Computing best bounds for nonlinear risk measures with partial information

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Abstract

Extreme events occur rarely, but these are often the circumstance where an insurance coverage is demanded. Given the first, say, n moments of the risk(s) in these events, one is able to compute or approximate the tight bounds for a risk measures in the form of $\mathbb{E}(\psi(x))$ through semidefinite programmings (SDP), via robust optimization formulations. Existing results in the literature have already demonstrated the power of this technique when $\psi(x)$ is linear or piecewise linear. In this paper, we extend the technique in the case where $\psi(x)$ is a polynomial or fractional polynomial.

Keywords: moment bounds, semidefinite programming (SDP), robust optimization, worst-case scenario, nonlinear risk, risk management

MSC: 60E05, 62P05, 90C22

1. Introduction

1.1. Motivation

Without knowing the distribution of a random variable x, is it possible to estimate the expectation of the variable $\psi(x)$? As we shall see later, such problems are pervasive in risk management and financial engineering. To a lesser extent, we are interested in a confidence interval $[a, b]$ where $\mathbb{E}[\psi(x)] \in [a, b]$. In case $\mathbb{E}[\psi(x)]$ refers to a risk measure, we raise a particular concern on its (worst) upper bound. If some partial information of x is available, say its moment, then we show in this paper that it is possible to compute such a 100% confidence interval.

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We believe that the partial knowledge is a practical assumption. As a matter of fact, extreme events are of great concern in insurance industry, as we need to worry about the risk or the loss in the worst situations, where an insurance coverage is in place. However, owing to its low frequency of occurrence, extreme events can only be modeled or predicted with limited information. This is in contrast to the normal (or non-extreme) situations at which more data or information is available for estimations and statistical analysis. Very often, if we cannot find an exact solution to the risk measures, then we try to approximate it by estimating its bounds. It is now established that there is a perfect matching between the estimation and the use of semidefinite programming (SDP), a computationally efficient model, given that we know the moments of the risk. This results in the approach of computing the moment bounds for the risk measures. (We will further elaborate on the approach in the next subsection.)

1.2. Robustness and moment bounds

On the other hand, "the worst extreme" often refers to robustness in the field of optimization. Ben-Tal et al. (2009) explored rather comprehensively in the general formulation and applications in robust optimizations. They showed that when it could be formulated or approximated by SDPs. When robustness is considered in terms of probability distributions, which means we can pick any distribution to optimize our objective, the context turns into the generalized moment bound problem, the formulation of which is described as follows:

Let x be a random vector in $\mathbf{R}^d$. Given the moment information of $x, \mathbb{E}[x^i] = m_i, i = 0, \cdots, n$ (recall that $\mathbb{E}[x^0] = \mathbb{E}[1] \equiv 1$ is the probability of any whole sample space. Therefore $m_0 \equiv 1$ by definition), we want to find the upper (*resp.* lower) bounds on the expectation of a related quantity, $\psi(r)$, with the optimization problem:

$$
\begin{aligned}
(GP) \quad \sup_{x\sim(m_0,\cdots,m_n)} \mathbb{E}[\psi(x)] &:= \sup_{x\in\mathfrak{P}} \mathbb{E}[\psi(x)] \\
& \quad s.t. \quad \mathbb{E}[x^i] = m_i, \quad i = 0, \cdots, n \\
(\textit{resp.}) \quad (GPdown) \quad \inf_{x\sim(m_0,\cdots,m_n)} \mathbb{E}[\psi(x)] &:= \inf_{x\in\mathfrak{P}} \mathbb{E}[\psi(x)] \\
& \quad s.t. \quad \mathbb{E}[x^i] = m_i, \quad i = 0, \cdots, n
\end{aligned}
$$

where the optimization is taken over all possible distributions of the random variable x in the class $\mathfrak{P}$. It can be easily seen that the objective in (GPdown) can be regarded as $-\sup \mathbb{E}[-\psi(x)]$, which is essentially the same form as the objective in (GP). Hence the following discussion will no longer mention the lower bounds, as they can be easily derived from its upper bound counterparts. (Note that when $i = 1$, the first moment $\mathbb{E}[x]$ and m_1 are vectors; when $i = 2$, $\mathbb{E}[x^2]$ and m_2 are (covariance) matrices; when $i \geq 3$, $\mathbb{E}[x^i]$

and m_i are tenors, a higher dimension version of matrices. We will avoid discussing tenors throughout this paper.)

Remark 1. *In this paper the SDP formulations for the moment bounds problems are exact, not approximations. In general, this may not always be possible. Namely, not every moment bound problem will have an exact SDP formulation.*

1.3. Literature review in general

The feasibility of Problem (GP) given standard moments is the classical moment problem. Some well known result in the literature include Chebyshev (1874); Markov (1884); Karlin and Studden (1966); Kemperman (1968). Given the first and second moment of a random variable in $\mathbf{R}$, Chebyshev's inequality gives a bound on the distribution function. Bertsimas and Popescu (2005) generalized the result by computing optimal bounds, using SDP, on arbitrary distributions given any finite number of generalized moments. He et al. (2010) strengthened the inequality bounds when the first, second and fourth moments are known. Popescu (2005) further generalized the SDP approach for convex classes of distributions. Zuluaga and Peña (2005) worked on numerical solutions and approximations for the generalized tchebycheff inequalities. Regarding the applications of moment bounds, Scarf (1958) first derived the bound given mean-variance demand information in an inventory control problem. Lo (1987) and Grundy (1991) applied the context in bounding the option price, where x is a stock price. In the meantime, Levy (1985) discussed the option bounds with stochastic dominance approach. Rodriguez (2003) developed a unified approach for several existing option bounds. Zuluaga et al. (2009) extended Lo's bound to third-order. In the field of actuarial science (more details in the next subsection), Jansen et al. (1986) computed the analytical upper bound for the stop-loss payment given up to forth moment. Brockett and Cox (1985) discussed the insurance calculations using incomplete information. Cox (1991) developed bounds under a bounded support with a variety of piecewise linear functions. Cox et al. (2008) computed and approximated the bounds of the ruin probabilities and Value-at-Risk, through the sum-of-squares (SOS) formulations. Their extension can be found in Tian (2008). De Schepper and Heijnen (2010) derived bounds on tail probabilities and Value-at-Risk with a small number of parameters. Recently, Liu and Li (2009) considered the bound by assuming a bounded support and a unimodal distribution for the univariate random variable. They also obtained a bound for the variance. Moment bounds are also applied in portfolio selection problems, in which $\psi(x)$ can be some utility function with x being the weight allocation of different assets (Chen et al.; El Ghaoui et al., 2003; Han et al., 2005). Negative components of x means short selling the corresponding asset(s).

1.4. More literature review in actuarial science

Specially in the field of actuarial science, estimating moment bounds was in fact a very ad hoc issue during 1980's. At that time, a host of research papers were published, mainly by Vylder, Kaas, Goovaerts, Taylor and their co-authors (see e.g. Jansen et al. (1986), Kaas and Goovaerts (1986); Goovaerts and Kaas (1985), De Vylder and Goovaerts (1982, 1983), De Vylder (1982, 1983b,a), Goovaerts et al. (1982), Taylor (1977, 1986) and the references therein). In most cases, the risk measure in question was $\psi(x) = (x - k)_+$, where k was a given constant, since this was the stop-loss expression often used in insurance and reinsurance payoffs calculations. Broadly speaking, there were two sub-streams in this line of research. While some were exclusively devoted to an analytical form of the bounds (e.g. De Vylder and Goovaerts (1982), Jansen et al. (1986), Cox (1991), De Vylder and Goovaerts (1983)), there were also studies focusing on numerical bounds and the theories behind. In particular, De Vylder (1982) linked the moment bounds problem with conic theory, and Goovaerts et al. (1982) suggested an approximation scheme using linear programming (LP). An advancement of this sub-stream led to robust optimization, which further led to semidefinite programming (SDP)– a fundamental tool underlying this paper. Let us emphasize that the proposed formulation in this paper is an exact solution method, not an approximation.

A serious limitation does exist in another sub-stream. It was criticized by Goovaerts et al. (1982) that the method for analytical forms "can only be worked out in practice if $n \leq 3$", where n refers to the number of integral constraints (i.e. amount of given partial information). Even so, there were some recent papers on this sub-stream. De Schepper and Heijnen (2007) found the explicit form of bounds provided up to the first three moments and the mode. Laurence and Wang (2009) derived in closed form some distribution-free bounds and optimal subreplicating strategies for spread options in a one-period static arbitrage setting. Most recently, Goovaerts et al. (2011) revisited Vylder and his co-authors' results and found best-possible upper bounds on a rich class of risk measures. They also extended the discussion for the multivariate case. These papers reflect an on-going research activities on the moment bounds problem or the distribution-free issues.

1.5. Our contribution

Specifically, in this paper we focus on an instance of Problem (GP) with $x \in \mathbf{R}$, where $\psi(x) = L\left(\frac{p(x)}{q(x)}\right)$ with $L(\cdot)$ being some linear or piecewise linear function, and $p(x)$ and $q(x)$ are polynomials. In words, we try to handle nonlinear risks in the form of fractional polynomials. Given the existing methodology of polynomial optimization with SDP characterizations (Nesterov, 2000), we introduce this modern powerful tool in the framework of nonlinear risk management, or more specifically, interest rate risk management. To our knowledge, there is no previous discussion on nonlinear risk management regarding the use of the moment bounds. In subsequent sections, we will discuss the methodology of calculating the moment bound for the nonlinear quantity's expectation, followed by that of two typical risk measures associated with it, namely, the worst-case probability and the worst-case downside risk. Imagine that x is an interest rate. In a broad sense, x may represent as bond yields, mortgage rates, or any rates used in discounting future cashflows. The interest rate linked products are in the form of $L\left(\frac{p(x)}{q(x)}\right)$ including annuity life products, bond options and mortgage payments. We may not always have close form expressions for the risk or the price. Even if we do, from the risk management point of view, it is crucial to know the worst case. We will discuss how these bounds can be obtained through SDP formulations.

Given the insurance literatures in 1980's and the follow-up papers, we would like to introduce the use of SDP. Not only because it is one of the most powerful models in the field of optimization, but also the SDP formulations, like the ones in this paper, can provide an exact numerical solution (subject to machine rounding errors) to the problems, whereas in those literatures Goovaerts et al. (1982), Goovaerts et al. (2011), only an approximation scheme was suggested.

The rest of this paper will be organized as follows. Some moment bounds, dualities, and some results on nonnegative polynomials are introduced in section 2. In section 3, the use of SDP in moment bound applications in nonlinear form of risk is presented, under the framework of managing the interest rate risk in mortgage business, followed by some numerical results and conclusions in section 4 and section 5 respectively.

2. Methodological fundamentals behind the moment bounds

A well known example of moment bounds is probably the option bound provided by Lo (1987):

$$\sup_{x\sim(m_1,m_2)_+} \mathbb{E}[x-k]_+ = \begin{cases} \mu - k\frac{m_1^2}{m_2} & \text{if } k \le \frac{m_2}{2m_1} \\ \frac{1}{2}(m_1 - k + \sqrt{k^2 - 2m_1 k + m_2}) & \text{otherwise} \end{cases}$$

Referring to (GP), he took $\psi(x) = (x-k)_+$ and assumed the knowledge of only the first two moments ($n = 2$). His close form solution was referred to Scarf (1958), who considered almost the same function $\psi(x) = \min(x, k)$. Jansen et al. (1986) also worked out an explicit form given the moments information up to the forth order, which appeared to be sophisticated. In general, given any n, it could be imagined the limitation to explore a closed form expression of (GP). One would then resort to numerical methods. It turns out that the conic optimization approach brings fruitful results into the context. In view of this, the dual formulations and semidefinite programming (SDP) need to be introduced.

2.1. Dual formulations, duality and tight bounds

(GP) is an infinite dimensional problem, since the decision variable is any (discrete or continuous) probability distribution satisfying the moment requirement. This makes the problem counter-intuitive and difficult to solve in general. However, if we shift our focus to its dual formulation, it may be solved numerically under some settings, depending on the dimension of x and the choice of $\psi(x)$. In this paper, we restrict our discussion to $x \in \mathbf{R}$ and let us name the restricted (GP) as (GP'). Then the dual formulation of (GP') is

$$
\begin{aligned}
(GD') \qquad & \inf_{z_0,\cdots,z_n} \sum_{i=0}^{n} z_i m_i \\
& s.t. \quad \sum_{i=0}^{n} z_i x^i \geq \psi(x) \quad \forall x \in \Omega
\end{aligned}
$$

where Ω is a pre-defined sample space. It is worth noting that the dual formulation is an upper bound of its primal problem (GP'), which is guaranteed by the weak duality.

Theorem 1. *(Weak Duality) Let v_d be the optimal value of (GD'), and v_p be the optimal value of (GP'). We have $v_p \leq v_d$.*

Proof. For any feasible distribution π in (GP'), and dual feasible solution $\bar{z}_0, \cdots, \bar{z}_n$ to (GD'), the dual constraint implies that

$$
\mathbb{E}^{\pi}[\psi(x)] \leq \mathbb{E}^{\pi}[\sum_{i=0}^{n} \bar{z}_i x^i] = \sum_{i=0}^{n} \bar{z}_i \mathbb{E}^{\pi}[x^i] = \sum_{i=0}^{n} \bar{z}_i m_i,
$$

where the first equality is given by the primal constraints. Taking the respective optimal, we have completed the proof. □

When the equity holds (i.e. $v_p = v_d$), we call v_d a tight (upper) bound. In optimization problems, the weak duality always holds. Nonetheless, the most informative upper bound is the tight one, and this is guaranteed available by the strong duality (see, e.g. Luo et al.; Shapiro (2001)).

Theorem 2. *(Strong Duality) When (GP') is feasible and the dual (GD') is strictly feasible, then $v_p = v_d$.*

Let us remark that (GP') being feasible refers to any distribution with the given moment, while a strict feasibility of (GD') means there exist $\bar{z}_0, \cdots, \bar{z}_n$ such that the inequality constraint is strict.

2.2. SDP and LMIs for some dual problems

By theorem 2, we can look into the dual problem (GD') for the optimal solution to (GP'). The next concern is when (GD') is computable. Practically, we mean to ask if the dual can be formulated by semidefinite programming (SDP), [1] or equivalently, if the dual constraint can be written as Linear Matrix Inequalities[2](LMIs) (For more information about SDP and LMI, we refer the interested reader to Boyd and Vandenberghe (2004); Nemirovski). Given that $x \in \mathbf{R}$ (univariate), the possibility of forming LMIs depends on the choice of $\psi(x)$ and Ω. When $\sum_{i=0}^{n} z_i x^i - \psi(x)$ is a polynomial over a bounded interval or semi-infinite interval, Nesterov (2000) provided an affirmative answer by giving an explicit description of the cones of polynomials that are representable as LMIs. Inspired by Nesterov (1997), Bertsimas and Popescu (2005) gave an explicit formulation for applications of moment problems. To summarize, when $\sum_{i=0}^{n} z_i x^i - \psi(x)$ is a polynomial, we conclude that each of the following three dual problems can be written as an SDP.

$$(D1) \qquad \inf_{z_0,\cdots,z_n} \sum_{i=0}^{n} z_i m_i$$
$$\text{s.t.} \sum_{i=0}^{n} z_i x^i \geq \psi(x) \quad \forall x \in \mathbb{R}$$

$$(D2) \qquad \inf_{z_0,\cdots,z_n} \sum_{i=0}^{n} z_i m_i$$
$$\text{s.t.} \sum_{i=0}^{n} z_i x^i \geq \psi(x) \quad \forall x \geq 0$$

$$(D3) \qquad \inf_{z_0,\cdots,z_n} \sum_{i=0}^{n} z_i m_i$$
$$\text{s.t.} \sum_{i=0}^{n} z_i x^i \geq \psi(x) \quad \forall x \in [a,b]$$

Our claims are back by Theorem 4, 5 and 6 respectively in Appendix. For the calculations of the nonlinear risk and its risk measures in this paper, we extend to claim that the dual problem with $\psi(x)$ being a fractional polynomial and Ω a union of (disjoint) intervals can also be written an SDP:

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[1]Informally speaking, SDP is a linear programming with LMI(s).

[2]Let $y = [y_1, \cdots, y_k]$ be a variable vector and $A_0, A_1, \cdots, A_k$ be some constant matrices. An LMI is of the form $A_0 + y_1 A_1 + \cdots + y_k A_k \succeq 0$, where "$\succeq 0$" means that the sum is a positive semidefinite matrix. Upon arriving at this form, the constraint can be handled computationally with Matlab free toolboxes, for instance, SeDuMi (Sturm, 2004) and cvx (Grant and Boyd, 2011).

Theorem 3. *Let $I_j = [a_j, b_j]$ for $j = 1, \cdots, k$, where $-\infty \leq a_1 < b_1 < a_2 < \cdots < b_k \leq \infty$. Consider*

$$
\begin{aligned}
(D4) \qquad & \inf_{z_0,\cdots,z_n} \sum_{i=0}^{n} z_0 m_i \\
& s.t. \ \sum_{i=0}^{n} z_i x^i \geq \psi(x) \quad \forall x \in \bigcup_{j=1}^{k} I_j
\end{aligned}
$$

When $\psi(x)$ is a fractional polynomial, (D4) is an SDP.

Proof. Let $\psi(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are some polynomials and $q(x) \neq 0$. Then

$$
\begin{aligned}
& \sum_{i=0}^{n} z_i x^i \geq \psi(x) \quad \forall x \in I_1 \cup \cdots \cup I_k \\
\Longleftrightarrow \ & q(x) \sum_{i=0}^{n} z_i x^i \geq p(x) \quad \forall x \in I_1 \cup \cdots \cup I_k \\
\Longleftrightarrow \ & \begin{cases} q(x) \sum_{i=0}^{n} z_i x^i \geq p(x) \quad \forall x \in I_1 \\ \qquad \vdots \\ q(x) \sum_{i=0}^{n} z_i x^i \geq p(x) \quad \forall x \in I_k \end{cases}
\end{aligned}
$$

By Theorem 5 and 6 in the Appendix, each nonnegative univariate polynomial here can be represented with an LMI. Hence (D4) is an SDP. □

Similarly, we can extend to handle the constraint of nonnegative polynomial over another:

Lemma 1. *Let $g(x)$ be a polynomial and $\psi(x)$ a fractional polynomial. Consider*

$$
\begin{aligned}
(D5) \qquad & \inf_{z_0,\cdots,z_n} \sum_{i=0}^{n} z_0 m_i \\
& s.t. \ \sum_{i=0}^{n} z_i x^i \geq \psi(x) \quad \forall g(x) \geq 0
\end{aligned}
$$

(D5) is equivalent to (D4).

Proof. Note that $g(x) \geq 0 \Leftrightarrow x \in I_1 \cup \cdots \cup I_k$ for some k. Hence (D5) is an SDP. □

3. Worst Expectation and Worst Risk Measures on Annuity Payments

As our work focuses on the management of nonlinear risk of a fractional polynomial form, we try to apply the moment bound on annuity payments, in particular on mortgages, which is a natural association of such a form where interest rate is considered the risk. With different choices of $\psi(\cdot)$, we will discuss the use of the dual forms in previous section in risk management.

3.1. The worst mortgage payments

Let us start with the most familiar annuity formula. Given the mortgage loan P, period t and interest rate r and the annuity A, we have

$$P = A\left(\frac{1}{1+r} + \cdots + \frac{1}{(1+r)^t}\right) = A\frac{(1+r)^t - 1}{r(1+r)^t}$$

$$f_{P,t}(r) := A = \frac{Pr(1+r)^t}{(1+r)^t - 1} \tag{1}$$

If we fix P and t, the annuity can be regarded as a nonlinear (fractional) polynomial in the interest rate. ~~We are led to wonder the possible values of the annuity.~~ Before we enter into a mortgage contract, we should take the first precautious step to estimate how worst the periodic payment can be. In other words, given the moment information of r, how much is charged in the worst case? We can formulate this as the moment bound problem with $\psi(r) = f_{P,t}(r)$ as follows:

$$(RM1) \qquad \sup_{r\sim(m_1,\cdots,m_n)_+} \mathbb{E}(f_{P,t}(r)) = \inf_{z_0,\cdots,z_n} \sum_{i=0}^{n} z_i m_i$$
$$\text{s.t.} \quad \sum_{i=0}^{n} z_i r^i \geq f_{P,t}(r) \quad \forall r \geq 0$$

By Theorem 5, (RM1) is an SDP and therefore can be computed efficiently.

3.2. The worst probability of repayment failure

Risk is typically viewed as an uncertainty, or a random variable, in the future. In order to get more information for what one is going to face, people naturally seek to know the chance of each scenario's occurrence. For example, both parties, the mortgagor and the mortgagee, typically worry about the ability of the former to pay a series of periodic obligated payments. If the mortgagor can only set aside a portion of his monthly income, let say h, how likely is his failure in the mortgage obligation when the interest rate moves against him? Mathematically, he needs to estimate the greatest probability that the mortgage payment exceeds h. When we know the moments of r, this probability can be calculated by formulating a moment bound problem (i.e. an instance of Problem (GP)) as follows.

$$
\begin{aligned}
(RM2) \qquad \sup_{r\sim(m_1,\cdots,m_n)_+} \mathbb{P}(f_{P,t}(r)\geq h) &= \inf_{z_0,\cdots,z_n} \sum_{i=0}^{n} z_i m_i \\
&\qquad \text{s.t. } \sum_{i=0}^{n} z_i r^i \geq \mathbf{1}_{\{f_{P,t}(r)\geq h\}} \quad \forall r\geq 0 \\
&= \inf_{z_0,\cdots,z_n} \sum_{i=0}^{n} z_i m_i \\
&\qquad \text{s.t. } \begin{cases} \sum_{i=0}^{n} z_i r^i \geq 1 & \forall f_{P,t}(r)\geq h, r\geq 0 \\ \sum_{i=0}^{n} z_i r^i \geq 0 & \forall f_{P,t}(r)< h, r\geq 0 \end{cases}
\end{aligned} \tag{2}
$$

Note that

$$
f_{P,t}(r) = h \Rightarrow P\cdot r(1+r)^t = h((1+r)^t - 1). \tag{3}
$$

Therefore, $f_{P,t}(r) \geq h$ can be represented by a union of intervals. Together with $r \geq 0$, the for-all condition above is still a union of intervals, thus representable as LMIs by Lemma 1. Another key to note is that $\mathbb{P}(f_{P,t}(r)\geq h) = \mathbb{E}[\mathbf{1}_{\{f_{P,t}(r)\geq h\}}]$, where $\mathbf{1}_{\{x\in\mathcal{A}\}}$ takes value 1 if $x \in \mathcal{A}$ and 0 otherwise.

3.3. The worst expected downside risk of exceeding the threshold

Treating the probability as a preliminary estimation on risk prevailing in the market, a financial institution (or mortgagee) needs to take more steps when accepting mortgage applications. In the subprime crisis, in order to diversify the risk of this mortgage pool, they securitized it into different tranches of bond-like hybrids, called collateralized mortgage obligations (CMOs). To make these products attractive enough, they are covered with an insurance (most of which were from AIG during the crisis) to become bonds of investment grades. As a result, most banks, pension funds and insurance firms, held with confidence a rather large portfolio of them. When the interest rate rallied, all such investment graded products became toxic assets poisoning quite a number of the entities and the disaster followed. While someone blamed the quants for facilitating the domino effect with complicated models and some blamed the human greed, we believed one of the fundamental reasons was due to the underestimation of the interest rate risk at stake. Taking into account the precautious attitudes in financial industry and the fact that it is always a difficult task to forecast the stochastic movements of interest rate, we urge for the concern about the worst expected risk of failing mortgage payments: If the payment $f_{P,t}(r)$ climbs beyond the homeowner's ability (or a certain threshold) h, what is the expected value of $[f_{P,t}(r) - h]_+$ based on the handy moment information of the interest rate? Again, we can consider the dual formulation of the moment bound problem:

$$
\begin{aligned}
(RM3) \quad \sup_{r\sim(m_1,\cdots,m_n)_+} \mathbb{E}(f_{P,t}(r)-h)_+ = & \inf_{z_0,\cdots,z_n} \sum_{i=0}^{n} z_i m_i \\
& \text{s.t.} \ \sum_{i=0}^{n} z_i r^i \geq (f_{P,t}(r)-h)_+ \quad \forall r \geq 0 \\
= & \inf_{z_0,\cdots,z_n} \sum_{i=0}^{n} z_i m_i \\
& \text{s.t.} \ \begin{cases} \sum_{i=0}^{n} z_i r^i \geq f_{P,t}(r) - h & \forall f_{P,t}(r) \geq h, r \geq 0 \\ \sum_{i=0}^{n} z_i r^i \geq 0 & \forall f_{P,t}(r) \leq h, r \geq 0 \end{cases}
\end{aligned} \tag{4}
$$

By Lemma 1, constraints in (4) can be re-formulated as LMIs.

We suggest that this risk measure could be put into practice in quite some places. For instance, nowadays, mortgage applicants need to submit their credit information, such as monthly salary and loan record over the past few years. The bank in charge then makes sure that they pass a certain stress test based on their credit profile before offering the loan. Our risk value can certainly play a role in setting the passing mark there. Suppose that, under the prevailing 1M-LIBOR rate, an applicant needs to repay USD\$1000 monthly, which is currently affordable based on his credit profile. Can his remaining salary and savings absorb the maximum expected high side $[f_{P,t}(r) - \$1000]_+$? If not, his application may probably be turned down or he may be required to purchase some facilities to enhance his credit.

Another application is insurance pricing, which is obvious from the very nature of its form $\mathbb{E}[f_{P,t}(r) - h]_+$. From the prospect of home buyers, floating rate mortgage plan is more attractive than the fixed rate plans in the low interest environment. But what if the interest rate soars? Although the mortgage plan usually includes a cap on the rate, this still possibly creates a serious financial burden for them. The reason is that this cap is usually referenced to another more stable and yet changing rate (e.g. PRIME rate in Hong Kong). Meanwhile, the cap may still be too high for protection. Being offered an insurance against the unwanted high side of the pre-specified amount, the home buyer can decide to pay the extra premium for the protection. In other words, we mean to calculate $\mathbb{E}[\sum_{j=1}^{J} \delta_j [f_{P,t}(r) - h]_+]$, where δ_j is some discount factor for the j-th repayment. Since $f_{P,t}(r)$ is nonlinear, there is no easy way to find its close form solution, but the upper bound can be estimated by our method with some common choices of δ_j:

1. If the discount factor δ_j is chosen to be independent of r, we can simply compute

$$
\sum_{j=1}^{J} \delta_j \sup_{r\sim(m_1,\cdots,m_n)_+} \mathbb{E}[[f_{P,t-j+1}(r) - h]_+].
$$

2. If the discount factor is $\delta_j = \frac{1}{(1+r)^j}$, we will have to handle some of piecewise fractional polynomials $\sum_{j=1}^{J} \frac{1}{(1+r)^j}[f_{P,t-j+1}(r) - h]_+$ as our objective. This is sophisticated, but still computable with our models.

3. If the discount factor is $\delta_j = \frac{1}{(1+r+s)^j}$, i.e. depending on r plus a given constant spread s, it is easy to see from 2 that our model still applies.

Therefore the numerical value of the bounds could be obtained.

4. Numerical Examples for Risk Management

4.1. A Mortgage Example

To demonstrate the use of our models, let us present an experimental scenario.

Consider an annuity for a loan \$1000 with annual payments in the next 20 years, charged for a floating interest rate. Suppose the latest reference rate is 2.5% p.a.. By (1), the annual payment

$$f_{1000,20}(0.0013) = \$50.69.$$

To mimic the trend of interest rate in real situation, we took the mean and standard deviation based on the historical 1-year Treasury rate[3]. In particular, we chose the most recent 5-year (2007-2011), 10-year (2002-2011) and 20-year (1992-2011) monthly samplings for comparisons, which reflected concerns on different risk horizons. According to our first model, each of the three sets of data gives a tight bound on $\sup \mathbb{E}[f_{1000,20}(r)]$:

Sampling period	μ	σ	$\sup \mathbb{E}[f_{1000,20}(r)]$	$\frac{\sup \mathbb{E}[f_{1000,20}(r)]}{f_{1000,20}(0.025)} - 1$
5-year	1.46%	1.70%	\$58.4817	15%
10-year	2.10%	1.68%	\$62.1876	23%
20-year	3.52%	2.00%	\$71.1213	40%

If this loan serves a period of 5 and 10 years, our model shows that the risk (potential increase) can be as much as 15% and 23% of the current level respectively. In case the loan is taken care until its maturity, our model based on a 20-year sampling suggests that there is a risk of increasing 40% of the current level.

[3] 1-year Treasury constant maturities is used in `http://www.federalreserve.gov/releases/h15/data.htm`

Meanwhile, one of the common risk management techniques is to study the risk at one or two standard deviation about the mean. We can convert these risk levels into corresponding thresholds h. In other words, we worry about the increasing payments due to interest rate fluctuation. When the payment reaches a certain threshold level h, we may need coverage or cede the unwanted risk to other parties. Based on our second and third models, we can calculate the maximum stop loss payment as well as probability with our models at the thresholds.

Sampling period	$\mu+\sigma$	eqv. threshold h^4	$\sup \mathbb{E}[f_{1000,20}(r)-h]_+$	$\sup \mathbb{P}(f_{1000,20}(r) \geq h)$
5-year	3.16%	\$61.2121	\$2.3312	0.4630
10-year	3.78%	\$72.1465	\$2.3666	0.5000
20-year	5.53%	\$83.8605	\$3.0463	0.5000

Sampling period	$\mu+2\sigma$	eqv. threshold h^5	$\sup \mathbb{E}[f_{1000,20}(r)-h]_+$	$\sup \mathbb{P}(f_{1000,20}(r) \geq h)$
5-year	4.86%	\$79.2686	\$1.4726	0.2000
10-year	5.46%	\$83.3836	\$1.4767	0.2000
20-year	7.53%	\$98.3171	\$1.8929	0.2000

Compare the results of the two risk levels $\mu+\sigma$ and $\mu+2\sigma$. If we manage to retain more risk, the coverage can be as cheap as 62% of otherwise, e.g. $\frac{1.4726}{2.3312} \approx 63\%$. If we consider the worst probabilities as the payoff of binary options $\mathbb{E}[\mathbf{1}_{f_{1000,20}(r)\geq h}]$, a similar conclusion results (40% in this case). Hinted by the calculated chance of happening and cost for protections, we may judge the acceptance level of risk.

4.2. A Reserve Example for an Endowment Insurance

When setting a reserve for a policy, an actuary needs to assume the interest rate charged. We can go through a sensitivity analysis with the following simplified example. Suppose a man of age 50 has just entered into an endowment life insurance with \$10000 coverage for 15 years. When he lives to 65 years old, a cash bonus of \$10000 is returned and the insurance contract ends.

Let us refer to the 2007 period life table in the U.S. Social Security Administration.[6] The actuarial present value for this policy is calculated to be $v = \$6273.07$ (calculated in the following table) assuming a 5% annual interest rate.

[4] $h = f_{1000,20}(\mu+\sigma)$
[5] $h = f_{1000,20}(\mu+2\sigma)$
[6] `http://www.ssa.gov/oact/STATS/table4c6.html`

policy year i	mortality rate at i (q_i)	$\alpha_i := q_i \times 10000$	$\frac{1}{1.05^i}$	discounted paid off
0	0.00551	55.12	1.0000	55.12
1	0.00597	59.75	0.9524	56.90
2	0.00643	64.25	0.9070	58.28
3	0.00685	68.52	0.8638	59.19
4	0.00727	72.71	0.8227	59.82
5	0.00772	77.20	0.7835	60.48
6	0.00821	82.11	0.7462	61.27
7	0.00873	87.34	0.7107	62.07
8	0.00929	92.90	0.6768	62.88
9	0.00989	98.86	0.6446	63.72
10	0.92412	105.41	0.6139	64.72
			Premium	\$6273.07

We further assume that this is a single premium without any loading. Then v is the theoretical reserve which is set against the man's mortality in the insurance company. The actuary can examine the risk of interest rate on v using the moment bounds, say, referencing to the 1-year market yield on U.S. Treasury[7] securities. If he want to know the worst expectation [8] of the reserve, he can consider (RM1) by taking $\tilde{f}(r) := \sum_{i=0}^{10} \frac{\alpha_i}{(1+r)^i}$ and assume the knowledge of mean μ and standard deviation σ. A transformation $x = 1 + r$ (so that we use $f(x) := \tilde{f}(x-1) = \sum_{i=0}^{10} \frac{\alpha_i}{x^i}$) will lead to this moment bound formulation:

$$\sup_{x \sim (1+\mu, (1+\mu)^2+\sigma^2)_+} \mathbb{E}(f(x)) = \inf_{z_0, z_1, z_2} z_0 + z_1(1+\mu) + z_2((1+\mu)^2 + \sigma^2)$$

$$\text{s.t. } z_0 + z_1 x + z_2 x^2 \geq f(x) \quad \forall x \geq 1$$

The actuary can compare the impact of interest rate against different horizons of history on the reserve. The calculation is in the following table:

Horizon of History	μ	σ	$\sup \mathbb{E}[f(x)]$	% of risk
5 years	1.466%	1.836%	\$8839.53	33%
10 years	2.102%	1.721%	\$8315.02	33%
20 years	3.524%	2.000%	\$7329.79	17%

In the table, the risk is regarded as the portion in excess of the reserve. The risk using 20 year's history is lower. This may be explained by the fact that the standard deviation is lower relative to the mean.

[7] `http://www.federalreserve.gov/releases/h15/data.htm`

[8] Here assumes that the randomness of interest rate is independent of mortality.

5. Conclusion

By Theorem 3, we manage to extend the SDP techniques to calculate moment bounds for nonlinear risk in the form of fractional polynomials. When we estimate a variety of risk measures or payments in such nonlinear forms, we can use the techniques to find their supremum (and infimum, if necessary). We believe that these estimations are essential in risk management, especially when we have limited information on the products.

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Appendix

Theorem 4. *(cf. Theorem 17.10 in Nesterov (2000)) Let* $p(x) = \sum_{i=1}^{2d} p_i x^i$ *be a univariate polynomial of degree 2d.* $p(x) \geq 0$ *for all* $x \in \mathbf{R}$ *if and only if there exists a positive semidefinite matrix Q (in symbol we write* $Q \succeq 0$*) such that*

$$p_i = \sum_{j+k=i} Q_{jk},$$

where Q_{jk} *is the entry of Q in the j's row and k's column.*

Theorem 5. *(cf. Theorem 17.11 in Nesterov (2000)) Let* $p(x) = \sum_{i=1}^{d} p_i x^i$ *be a univariate polynomial of degree d.* $p(x) \geq 0$ *for all* $x \in \mathbf{R}_+$ *can be represented by an LMI.*

Proof. Take $x = y^2$ and apply Theorem 4. □

Theorem 6. *(cf. Theorem 17.12 in Nesterov (2000)) Let* $p(x) = \sum_{i=1}^{d} p_i x^i$ *be a univariate polynomial of degree d.* $p(x) \geq 0$ *for all* $x \in [a, b]$ *can be represented by an LMI.*

Proof. Take $x = (b-a)\frac{y^2}{y^2+1} + a$ and apply Theorem 4 on $(y^2+1)p((b-a)\frac{y^2}{y^2+1} + a) \geq 0$. □