LC_m) and generating its approximate solution $(x^1,x^2,\cdots,x^{2d})$ from Theorem 3.4, we know $x^k \in \boldsymbol{\Omega}_m^n$ for $k=1,2,\ldots,2d$. Observe that each component of x_β defined by (8) is a convex combination of the elements in $\boldsymbol{\Omega}_m$, and is thus in $\mathrm{conv}\,(\boldsymbol{\Omega}_m)$. Though x_β may not be feasible to (C_m), a vertex solution (in $\boldsymbol{\Omega}_m$) can be found under certain conditions.

Lemma 5.2 *Let $m \in \{3, 4, \cdots, \infty\}$. Suppose $x \in \mathbb{C}^n$ with $x_i \in \operatorname{conv}(\mathbf{\Omega}_m)$ for all $1 \le i \le n$.*
(1) If $C(\overline{x}, x)$ is a square-free conjugate form, then $y, z \in \mathbf{\Omega}_m^n$ can be found in polynomial-time, such that $C(\overline{y}, y) \le C(\overline{x}, x) \le C(\overline{z}, z)$.
(2) If $C(\overline{x}, x)$ is convex, then $z \in \mathbf{\Omega}_m^n$ can be found in polynomial-time, such that $C(\overline{x}, x) \le C(\overline{z}, z)$.

The proof is similar to that of Lemma 4.2, and is thus omitted. Basically, the algorithm optimizes one variable x_i over $\mathbf{\Omega}_m$ while fixing other $n-1$ variables, alternatively for $i = 1, 2, \ldots, n$. The condition of square-free or convexity guarantees that each step of optimization can be done in polynomial-time. With all these preparations in place, we are ready to present the first approximation result for conjugate form optimization.

Theorem 5.3 *If $C(\overline{x}, x)$ is convex, then (C_m) admits a polynomial-time randomized approximation algorithm with approximation ratio $\tau(C_m) := \tau_m^{2d-2}(2\tau_m - 1)(d!)^2(2d)^{-2d}n^{-(d-1)}$.*

Proof. By relaxing (C_m) to (LC_m) and getting its approximate solution $(x^1, x^2, \cdots, x^{2d})$, we have

$$\operatorname{Re} L(x^1, x^2, \cdots, x^{2d}) \ge \tau_m^{2d-2}(2\tau_m - 1)n^{-(d-1)}v(LC_m) \ge \tau_m^{2d-2}(2\tau_m - 1)n^{-(d-1)}v(C_m). \quad (9)$$

Applying Lemma 5.1, we further get x_β defined by (8), satisfying (7), i.e.,

$$\operatorname{Re}\left(\prod_{i=1}^{d}\beta_i\right)\left(\prod_{i=d+1}^{2d}\overline{\beta_i}\right)C\left(\overline{x_\beta}, x_\beta\right) \ge (d!)^2 2d^{-2d}\operatorname{Re} L(x^1, x^2, \cdots, x^{2d}) \ge \tau(C_m)v(C_m).$$

Next we notice that any convex conjugate form is always nonnegative [15], i.e., $C(\overline{x}, x) \ge 0$ for all $x \in \mathbb{C}^n$. This further leads to

$$C\left(\overline{x_\beta}, x_\beta\right) \ge \operatorname{Re}\left(\prod_{i=1}^{d}\beta_i\right)\left(\prod_{i=d+1}^{2d}\overline{\beta_i}\right)C\left(\overline{x_\beta}, x_\beta\right) \ge \tau(C_m)v(C_m).$$

Finally, as each component of x_β belongs to $\operatorname{conv}(\mathbf{\Omega}_m)$, applying Lemma 5.2, we find $z \in \mathbf{\Omega}_m^n$ with $C(\overline{z}, z) \ge C\left(\overline{x_\beta}, x_\beta\right) \ge \tau(C_m)v(C_m)$. $\square$

As seen from the proof in Theorem 5.3, the nonnegativity of convex conjugate form plays an essential role in preserving approximation guarantee. For the general case, this approximation is not possible, since a conjugate form may be negative definite. However under the square-free condition, relative approximation is doable.

Theorem 5.4 *If $C(\overline{x}, x)$ is square-free, then (C_m) admits a polynomial-time randomized approximation algorithm with relative approximation ratio $\frac{1}{2}\tau(C_m)$.*

Proof. The main structure of the proof is similar to that of Theorem 4.4, based on two complementary cases: $v(C_m) \ge \frac{1}{2}\left(v(C_m) - \underline{v}(C_m)\right)$ and $-\underline{v}(C_m) \ge \frac{1}{2}\left(v(C_m) - \underline{v}(C_m)\right)$. For the latter case, it is obvious that

$$C(\overline{0}, 0) - \underline{v}(C_m) = 0 - \underline{v}(C_m) \ge \frac{1}{2}\left(v(C_m) - \underline{v}(C_m)\right) \ge \frac{1}{2}\tau(C_m)\left(v(C_m) - \underline{v}(C_m)\right). \quad (10)$$

For the former case, we relax (C_m) to (LC_m) and get its approximate solution $(x^1, x^2, \cdots, x^{2d})$. By (9) it follow that

$$
\begin{aligned}
(d!)^2(2d)^{-2d}\mathrm{Re}\, L(x^1, x^2, \cdots, x^{2d}) &\geq (d!)^2(2d)^{-2d}\tau_m^{2d-2}(2\tau_m - 1)n^{-(d-1)}v(C_m) \\
&\geq \frac{1}{2}\tau(C_m)\left(v(C_m) - \underline{v}(C_m)\right). \qquad (11)
\end{aligned}
$$

Assume $\xi \in \mathbf{\Omega}_m^{2d}$, whose components are i.i.d. uniform distribution on $\mathbf{\Omega}_m$. As each component of x_ξ defined by (8) belongs to $\mathrm{conv}\,(\mathbf{\Omega}_m)$, by Lemma 5.2, there exists $y \in \mathbf{\Omega}_m^n$ such that

$$
C(\overline{x_\xi}, x_\xi) \geq C(\overline{y}, y) \geq \underline{v}(C_m). \qquad (12)
$$

Applying Lemma 5.1, (11) further leads to

$$
\begin{aligned}
\frac{1}{2}\tau(C_m)\left(v(C_m) - \underline{v}(C_m)\right) &\leq (d!)^2(2d)^{-2d}\,\mathrm{Re}\, L(x^1, x^2, \cdots, x^{2d}) \\
&= \mathsf{E}\left[\mathrm{Re}\left(\prod_{i=1}^{d}\xi_i\right)\left(\prod_{i=d+1}^{2d}\overline{\xi_i}\right)C(\overline{x_\xi}, x_\xi)\right] \\
&= \mathsf{E}\left[\mathrm{Re}\left(\prod_{i=1}^{d}\xi_i\right)\left(\prod_{i=d+1}^{2d}\overline{\xi_i}\right)\left(C(\overline{x_\xi}, x_\xi) - \underline{v}(C_m)\right)\right] \\
&\leq \mathsf{E}\left[\left|\left(\prod_{i=1}^{d}\xi_i\right)\left(\prod_{i=d+1}^{2d}\overline{\xi_i}\right)\right| \cdot \left|C(\overline{x_\xi}, x_\xi) - \underline{v}(C_m)\right|\right] \\
&= \mathsf{E}\left[C(\overline{x_\xi}, x_\xi) - \underline{v}(C_m)\right],
\end{aligned}
$$

where the third step is due to $\mathsf{E}\left[\left(\prod_{i=1}^{d}\xi_i\right)\left(\prod_{i=d+1}^{2d}\overline{\xi_i}\right)\right] = 0$, and the last step is due to (12). Therefore by randomization, we are able to find $\beta \in \mathbf{\Omega}_m^{2d}$, such that

$$
C(\overline{x_\beta}, x_\beta) - \underline{v}(C_m) \geq \mathsf{E}\left[C(\overline{x_\xi}, x_\xi) - \underline{v}(C_m)\right] \geq \frac{1}{2}\tau(C_m)\left(v(C_m) - \underline{v}(C_m)\right).
$$

Combining (10), if we let $x' = \arg\max\left\{C(\overline{0}, 0), C(\overline{x_\beta}, x_\beta)\right\}$, then we shall uniformly have $C(\overline{x'}, x') - \underline{v}(C_m) \geq \frac{1}{2}\tau(C_m)\left(v(C_m) - \underline{v}(C_m)\right)$. Finally, as each component of x' belongs to $\mathrm{conv}\,(\mathbf{\Omega}_m)$ and $C(\overline{x}, x)$ is square-free, by Lemma 5.2, we are able to find $z \in \mathbf{\Omega}_m^n$ in polynomial-time, such that

$$
C(\overline{z}, z) - \underline{v}(C_m) \geq C(\overline{x'}, x') - \underline{v}(C_m) \geq \frac{1}{2}\tau(C_m)\left(v(C_m) - \underline{v}(C_m)\right).
$$

□

5.2 Conjugate form with unity constraints or spherical constraint

The discussion in Section 5.1 can be extended to conjugate form optimization over unity constraints, and the complex spherical constraint: (C_∞) and (C_S). Due to its similar nature, here we shall skip the details and only provide the main approximation results; the details can be easily supplemented by the interested reader. Essentially, the main steps are: (1) relax to multilinear form optimization models and find their approximate solutions as discussed in Section 3; (2) conduct randomization based on the link provided in Lemma 5.1; (3) search for the best vertex solution. For the complex unity constrained (C_∞), a vertex solution is guaranteed by Lemma 5.2, and for the spherically constrained (C_S), a vertex solution is obtained by scaling to $\mathbf{S}^n$: $x_\beta/\|x_\beta\|$.

Theorem 5.5 *(1) If $C(\overline{x}, x)$ is convex, then (C_∞) admits a polynomial-time randomized approximation algorithm with approximation ratio $\tau(C_\infty) := 0.7118\left(\frac{\pi}{4}\right)^{2d-2}(d!)^2(2d)^{-2d}n^{-(d-1)}$.*
(2) If $C(\overline{x}, x)$ is square-free, then (C_∞) admits a polynomial-time randomized approximation algorithm with relative approximation ratio $\frac{1}{2}\tau(C_\infty)$.

Theorem 5.6 *(1) If $C(\overline{x}, x)$ is nonnegative (including convex as its special case), then (C_S) admits a deterministic polynomial-time approximation algorithm with approximation ratio $\tau(C_S) := (d!)^2(2d)^{-2d}n^{-(d-1)}$.*
(2) For general $C(\overline{x}, x)$, (C_S) admits a deterministic polynomial-time approximation algorithm with relative approximation ratio $\frac{1}{2}\tau(C_S)$.

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A Proofs of the lemmas

Lemma 3.2 *Define* $F_m : \mathbb{C} \mapsto \mathbb{C}$ *with* $F_m(x) := \frac{m(2-\omega_m-\omega_m^{-1})}{8\pi^2} \sum_{\ell=0}^{m-1} \omega_m^\ell \left(\arccos\left(-\mathrm{Re}\,\omega_m^{-\ell}x\right)\right)^2$.
(1) If $a \in \mathbb{C}$ *and* $b \in \mathbf{\Omega}_m$, *then* $F_m(ab) = bF_m(a)$.
(2) If $a \in \mathbb{R}$, *then* $F_m(a) \in \mathbb{R}$.
Proof. (1) If $b \in \mathbf{\Omega}_m$, let $b = \omega_m^k$ for some $k \in \mathbb{Z}$. It holds that

$$\begin{aligned}
F_m(ab) = F_m(\omega_m^k a) &= \frac{m(2-\omega_m-\omega_m^{-1})}{8\pi^2} \sum_{\ell=0}^{m-1} \omega_m^\ell \left(\arccos\left(-\mathrm{Re}\,\omega_m^{-\ell}\omega_m^k a\right)\right)^2 \\
&= \omega_m^k \frac{m(2-\omega_m-\omega_m^{-1})}{8\pi^2} \sum_{\ell=0}^{m-1} \omega_m^{\ell-k} \left(\arccos\left(-\mathrm{Re}\,\omega_m^{-(\ell-k)} a\right)\right)^2 \\
&= b\frac{m(2-\omega_m-\omega_m^{-1})}{8\pi^2} \sum_{j=-k}^{m-1-k} \omega_m^j \left(\arccos\left(-\mathrm{Re}\,\omega_m^{-j} a\right)\right)^2 \\
&= bF_m(a).
\end{aligned}$$

(2) If $a \in \mathbb{R}$, then $\mathrm{Re}\,\omega_m^{-k}a = a\mathrm{Re}\,\omega_m^{-k} = a\mathrm{Re}\,\omega_m^k = \mathrm{Re}\,\omega_m^k a$ for any $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned}
\overline{F_m(a)} &= \frac{m(2-\omega_m^{-1}-\omega_m)}{8\pi^2} \sum_{\ell=0}^{m-1} \omega_m^{-\ell} \left(\arccos\left(-\mathrm{Re}\,\omega_m^{-\ell} a\right)\right)^2 \\
&= \frac{m(2-\omega_m-\omega_m^{-1})}{8\pi^2} \sum_{\ell=0}^{m-1} \omega_m^{-\ell} \left(\arccos\left(-\mathrm{Re}\,\omega_m^{\ell} a\right)\right)^2 \\
&= \frac{m(2-\omega_m-\omega_m^{-1})}{8\pi^2} \sum_{j=1-m}^{0} \omega_m^{j} \left(\arccos\left(-\mathrm{Re}\,\omega_m^{-j} a\right)\right)^2 \\
&= F_m(a),
\end{aligned}$$

implying that $F_m(a) \in \mathbb{R}$. $\square$

Lemma 4.1 *Let $m \in \{3,4,\cdots,\infty\}$. Suppose $x^1, x^2, \cdots, x^d \in \mathbb{C}^n$, and $\mathcal{F} \in \mathbb{C}^{n^d}$ is a super-symmetric complex tensor with its associated multilinear form L and homogeneous polynomial H. If $\xi_1, \xi_2, \cdots, \xi_d$ are i.i.d. uniform distribution on $\mathbf{\Omega}_m$, then*

$$\mathsf{E}\left[\prod_{i=1}^d \overline{\xi_i} H\left(\sum_{k=1}^d \xi_k x^k\right)\right] = d! L(x^1, x^2 \cdots, x^d) \text{ and } \mathsf{E}\left[\prod_{i=1}^d \xi_i H\left(\sum_{k=1}^d \xi_k x^k\right)\right] = 0.$$

Proof. First we observe that

$$\begin{aligned}\mathsf{E}\left[\prod_{i=1}^d \overline{\xi_i} H\left(\sum_{k=1}^d \xi_k x^k\right)\right] &= \mathsf{E}\left[\prod_{i=1}^d \overline{\xi_i} \sum_{1\le k_1,k_2,\cdots,k_d\le d} L\left(\xi_{k_1}x^{k_1}, \xi_{k_2}x^{k_2}, \cdots, \xi_{k_d}x^{k_d}\right)\right] \\ &= \sum_{1\le k_1,k_2,\cdots,k_d\le d} \mathsf{E}\left[\left(\prod_{i=1}^d \overline{\xi_i}\right)\left(\prod_{j=1}^d \xi_{k_j}\right) L\left(x^{k_1}, x^{k_2}, \cdots, x^{k_d}\right)\right].\end{aligned}$$

If $(k_1, k_2, \cdots, k_d) \in \Pi(1, 2, \ldots, d)$, i.e., a permutation of $\{1, 2, \ldots, d\}$, then

$$\mathsf{E}\left[\left(\prod_{i=1}^d \overline{\xi_i}\right)\left(\prod_{j=1}^d \xi_{k_j}\right)\right] = \mathsf{E}\left[\prod_{i=1}^d \overline{\xi_i}\xi_i\right] = 1;$$

otherwise, there exists k_0 $(1 \le k_0 \le d)$ such that and $k_0 \ne k_j$ for all $j = 1, 2, \ldots, d$. In the latter case,

$$\mathsf{E}\left[\left(\prod_{i=1}^d \overline{\xi_i}\right)\left(\prod_{j=1}^d \xi_{k_j}\right)\right] = \mathsf{E}\left[\overline{\xi_{k_0}}\right] \mathsf{E}\left[\left(\prod_{1\le i\le d, i\ne k_0} \overline{\xi_i}\right)\left(\prod_{j=1}^d \xi_{k_j}\right)\right] = 0.$$

Since the number of different permutations of $\{1, 2, \ldots, d\}$ is $d!$, by taking into account the super-symmetric property of L, the first identity follows.

For the second identity, similarly we have

$$\mathsf{E}\left[\prod_{i=1}^d \xi_i H\left(\sum_{k=1}^d \xi_k x^k\right)\right] = \sum_{1\le k_1,k_2,\cdots,k_d\le d} \mathsf{E}\left[\left(\prod_{i=1}^d \xi_i\right)\left(\prod_{j=1}^d \xi_{k_j}\right) L\left(x^{k_1}, x^{k_2}, \cdots, x^{k_d}\right)\right].$$

There exists k_0 $(1 \le k_0 \le d)$ such that ξ_{k_0} appears once or twice in $\left(\prod_{i=1}^d \xi_i\right)\left(\prod_{j=1}^d \xi_{k_j}\right)$. For $m \in \{3, 4, \cdots, \infty\}$, we notice that $\mathsf{E}[\xi_i] = 0$ and $\mathsf{E}[\xi_i^2] = 0$ for $i = 1, 2, \ldots, d$. By independence of ξ_i's, $\mathsf{E}\left[\left(\prod_{i=1}^d \xi_i\right)\left(\prod_{j=1}^d \xi_{k_j}\right)\right]$ is always zero, leading to the second identity. $\square$

Lemma 5.1 *Let $m \in \{3,4,\cdots,\infty\}$. Suppose $x^1, x^2, \cdots, x^{2d} \in \mathbb{C}^n$, and $\mathcal{F} \in \mathbb{C}^{n^{2d}}$ is a conjugate partial-symmetric tensor with its associated multilinear form L and conjugate form C. If $\xi_1, \xi_2, \cdots, \xi_{2d}$ are i.i.d. uniform distribution on $\mathbf{\Omega}_m$, then*

$$\mathsf{E}\left[\left(\prod_{i=1}^d \xi_i\right)\left(\prod_{i=d+1}^{2d} \overline{\xi_i}\right) C\left(\sum_{k=1}^d \overline{\xi_k} x^k + \sum_{k=d+1}^{2d} \overline{\xi_k x^k}, \sum_{k=1}^d \xi_k \overline{x^k} + \sum_{k=d+1}^{2d} \xi_k x^k\right)\right] = (d!)^2 L(x^1, x^2, \cdots, x^{2d}).$$

Proof. We first consider the following

$$
\begin{aligned}
&\mathsf{E}\left[\left(\prod_{i=1}^{d}\xi_i\right)\left(\prod_{i=d+1}^{2d}\overline{\xi_i}\right)C\left(\sum_{k=1}^{2d}\overline{\xi_k x^k},\sum_{k=1}^{2d}\xi_k x^k\right)\right]\\
&=\mathsf{E}\left[\left(\prod_{i=1}^{d}\xi_i\right)\left(\prod_{i=d+1}^{2d}\overline{\xi_i}\right)\sum_{1\le k_1,\cdots,k_{2d}\le 2d}L\left(\overline{\xi_{k_1}x^{k_1}},\cdots,\overline{\xi_{k_d}x^{k_d}},\xi_{k_{d+1}}x^{k_{d+1}},\cdots,\xi_{k_{2d}}x^{k_{2d}}\right)\right]\\
&=\sum_{1\le k_1,\cdots,k_{2d}\le 2d}\mathsf{E}\left[\left(\prod_{i=1}^{d}\xi_i\right)\left(\prod_{i=d+1}^{2d}\overline{\xi_i}\right)\left(\prod_{j=1}^{d}\overline{\xi_{k_j}}\right)\left(\prod_{j=d+1}^{2d}\xi_{k_j}\right)\right]L\left(\overline{x^{k_1}},\cdots,\overline{x^{k_d}},x^{k_{d+1}},\cdots,x^{k_{2d}}\right).
\end{aligned}
$$

For $m\in\{3,4,\cdots,\infty\}$, we observe that $\mathsf{E}[\xi_i]=0$ and $\mathsf{E}[\xi_i^2]=0$ for $i=1,2,\ldots,2d$. Using a similar argument in the proof of Lemma 4.1, we have

$$
\mathsf{E}\left[\left(\prod_{i=1}^{d}\xi_i\right)\left(\prod_{i=d+1}^{2d}\overline{\xi_i}\right)\left(\prod_{j=1}^{d}\overline{\xi_{k_j}}\right)\left(\prod_{j=d+1}^{2d}\xi_{k_j}\right)\right]=\begin{cases}1 & (k_1,\cdots,k_d)\in\Pi(1,\cdots,d)\text{ and}\\ & (k_{d+1},\cdots,k_{2d})\in\Pi(d+1,\cdots,2d);\\ 0 & \text{otherwise.}\end{cases}
$$

By noticing that $\mathcal{F}$ is conjugate partial-symmetric (see Definition 2.2), and considering numbers of permutations, it follows that

$$
\mathsf{E}\left[\left(\prod_{i=1}^{d}\xi_i\right)\left(\prod_{i=d+1}^{2d}\overline{\xi_i}\right)C\left(\sum_{k=1}^{2d}\overline{\xi_k x^k},\sum_{k=1}^{2d}\xi_k x^k\right)\right]=(d!)^2L\left(\overline{x^1},\cdots,\overline{x^d},x^{d+1},\cdots,x^{2d}\right).
$$

Finally, replacing $\overline{x^k}$ by x^k for $k=1,2,\ldots,d$ in the above identity leads to the desired result. $\square$