ON NEW CLASSES OF NONNEGATIVE SYMMETRIC TENSORS

BILIAN CHEN *, SIMAI HE †, ZHENING LI ‡, AND SHUZHONG ZHANG §

Abstract. In this paper we introduce three new classes of nonnegative forms (or equivalently, symmetric tensors) and their extensions. The newly identified nonnegative symmetric tensors constitute distinctive convex cones in the space of general symmetric tensors (order 6 or above). For the special case of quartic forms, they collapse into the set of convex quartic homogeneous polynomial functions. We discuss the properties and applications of the new classes of nonnegative symmetric tensors in the context of polynomial and tensor optimization.

Key words. symmetric tensors, nonnegative forms, polynomial and tensor optimization

AMS subject classifications. 15A69, 12Y05, 90C26

1. Introduction.

A classical result originally due to Banach [2] states that if $L(\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^m)$ is a continuous symmetric m-linear form, then

$$
\begin{aligned}
&\sup\{|L(\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^m)| \mid \|\boldsymbol{x}^1\| \le 1, \|\boldsymbol{x}^2\| \le 1, \ldots, \|\boldsymbol{x}^m\| \le 1\} \\
=& \sup\{|L(\boldsymbol{x}, \boldsymbol{x}, \ldots, \boldsymbol{x})| \mid \|\boldsymbol{x}\| \le 1\}. \qquad (1.1)
\end{aligned}
$$

One is referred to [27] for a recent proof. An extension of the above result can be found in [7]. Although the result holds for Banach space (where $\boldsymbol{x}$ resides) in general, for our purpose in this paper it is useful to consider $\boldsymbol{x}$ to be in an n-dimensional Euclidean space. In the latter case, $L(\boldsymbol{x}, \boldsymbol{x}, \ldots, \boldsymbol{x})$ becomes a homogenous polynomial of n variables and degree m, whose coefficients form nothing but a symmetric tensor in $\mathbb{R}^{n^m}$. By symmetry (aka super-symmetry in some papers in the literature), we mean that the entries of the tensor are invariant under the permutation of its indices. In the tensor setting, (1.1) is equivalent to the fact that the best rank-one approximation of a symmetric tensor can be obtained at a symmetric rank-one tensor, which was recently rediscovered in [7, 33].

Two natural questions arise in Banach's result: (i) Can the absolute value sign be removed? (ii) Can the constraint $\|\boldsymbol{x}\| \le 1$ be replaced by some more general constraints? The answer to these questions would be negative without further conditions. This in fact triggers an interesting question: What kind of m-form L will naturally ensure that

$$
\begin{aligned}
&\sup\{L(\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^m) \mid \boldsymbol{x}^1 \in S, \boldsymbol{x}^2 \in S, \ldots, \boldsymbol{x}^m \in S\} \\
=& \sup\{L(\boldsymbol{x}, \boldsymbol{x}, \ldots, \boldsymbol{x}) \mid \boldsymbol{x} \in S\}?
\end{aligned}
$$

By taking $m = 2$ as an example, one quickly finds out that if $L(\boldsymbol{x}^1, \boldsymbol{x}^2) = (\boldsymbol{x}^1)^{\mathrm{T}} Q \boldsymbol{x}^2$ then the above holds for all $S \subseteq \mathbb{R}^n$ if and only if Q is positive semidefinite. In case

*Department of Automation, Xiamen University, Xiamen 361005, China. Email: blchen@xmu.edu.cn. Research of this author was supported in part by National Science Foundation of China (Grant 11301436).

†School of Information Management and Engineering, Shanghai University of Finance and Economics, Shanghai 200433, China. Email: simaihe@mail.shufe.edu.cn.

‡Department of Mathematics, University of Portsmouth, Portsmouth PO1 3HF, United Kingdom. Email: zheningli@gmail.com. Research of this author was supported in part by the Faculty of Technology RDI 2014, University of Portsmouth.

§Department of Industrial and Systems Engineering, University of Minnesota, Minneapolis, MN 55455, United States. Email: zhangs@umn.edu. Research of this author was supported in part by National Science Foundation of USA (Grant CMMI-1161242).

$m = 4$, as we will later discuss in the paper, the necessary and sufficient condition for the equivalence is that the quartic form $L(\boldsymbol{x}, \boldsymbol{x}, \boldsymbol{x}, \boldsymbol{x})$ is convex. Surprisingly, further investigation into the case $m \geq 6$ reveals a new structure of the forms or symmetric tensors, which will be termed *M-quasiconvex* in this paper.

The new notion of M-quasiconvexity is not only theoretically interesting, but also useful for solving practical polynomial and tensor optimization models. For instance, polynomial optimization with spherical constraint is important and widely studied in the literature (see e.g. [10, 7]), and is closely related to the computation of eigenvalues and singular values of tensors proposed by Lim [20] and Qi [29] independently. In [7], we proposed a new convergent solution method for block optimization models, known as the *maximum block improvement* (MBI). It turns out that the MBI method works well for polynomial optimization with spherical constraint. Essentially, in [7] homogeneous polynomial optimization over spherical constraint is relaxed to the corresponding multilinear form optimization. Two questions arise in this context: (i) When will the relaxation be exact? (ii) After one finds a stationary solution to the relaxed problem, how can a solution to the original problem be obtained? The results in the current paper help to address these issues.

Our analysis relies on the notion of nonnegative polynomials or nonnegative tensors. In fact, there is a known intrinsic connection between the optimization of a polynomial function and the description of all polynomial functions which are nonnegative over a given domain. This connection was explored earlier by Sturm and Zhang [32] for the case of quadratic polynomials, and Luo et al. [21] for the case of bi-quadratic functions. We also refer to [13, 31, 11, 15, 1] and a recent book [4] for investigating the relationship between nonnegative polynomials and sum of squares of polynomials. The final interesting connection that will be established in this study is the link to the convexity of a polynomial function. In fact, convexity of an even degree form is a stronger notion than mere nonnegativity. For instance, it is well known that a quartic form $L(\boldsymbol{x}, \boldsymbol{x}, \boldsymbol{x}, \boldsymbol{x})$ is convex if and only if it is bi-quadratically nonnegative, i.e., $L(\boldsymbol{x}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}) \geq 0$, hence nonnegative. In the world of higher (than 2) degree forms, checking the nonnegativity and convexity are both difficult tasks. For instance, Ahmadi et al. [1] proved that checking the convexity of a quartic form is actually strongly NP-hard in general, highlighting a crucial difference between quadratic forms and quartic forms. Speaking of quartic polynomials, in a recent study, Jiang et al. [15] discussed six fundamentally important convex cones of quartic forms, including the cone of nonnegative quartic forms, the sum of squared quartic forms, the convex quartic forms, and the sum of fourth powered linear forms. The complexity status of these nonnegative tensor cones are discussed as well.

In this paper, we extend the study beyond quartics. We show that the world of quartic tensors are still special, in that an interesting structure known as M-quasiconvexity is still hidden under the usual convexity. Applying the new notion of M-quasiconvexity and its properties, we establish an equivalence between any constrained homogeneous polynomial optimization and its multilinear tensor relaxation model. This enables us to apply some block coordinate search methods to solve the multilinear model, and also suggests a simple way to find a stationary solution for the original model. The paper is organized as follows. In Section 2, we introduce the notations for tensors and polynomials, and define new classes of nonnegative tensors with illustrating examples. Sections 3, 4 and 5 are devoted to the theoretical study of these classes of tensors, i.e., their equivalent definitions, relationships and proper containments. Applications of the classes of tensors in polynomial and tensor optimization are discussed in Section 6 via some practical examples. Finally we conclude the paper in Section 7 by discussing a generalization of the newly introduced nonnegative tensors.

2. Preparations. Throughout this paper we uniformly use nonbold lowercase letters, boldface lowercase letters, capital letters and calligraphic letters to denote scalars, vectors, matrices and tensors, respectively; e.g. a scalar i, a vector $\boldsymbol{x}$, a matrix A and a tensor $\mathcal{F}$. We use subscripts to denote their components; e.g. x_i is the i-th entry of a vector $\boldsymbol{x}$, A_{ij} is the (i,j)-th entry of a matrix A, and $\mathcal{F}_{ijk}$ is the (i,j,k)-th entry of a third order tensor $\mathcal{F}$. As previously mentioned, an m-th order tensor $\mathcal{F} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_m}$ is *symmetric* if $n_1 = n_2 = \cdots = n_m (= n)$ and every component $\mathcal{F}_{i_1 i_2 \dots i_m}$ is invariant under all permutations of $\{i_1, i_2, \dots, i_m\}$; the set of such symmetric tensors is denoted by $\mathbb{S}^{n^m}$.

The symbol $\circ$ denotes the *vector outer product*. For example, for vectors $\boldsymbol{x} \in \mathbb{R}^{n_1}, \boldsymbol{y} \in \mathbb{R}^{n_2}, \boldsymbol{z} \in \mathbb{R}^{n_3}$, the notion $\boldsymbol{x} \circ \boldsymbol{y} \circ \boldsymbol{z}$ represents a third order tensor $\mathcal{F} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$, where $\mathcal{F}_{ijk} = x_i y_j z_k$ for all (i,j,k). The symbol $\otimes$ represents the *matrix outer product*. If tensor $\mathcal{F} = X \otimes X$ for some $X \in \mathbb{R}^{n \times n}$, then $\mathcal{F}_{ijk\ell} = X_{ij} X_{k\ell}$ for all (i,j,k,ℓ).

Given any m-th order tensor $\mathcal{F} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_m}$, we define F to be its associated multilinear form, i.e.,

$$
\begin{aligned}
F(\boldsymbol{x}^1, \boldsymbol{x}^2, \dots, \boldsymbol{x}^m) &:= \sum_{1 \le i_1 \le n_1, 1 \le i_2 \le n_2, \dots, 1 \le i_m \le n_m} \mathcal{F}_{i_1 i_2 \dots i_m} x^1_{i_1} x^2_{i_2} \dots x^m_{i_m} \\
&= \langle \mathcal{F}, \boldsymbol{x}^1 \circ \boldsymbol{x}^2 \circ \cdots \circ \boldsymbol{x}^m \rangle,
\end{aligned}
$$

where $\boldsymbol{x}^k \in \mathbb{R}^{n_k}$ for $k = 1, 2, \dots, m$ and $\langle \cdot, \cdot \rangle$ denotes the Frobenius inner product. Closely related to the multilinear form is a general m-th degree homogeneous polynomial $f(\boldsymbol{x})$ of variable $\boldsymbol{x} \in \mathbb{R}^n$, with its associated symmetric tensor $\mathcal{F} \in \mathbb{S}^{n^m}$. In fact, symmetric tensors are bijectively related to homogeneous polynomials; see [18]. Denote F to be the multilinear form associated with the symmetric tensor $\mathcal{F}$, we then have

$$
f(\boldsymbol{x}) = F(\underbrace{\boldsymbol{x}, \boldsymbol{x}, \dots, \boldsymbol{x}}_{m}) = \sum_{1 \le i_1, i_2, \dots, i_m \le n} \mathcal{F}_{i_1 i_2 \dots i_m} x_{i_1} x_{i_2} \dots x_{i_m} = \langle \mathcal{F}, \underbrace{\boldsymbol{x} \circ \boldsymbol{x} \circ \cdots \circ \boldsymbol{x}}_{m} \rangle.
$$

In this paper we uniformly use the 2-norm for vectors, matrices and tensors in general, which is the usual Euclidean norm or the Frobenius norm. For example, the norm of an m-th order tensor $\mathcal{F} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_m}$ is defined as

$$
\|\mathcal{F}\| := \sqrt{\langle \mathcal{F}, \mathcal{F} \rangle} = \sqrt{\sum_{1 \le i_1 \le n_1, 1 \le i_2 \le n_2, \dots, 1 \le i_m \le n_m} {\mathcal{F}_{i_1 i_2 \dots i_m}}^2}.
$$

2.1. Nonnegativity and co-quadratic nonnegativity. All the discussion in this paper is focused on nonnegative tensors, which is the following.

DEFINITION 2.1. *Suppose F is a multilinear form associated with a $2m$-th order symmetric tensor $\mathcal{F} \in \mathbb{S}^{n^{2m}}$. The tensor $\mathcal{F}$ (or the form F) is called nonnegative if*

$$
F(\underbrace{\boldsymbol{x}, \boldsymbol{x}, \dots, \boldsymbol{x}}_{2m}) \ge 0 \quad \forall \boldsymbol{x} \in \mathbb{R}^n.
$$

We emphasize that a symmetric tensor is exactly a symmetric multilinear form; the former is denoted by a calligraphic letter and the latter is denoted by its corresponding capital letter, for the easy presentation whenever appropriate. Therefore, any terminology related to a form is also related to its tensor representation, e.g. a nonnegative form is simply a nonnegative tensor.

As a notation, we denote the set of nonnegative symmetric tensors

$$\mathbb{S}_+^{n^{2m}} := \{\mathcal{F} \in \mathbb{S}^{n^{2m}} \mid F(\underbrace{\boldsymbol{x}, \boldsymbol{x}, \ldots, \boldsymbol{x}}_{2m}) \geq 0 \; \forall \, \boldsymbol{x} \in \mathbb{R}^n\}.$$

Obviously, the degree of a nonnegative form has to be even. It is well known that checking the nonnegativity of a quadratic form can be done in polynomial-time, which amounts to checking the positive semidefiniteness of its associated symmetric matrix. However, it is NP-hard to check the nonnegativity of a form whose degree is larger than 2; see e.g. [15]. To extend the definition of nonnegativity, let us introduce the following.

Definition 2.2. *Suppose F is a multilinear form associated with a $2m$-th order symmetric tensor $\mathcal{F} \in \mathbb{S}^{n^{2m}}$. The tensor $\mathcal{F}$ is called co-quadratic nonnegative if*

$$F(\boldsymbol{x}_1, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m, \boldsymbol{x}_m) \geq 0 \quad \forall \, \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m \in \mathbb{R}^n.$$

Let us denote the set of co-quadratic nonnegative tensors to be

$$\mathbb{S}_{2+}^{n^{2m}} := \{\mathcal{F} \in \mathbb{S}^{n^{2m}} \,|\, F(\boldsymbol{x}_1, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m, \boldsymbol{x}_m) \geq 0 \; \forall \, \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m \in \mathbb{R}^n\}.$$

Clearly, a co-quadratic nonnegative tensor is always nonnegative; i.e., $\mathbb{S}_{2+}^{n^{2m}} \subseteq \mathbb{S}_+^{n^{2m}}$. In particular, when $m = 1$, the co-quadratic nonnegativity is equivalent to the nonnegativity of a quadratic form, and when $m = 2$, the co-quadratic nonnegativity is equivalent to the convexity of a quartic form (see below).

Proposition 2.3. *A co-quadratic nonnegative form is always convex. In particular, a quartic form is co-quadratic nonnegative if and only if it is convex.*

Proof. It is straightforward to compute the Hessian matrix of a form $F(\underbrace{\boldsymbol{x}, \boldsymbol{x}, \ldots, \boldsymbol{x}}_{2m})$, which is $2m(2m-1)F(\underbrace{\boldsymbol{x}, \boldsymbol{x}, \ldots, \boldsymbol{x}}_{2m-2}, \cdot, \cdot)$. Therefore, the form is convex if and only if $F(\underbrace{\boldsymbol{x}, \boldsymbol{x}, \ldots, \boldsymbol{x}}_{2m-2}, \cdot, \cdot)$ is a positive semidefinite matrix for all $\boldsymbol{x} \in \mathbb{R}^n$, which is equivalent to

$$F(\underbrace{\boldsymbol{x}, \boldsymbol{x}, \ldots, \boldsymbol{x}}_{2m-2}, \boldsymbol{y}, \boldsymbol{y}) \geq 0 \quad \forall \, \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n. \tag{2.1}$$

By Definition 2.2, it is obvious that co-quadratic nonnegativity implies (2.1), which is convexity. In particular, for the case of quartic form ($m = 2$), co-quadratic nonnegative is equivalent to convexity. ☐

Therefore, when $m = 2$, the usual nonnegative quartic form $F(\boldsymbol{x}, \boldsymbol{x}, \boldsymbol{x}, \boldsymbol{x})$ is not necessarily convex, while convexity implies nonnegativity. As a result, co-quadratic nonnegativity is indeed stronger than nonnegativity. When $m \geq 3$, co-quadratic nonnegativity is even stronger than convexity. Unfortunately, checking the co-quadratic nonnegativity for $m = 2$ is also NP-hard [1]. Jiang et al. [15] presented a study on different classes of nonnegative quartic forms, the case when $m = 2$. Below we present two examples of co-quadratic nonnegative tensors for general m.

EXAMPLE 2.4. *If $\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_s \in \mathbb{R}^n$, then the tensor $\mathcal{F} = \sum_{i=1}^{s} \underbrace{\boldsymbol{a}_i \circ \boldsymbol{a}_i \circ \cdots \circ \boldsymbol{a}_i}_{2m}$ is co-quadratic nonnegative.*

The reason for this is that for all $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_m \in \mathbb{R}^n$,

$$F(\boldsymbol{x}_1, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_2, \dots, \boldsymbol{x}_m, \boldsymbol{x}_m) = \sum_{i=1}^{s} (\boldsymbol{a}_i^{\mathrm{T}} \boldsymbol{x}_1)^2 (\boldsymbol{a}_i^{\mathrm{T}} \boldsymbol{x}_2)^2 \dots (\boldsymbol{a}_i^{\mathrm{T}} \boldsymbol{x}_m)^2 \geq 0.$$

EXAMPLE 2.5. *The symmetric tensor associated with homogeneous polynomial $(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x})^m$ is co-quadratic nonnegative. Explicitly, its multilinear form is*

$$F(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{2m}) = \frac{1}{|\Pi|} \sum_{(i_1 i_2 \dots i_{2m}) \in \Pi} (\boldsymbol{x}_{i_1}^{\mathrm{T}} \boldsymbol{x}_{i_2})(\boldsymbol{x}_{i_3}^{\mathrm{T}} \boldsymbol{x}_{i_4}) \dots (\boldsymbol{x}_{i_{2m-1}}^{\mathrm{T}} \boldsymbol{x}_{i_{2m}}), \tag{2.2}$$

where Π is the set of all permutations of $\{1, 2, \dots, 2m\}$.

In fact for $m = 1$, this symmetric tensor is nothing but an identity matrix, which is clearly co-quadratic nonnegative. For $m = 2$, it is easy to verify that

$$F(\boldsymbol{x}_1, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_2) = \frac{1}{3}(\boldsymbol{x}_1^{\mathrm{T}} \boldsymbol{x}_1)(\boldsymbol{x}_2^{\mathrm{T}} \boldsymbol{x}_2) + \frac{2}{3}(\boldsymbol{x}_1^{\mathrm{T}} \boldsymbol{x}_2)^2 \geq 0 \quad \forall\, \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n.$$

For general $m \geq 3$, it is not an easy work to directly check the nonnegativity of $F(\boldsymbol{x}_1, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_2, \dots, \boldsymbol{x}_m, \boldsymbol{x}_m)$ using (2.2). However, this property can be easily proven thanks to the so-called Hilbert's identity (see e.g. [3, 14]), which states that there exist vectors $\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_s \in \mathbb{R}^n$ such that

$$(\boldsymbol{x}^{\mathrm{T}} \boldsymbol{x})^m = \sum_{i=1}^{s} (\boldsymbol{b}_i^{\mathrm{T}} \boldsymbol{x})^{2m} \quad \forall\, \boldsymbol{x} \in \mathbb{R}^n.$$

Thus, the symmetric tensor $\mathcal{F}$ can be expressed by $\mathcal{F} = \sum_{i=1}^{s} \underbrace{\boldsymbol{b}_i \circ \boldsymbol{b}_i \circ \cdots \circ \boldsymbol{b}_i}_{2m}$, which is co-quadratic nonnegative as Example 2.4 stipulates.

2.2. M-quasiconvexity and co-quadratic M-quasiconvexity.

For a given set of entry vectors, a tensor form naturally lends to some degree of freedom in distributing the *multiplicities* among these entry vectors. To capture the characteristic of how the distribution of the multiplicities affects the function values, let us introduce the following notion of M-quasiconvexity. Here the terminology is to be understood as "quasiconvexity in the multiplicity of its entries"; one should distinguish this notion from the so-called M-convex functions in the theory of discrete convex functions.

DEFINITION 2.6. *Suppose F is a nonnegative form associated with a $2m$-th order symmetric tensor $\mathcal{F} \in \mathbb{S}^{n^{2m}}$. The tensor $\mathcal{F}$ is called M-quasiconvex if*

$$\begin{aligned} &F(\underbrace{\boldsymbol{x}_1, \dots, \boldsymbol{x}_1}_{\lambda_1}, \underbrace{\boldsymbol{x}_2, \dots, \boldsymbol{x}_2}_{\lambda_2}, \dots, \underbrace{\boldsymbol{x}_s, \dots, \boldsymbol{x}_s}_{\lambda_s}) \\ &\leq \max_{1 \leq i \leq s} \{F(\underbrace{\boldsymbol{x}_i, \boldsymbol{x}_i, \dots, \boldsymbol{x}_i}_{2m})\} \qquad \forall\, \boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_s \in \mathbb{R}^n \end{aligned} \tag{2.3}$$

for any positive integers s and λ_i $(i = 1, 2, \ldots, s)$ with $\sum_{i=1}^{s} \lambda_i = 2m$.

It is easy to observe that an M-quasiconvex $\mathcal{F}$ is actually equivalent to:

$$F(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{2m}) \le \max_{1 \le i \le 2m} \{F(\underbrace{\boldsymbol{x}_i, \boldsymbol{x}_i, \ldots, \boldsymbol{x}_i}_{2m})\} \quad \forall\, \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{2m} \in \mathbb{R}^n, \tag{2.4}$$

i.e., (2.3) holds for a special case when $s = 2m$ and $\lambda_i = 1$ for $i = 1, 2, \ldots, 2m$. Thus to verify the M-quasiconvexity, it is convenient to check only (2.4) rather than (2.3) for all combinations of λ_i's. For ease of referencing, let us denote

$$\mathbb{M}^{n^{2m}} := \left\{ \mathcal{F} \in \mathbb{S}_+^{n^{2m}} \,\middle|\, F(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{2m}) \le \max_{1 \le i \le 2m} \{F(\underbrace{\boldsymbol{x}_i, \boldsymbol{x}_i, \ldots, \boldsymbol{x}_i}_{2m})\} \; \forall\, \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{2m} \in \mathbb{R}^n \right\}. \tag{2.5}$$

If (2.3) holds for a special case when $s = m$ and $\lambda_i = 2$ for $i = 1, 2, \ldots, m$, then $\mathcal{F}$ is called co-quadratic M-quasiconvex to be introduced below.

DEFINITION 2.7. *Suppose F is a nonnegative form associated with a $2m$-th order symmetric tensor $\mathcal{F} \in \mathbb{R}^{n^{2m}}$. The tensor $\mathcal{F}$ is called co-quadratic M-quasiconvex if*

$$\begin{aligned} & F(\boldsymbol{x}_1, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m, \boldsymbol{x}_m) \\ \le & \max_{1 \le i \le m} \{F(\underbrace{\boldsymbol{x}_i, \boldsymbol{x}_i, \ldots, \boldsymbol{x}_i}_{2m})\} \qquad \forall\, \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m \in \mathbb{R}^n. \end{aligned} \tag{2.6}$$

We denote the set of co-quadratic M-quasiconvex tensors to be

$$\mathbb{M}_2^{n^{2m}} := \left\{ \mathcal{F} \in \mathbb{S}_+^{n^{2m}} \,\middle|\, F(\boldsymbol{x}_1, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m, \boldsymbol{x}_m) \le \max_{1 \le i \le m} \{F(\underbrace{\boldsymbol{x}_i, \boldsymbol{x}_i, \ldots, \boldsymbol{x}_i}_{2m})\} \; \forall\, \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m \in \mathbb{R}^n \right\}.$$

Trivially we have $\mathbb{M}^{n^{2m}} \subseteq \mathbb{M}_2^{n^{2m}}$.

We remark that in Definition 2.6, (2.3) already implies that $\mathcal{F}$ is nonnegative. This is because if we let $\boldsymbol{x}_1 = -\boldsymbol{x}$ and $\boldsymbol{x}_i = \boldsymbol{x}$ for $i = 2, 3, \ldots, 2m$ in (2.4), then

$$F(-\boldsymbol{x}, \underbrace{\boldsymbol{x}, \boldsymbol{x}, \ldots, \boldsymbol{x}}_{2m-1}) \le F(\underbrace{\boldsymbol{x}, \boldsymbol{x}, \ldots, \boldsymbol{x}}_{2m}),$$

implying the nonnegativity of $\mathcal{F}$. However, in Definition 2.7, (2.6) does not imply the nonnegativity of $\mathcal{F}$, as shown by the following example.

EXAMPLE 2.8. *Let $\mathcal{F} = -1 \in \mathbb{R}^{1^4}$ be a 4th order symmetric tensor and its associated multilinear form $F(x, y, z, w) = -xyzw$ for $x, y, z, w \in \mathbb{R}$. We have*

$$F(x, x, y, y) = -x^2 y^2 \le \max\{-x^4, -y^4\} = \max\{F(x, x, x, x), F(y, y, y, y)\},$$

implying (2.6). However $\mathcal{F}$ is clearly not nonnegative.

We also remark that $\mathcal{F}$ being nonnegative is important for our discussion. Without nonnegativity, the cone consisting all co-quadratic M-quasiconvex tensors in $\mathbb{S}^{n^{2m}}$

is not even convex; see Example 2.9. However, with nonnegativity this cone is convex as Corollary 3.4 shall stipulate. The nonnegativity of $\mathcal{F}$ also ensures that the right hand sides of (2.3) and (2.6) will always be nonnegative. For consistency of expression we still keep the requirement of nonnegativity in Definition 2.6 although it is actually redundant in that case. Throughout this paper, the cone of nonnegative symmetric tensors (or the cone of nonnegative forms) is the ground set for our discussion in this paper.

EXAMPLE 2.9. *Let $\mathcal{F}_1, \mathcal{F}_2 \in \mathbb{R}^{2^4}$ be 4th order symmetric tensors, with their associated multilinear forms being $F_1(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{w}) = -2x_1y_1z_1w_1$ and $F_2(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{w}) = -2x_2y_2z_2w_2$. According to Example 2.8, both F_1 and F_2 satisfy (2.6). However $F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{w}) = -x_1y_1z_1w_1 - x_2y_2z_2w_2$ with its associated tensor $\mathcal{F} = \frac{1}{2}(\mathcal{F}_1 + \mathcal{F}_2)$ does not satisfy (2.6), since*

$$
\begin{aligned}
F(\boldsymbol{x},\boldsymbol{x},\boldsymbol{y},\boldsymbol{y}) &= -x_1^2y_1^2 - x_2^2y_2^2 \\
&> \max\{-x_1^4 - x_2^4, -y_1^4 - y_2^4\} \\
&= \max\{F(\boldsymbol{x},\boldsymbol{x},\boldsymbol{x},\boldsymbol{x}), F(\boldsymbol{y},\boldsymbol{y},\boldsymbol{y},\boldsymbol{y})\}
\end{aligned}
$$

when $\boldsymbol{x} = (1,2)^{\mathrm{T}}$ and $\boldsymbol{y} = (2,1)^{\mathrm{T}}$.

To simplify the notation, whenever appropriate we now use superscripts to simplify the form $F(\underbrace{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_1}_{\lambda_1},\underbrace{\boldsymbol{x}_2,\ldots,\boldsymbol{x}_2}_{\lambda_2},\ldots,\underbrace{\boldsymbol{x}_s,\ldots,\boldsymbol{x}_s}_{\lambda_s})$, i.e., $F(\boldsymbol{x}_1^{\lambda_1}\boldsymbol{x}_2^{\lambda_2}\ldots\boldsymbol{x}_s^{\lambda_s})$. For example, $F(\boldsymbol{x}^2\boldsymbol{y}^2)$ denotes $F(\boldsymbol{x},\boldsymbol{x},\boldsymbol{y},\boldsymbol{y})$, which is equal to $F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{x},\boldsymbol{y})$ (or any possible permutation of the position of the entry vectors due to the symmetry). With this simplified notation, (2.3) is then

$$
F(\boldsymbol{x}_1^{\lambda_1}\boldsymbol{x}_2^{\lambda_2}\ldots\boldsymbol{x}_s^{\lambda_s}) \le \max_{1\le i\le s}\{F(\boldsymbol{x}_i^{2m})\}.
$$

In the same spirit, the homogeneous polynomial $f(\boldsymbol{x})$ associated with the multilinear form F or the symmetric tensor $\mathcal{F}$ can be simply written as $F(\boldsymbol{x}^{2m})$.

We conclude this section by pointing out that Definitions 2.1, 2.2, 2.6 and 2.7 are actually identical in the matrix setting, i.e., when $m = 1$ we have $\mathbb{S}_+^{n^2} = \mathbb{S}_{2+}^{n^2} = \mathbb{M}^{n^2} = \mathbb{M}_2^{n^2}$.

PROPOSITION 2.10. *If $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^n$ and symmetric matrix $Q \in \mathbb{R}^{n\times n}$ is positive semidefinite, then the bilinear form $F(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{x}^{\mathrm{T}}Q\boldsymbol{y}$ is M-quasiconvex. Conversely, if $\boldsymbol{x}^{\mathrm{T}}Q\boldsymbol{y}$ is M-quasiconvex, then Q is positive semidefinite.*

Proof. For all $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^n$, Q being positive semidefinite implies $(\boldsymbol{x}-\boldsymbol{y})^{\mathrm{T}}Q(\boldsymbol{x}-\boldsymbol{y}) \ge 0$, which is $2\boldsymbol{x}^{\mathrm{T}}Q\boldsymbol{y} \le \boldsymbol{x}^{\mathrm{T}}Q\boldsymbol{x} + \boldsymbol{y}^{\mathrm{T}}Q\boldsymbol{y}$. This further leads to

$$
\boldsymbol{x}^{\mathrm{T}}Q\boldsymbol{y} \le \max\{\boldsymbol{x}^{\mathrm{T}}Q\boldsymbol{x}, \boldsymbol{y}^{\mathrm{T}}Q\boldsymbol{y}\},
$$

implying F is M-quasiconvex. Conversely, M-quasiconvexity of F implies that

$$
F(-\boldsymbol{x},\boldsymbol{x}) \le \max\{F(-\boldsymbol{x},-\boldsymbol{x}), F(\boldsymbol{x},\boldsymbol{x})\} = F(\boldsymbol{x},\boldsymbol{x}).
$$

Therefore $\boldsymbol{x}^{\mathrm{T}}Q\boldsymbol{x} = F(\boldsymbol{x},\boldsymbol{x}) \ge 0$, implying Q is positive semidefinite. ☐

3. Equivalent definitions. In this section we present a curious fact that the newly introduced M-quasiconvexity is actually equivalent to some of the seemingly more restrictive definitions.

LEMMA 3.1. *Suppose F is a nonnegative form associated with a $2m$-th order symmetric tensor $\mathcal{F} \in \mathbb{S}^{n^{2m}}$. The following three statements are equivalent:*

$$F(\boldsymbol{x}_1\boldsymbol{x}_2\dots\boldsymbol{x}_{2m}) \le \max_{1\le i\le 2m}\{F(\boldsymbol{x}_i^{2m})\} \quad \forall\, \boldsymbol{x}_1,\boldsymbol{x}_2,\dots,\boldsymbol{x}_{2m}\in\mathbb{R}^n, \tag{3.1}$$

$$F(\boldsymbol{x}_1\boldsymbol{x}_2\dots\boldsymbol{x}_{2m}) \le \frac{1}{2m}\sum_{i=1}^{2m} F(\boldsymbol{x}_i^{2m}) \quad \forall\, \boldsymbol{x}_1,\boldsymbol{x}_2,\dots,\boldsymbol{x}_{2m}\in\mathbb{R}^n, \tag{3.2}$$

$$F(\boldsymbol{x}_1\boldsymbol{x}_2\dots\boldsymbol{x}_{2m}) \le \left(\prod_{i=1}^{2m} F(\boldsymbol{x}_i^{2m})\right)^{\frac{1}{2m}} \quad \forall\, \boldsymbol{x}_1,\boldsymbol{x}_2,\dots,\boldsymbol{x}_{2m}\in\mathbb{R}^n. \tag{3.3}$$

Proof. Since $F(\boldsymbol{x}_i^{2m})\ge 0$ for all $\boldsymbol{x}^i\in\mathbb{R}^n$, it follows immediately by the mean inequalities that (3.3)⇒(3.2)⇒(3.1). It remains to prove that (3.1)⇒(3.3).

For any $a_1,a_2,\dots,a_{2m}\in\mathbb{R}$ and $\boldsymbol{x}_1,\boldsymbol{x}_2,\dots,\boldsymbol{x}_{2m}\in\mathbb{R}^n$, by the multilinearity of F, it follows from (3.1) that

$$\begin{aligned} a_1a_2\dots a_{2m}F(\boldsymbol{x}_1\boldsymbol{x}_2\dots\boldsymbol{x}_{2m}) &= F(a_1\boldsymbol{x}_1,a_2\boldsymbol{x}_2,\dots,a_{2m}\boldsymbol{x}_{2m})\\ &\le \max_{1\le i\le 2m}\{F\left((a_i\boldsymbol{x}_i)^{2m}\right)\}\\ &= \max_{1\le i\le 2m}\{a_i^{2m}F(\boldsymbol{x}_i^{2m})\}. \end{aligned} \tag{3.4}$$

If $F(\boldsymbol{x}_i^{2m})>0$ for all $i=1,2,\dots,2m$, we choose $a_i>0$ such that $a_i^{2m}F(\boldsymbol{x}_i^{2m})=1$ for all $i=1,2,\dots,2m$. Then (3.4) leads to

$$\left(\prod_{i=1}^{2m}\left(F(\boldsymbol{x}_i^{2m})\right)^{-\frac{1}{2m}}\right)F(\boldsymbol{x}_1\boldsymbol{x}_2\dots\boldsymbol{x}_{2m})\le 1,$$

which implies (3.3).

Otherwise, there exists some $1\le i\le 2m$ such that $F(\boldsymbol{x}_i^{2m})=0$. Without loss of generality we assume $F(\boldsymbol{x}_1^{2m})=0$. Letting $a_2=a_3=\dots=a_{2m}=1$ in (3.4) leads to for all $a_1\in\mathbb{R}$

$$a_1F(\boldsymbol{x}_1\boldsymbol{x}_2\dots\boldsymbol{x}_{2m})\le\max\left\{\max_{2\le i\le 2m}\{F(\boldsymbol{x}_i^{2m})\},a_1^{2m}F(\boldsymbol{x}_1^{2m})\right\}=\max_{2\le i\le 2m}\{F(\boldsymbol{x}_i^{2m})\}.$$

The above clearly implies that $F(\boldsymbol{x}_1\boldsymbol{x}_2\dots\boldsymbol{x}_{2m})=0$. Therefore (3.3) is satisfied as both sides are now zeros. □

In fact, the above equivalence indicates that any generalized mean with exponent $p\in(0,+\infty]$ of $F(\boldsymbol{x}_1^{2m}),F(\boldsymbol{x}_2^{2m}),\dots,F(\boldsymbol{x}_{2m}^{2m})$ on the right hand side can serve as the definition for M-quasiconvexity. Specifically, a symmetric tensor $\mathcal{F}$ is M-quasiconvex if and only if

$$F(\boldsymbol{x}_1\boldsymbol{x}_2\dots\boldsymbol{x}_{2m})\le\left(\frac{1}{2m}\sum_{i=1}^{2m}\left(F(\boldsymbol{x}_i^{2m})\right)^p\right)^{\frac{1}{p}}\quad\forall\,\boldsymbol{x}_1,\boldsymbol{x}_2,\dots,\boldsymbol{x}_{2m}\in\mathbb{R}^n,$$

for any fixed $p\in(0,+\infty]$. By Lemma 3.1, we may equivalently write

$$\mathbb{M}^{n^{2m}}=\left\{\mathcal{F}\in\mathbb{S}_+^{n^{2m}}\,\middle|\,F(\boldsymbol{x}_1\boldsymbol{x}_2\dots\boldsymbol{x}_{2m})\le\frac{1}{2m}\sum_{i=1}^{2m}F(\boldsymbol{x}_i^{2m})\ \forall\,\boldsymbol{x}_1,\boldsymbol{x}_2,\dots,\boldsymbol{x}_{2m}\in\mathbb{R}^n\right\}.$$

This formulation essentially implies the convexity of the set $\mathbb{M}^{n^{2m}}$, which does not follow straightforwardly from its original definition (2.5). Therefore we have

COROLLARY 3.2. *The set of all M-quasiconvex tensors (i.e., $\mathbb{M}^{n^{2m}}$) is a closed convex cone.*

Similarly we have the following equivalent definitions for co-quadratic M-quasiconvex tensors, whose proof is similar and is omitted here.

LEMMA 3.3. *Suppose F is a nonnegative form associated with a $2m$-th order symmetric tensor $\mathcal{F} \in \mathbb{S}^{n^{2m}}$. The following three statements are equivalent:*

$$\begin{aligned} F(\boldsymbol{x}_1^2\boldsymbol{x}_2^2\dots\boldsymbol{x}_m^2) &\le \max_{1\le i\le m}\{F(\boldsymbol{x}_i^{2m})\} \quad \forall \boldsymbol{x}_1,\boldsymbol{x}_2,\dots,\boldsymbol{x}_m \in \mathbb{R}^n,\\ F(\boldsymbol{x}_1^2\boldsymbol{x}_2^2\dots\boldsymbol{x}_m^2) &\le \frac{1}{m}\sum_{i=1}^m F(\boldsymbol{x}_i^{2m}) \quad \forall \boldsymbol{x}_1,\boldsymbol{x}_2,\dots,\boldsymbol{x}_m \in \mathbb{R}^n, \qquad (3.5)\\ F(\boldsymbol{x}_1^2\boldsymbol{x}_2^2\dots\boldsymbol{x}_m^2) &\le \left(\prod_{i=1}^m F(\boldsymbol{x}_i^{2m})\right)^{\frac{1}{m}} \quad \forall \boldsymbol{x}_1,\boldsymbol{x}_2,\dots,\boldsymbol{x}_m \in \mathbb{R}^n. \end{aligned}$$

We also have

$$\mathbb{M}_2^{n^{2m}} = \left\{\mathcal{F} \in \mathbb{S}_+^{n^{2m}} \,\middle|\, F(\boldsymbol{x}_1^2\boldsymbol{x}_2^2\dots\boldsymbol{x}_m^2) \le \frac{1}{m}\sum_{i=1}^m F(\boldsymbol{x}_i^{2m}) \ \forall \boldsymbol{x}_1,\boldsymbol{x}_2,\dots,\boldsymbol{x}_m \in \mathbb{R}^n\right\}$$

and consequently:

COROLLARY 3.4. *The set of all co-quadratic M-quasiconvex tensors (i.e., $\mathbb{M}_2^{n^{2m}}$) is a closed convex cone.*

4. The relationships. As mentioned at the end of Section 2, when the degree of the form is 2, then M-quasiconvex tensors, co-quadratic M-quasiconvex tensors and the co-quadratic nonnegative tensors are all the same, i.e., $\mathbb{S}_{2+}^{n^2} = \mathbb{M}^{n^2} = \mathbb{M}_2^{n^2}$. In this section we establish the relationship for general degree $2m$. Before presenting the main results, let us first study an important property for the co-quadratic nonnegative tensors.

LEMMA 4.1. *If F is a co-quadratic nonnegative form associated with a $2m$-th order symmetric tensor $\mathcal{F} \in \mathbb{S}^{n^{2m}}$, then*

$$F(\boldsymbol{y}^{2m}) + F(\boldsymbol{z}^{2m}) \ge F(\boldsymbol{y}^{2m-2}\boldsymbol{z}^2) + F(\boldsymbol{y}^2\boldsymbol{z}^{2m-2}) \quad \forall \boldsymbol{y},\boldsymbol{z} \in \mathbb{R}^n. \qquad (4.1)$$

Proof. The proof is based on induction on m. Obvious (4.1) holds when $m=1$. For the case $m=2$, due to the co-quadratic nonnegativity of $\mathcal{F}$, we have

$$F(\boldsymbol{y}+\boldsymbol{z},\boldsymbol{y}+\boldsymbol{z},\boldsymbol{y}-\boldsymbol{z},\boldsymbol{y}-\boldsymbol{z}) \ge 0 \quad \forall \boldsymbol{y},\boldsymbol{z} \in \mathbb{R}^n;$$

that is,

$$F(\boldsymbol{y}^4) + F(\boldsymbol{z}^4) \ge 2F(\boldsymbol{y}^2\boldsymbol{z}^2) \quad \forall \boldsymbol{y},\boldsymbol{z} \in \mathbb{R}^n,$$

which implies that (4.1) holds when $m=2$.

Suppose that (4.1) holds for tensors of order no more than $2m$. For the sake of induction, we next consider tensors of order $2(m+1)$, and we wish to show the following inequality

$$F(\boldsymbol{y}^{2m+2}) + F(\boldsymbol{z}^{2m+2}) \ge F(\boldsymbol{y}^{2m}\boldsymbol{z}^2) + F(\boldsymbol{y}^2\boldsymbol{z}^{2m}) \quad \forall \boldsymbol{y},\boldsymbol{z} \in \mathbb{R}^n. \qquad (4.2)$$

For any fixed $\boldsymbol{x} \in \mathbb{R}^n$ and $k = 1, 2, \ldots, m$, we define the following multilinear form

$$G_{\boldsymbol{x}^{2k}}(\boldsymbol{y}_1 \boldsymbol{y}_2 \ldots \boldsymbol{y}_{2m+2-2k}) := F(\boldsymbol{x}^{2k} \boldsymbol{y}_1 \boldsymbol{y}_2 \ldots \boldsymbol{y}_{2m+2-2k}).$$

By the co-quadratic nonnegativity of F, $G_{\boldsymbol{x}^{2k}}$ is a co-quadratic nonnegative form associated with a $(2m+2-2k)$-th order symmetric tensor. Therefore by the induction assumption, we have for $k = 1, 2, \ldots, m$,

$$G_{\boldsymbol{x}^{2k}}(\boldsymbol{y}^{2m+2-2k}) + G_{\boldsymbol{x}^{2k}}(\boldsymbol{z}^{2m+2-2k}) \geq G_{\boldsymbol{x}^{2k}}(\boldsymbol{y}^{2m-2k} \boldsymbol{z}^2) + G_{\boldsymbol{x}^{2k}}(\boldsymbol{y}^2 \boldsymbol{z}^{2m-2k}),$$

i.e.,

$$F(\boldsymbol{x}^{2k} \boldsymbol{y}^{2m+2-2k}) + F(\boldsymbol{x}^{2k} \boldsymbol{z}^{2m+2-2k}) \geq F(\boldsymbol{x}^{2k} \boldsymbol{y}^{2m-2k} \boldsymbol{z}^2) + F(\boldsymbol{x}^{2k} \boldsymbol{y}^2 \boldsymbol{z}^{2m-2k}).$$

Summing over $k = 1, 2, \ldots, m$ in the above inequality, and letting $\boldsymbol{x} = \boldsymbol{y}$ and $\boldsymbol{x} = \boldsymbol{z}$ respectively, we have

$$\sum_{k=1}^{m} \left(F(\boldsymbol{y}^{2m+2}) + F(\boldsymbol{y}^{2k} \boldsymbol{z}^{2m+2-2k}) \right) \geq \sum_{k=1}^{m} \left(F(\boldsymbol{y}^{2m} \boldsymbol{z}^2) + F(\boldsymbol{y}^{2k+2} \boldsymbol{z}^{2m-2k}) \right),$$
$$\sum_{k=1}^{m} \left(F(\boldsymbol{y}^{2m+2-2k} \boldsymbol{z}^{2k}) + F(\boldsymbol{z}^{2m+2}) \right) \geq \sum_{k=1}^{m} \left(F(\boldsymbol{y}^{2m-2k} \boldsymbol{z}^{2k+2}) + F(\boldsymbol{y}^2 \boldsymbol{z}^{2m}) \right).$$

Summing up these two inequalities and canceling out the same terms on both sides leads to

$$(m-1) \left(F(\boldsymbol{y}^{2m+2}) + F(\boldsymbol{z}^{2m+2}) \right) \geq (m-1) \left(F(\boldsymbol{y}^{2m} \boldsymbol{z}^2) + F(\boldsymbol{y}^2 \boldsymbol{z}^{2m}) \right),$$

which establishes the inductive step (4.2). □

Our first main result says that co-quadratic nonnegativity implies co-quadratic M-quasiconvexity.

THEOREM 4.2. *If F is a co-quadratic nonnegative form associated with a $2m$-th order symmetric tensor $\mathcal{F} \in \mathbb{S}^{n^{2m}}$, then it is also co-quadratic M-quasiconvex, i.e.,*

$$F(\boldsymbol{y}_1^2 \boldsymbol{y}_2^2 \ldots \boldsymbol{y}_m^2) \geq 0 \ \forall \boldsymbol{y}_i \in \mathbb{R}^n \Longrightarrow F(\boldsymbol{x}_1^2 \boldsymbol{x}_2^2 \ldots \boldsymbol{x}_m^2) \leq \max_{1 \leq i \leq m} \{F(\boldsymbol{x}_i^{2m})\} \ \forall \boldsymbol{x}_i \in \mathbb{R}^n.$$

In other words, we have $\mathbb{S}_{2+}^{n^{2m}} \subseteq \mathbb{M}_2^{n^{2m}}$.

Proof. We prove the co-quadratic M-quasiconvexity using (3.5), i.e.,

$$\sum_{i=1}^{m} F(\boldsymbol{x}_i^{2m}) \geq m\, F(\boldsymbol{x}_1^2 \boldsymbol{x}_2^2 \ldots \boldsymbol{x}_m^2) \quad \forall \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m \in \mathbb{R}^n$$

by induction on m. It is trivially true when $m = 1$. If it holds for the case m, then for the case $m+1$, we need to show that

$$\sum_{i=1}^{m+1} F(\boldsymbol{x}_i^{2m+2}) \geq (m+1)\, F(\boldsymbol{x}_1^2 \boldsymbol{x}_2^2 \ldots \boldsymbol{x}_m^2 \boldsymbol{x}_{m+1}^2) \quad \forall \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m, \boldsymbol{x}_{m+1} \in \mathbb{R}^n. \tag{4.3}$$

According to Lemma 4.1 we have

$$F(\boldsymbol{x}_i^{2m+2}) + F(\boldsymbol{x}_j^{2m+2}) \geq F(\boldsymbol{x}_i^2 \boldsymbol{x}_j^{2m}) + F(\boldsymbol{x}_i^{2m} \boldsymbol{x}_j^2) \quad \forall 1 \leq i, j \leq m+1.$$

Summing over all $i < j$ further leads to

$$\sum_{1\le i<j\le m+1} \left(F(\boldsymbol{x}_i^{2m+2}) + F(\boldsymbol{x}_j^{2m+2})\right) \ge \sum_{1\le i<j\le m+1} \left(F(\boldsymbol{x}_i^2\boldsymbol{x}_j^{2m}) + F(\boldsymbol{x}_i^{2m}\boldsymbol{x}_j^2)\right),$$

which implies that

$$\begin{aligned} m\sum_{i=1}^{m+1} F(\boldsymbol{x}_i^{2m+2}) &\ge \sum_{i=1}^{m+1}\sum_{j\ne i} F(\boldsymbol{x}_i^2\boldsymbol{x}_j^{2m}) \\ &\ge \sum_{i=1}^{m+1} mF\left(\boldsymbol{x}_i^2\prod_{j\ne i}\boldsymbol{x}_j^2\right) \\ &= m(m+1)F\left(\prod_{j=1}^{m+1}\boldsymbol{x}_j^2\right), \end{aligned}$$

where in the second inequality, the induction assumption on m is applied, since for any $\boldsymbol{x}_i \in \mathbb{R}^n$, the multilinear form $G_{\boldsymbol{x}_i}$ satisfies

$$G_{\boldsymbol{x}_i}(\boldsymbol{y}_1^2\boldsymbol{y}_2^2\ldots\boldsymbol{y}_m^2) := F(\boldsymbol{x}_i^2\boldsymbol{y}_1^2\boldsymbol{y}_2^2\ldots\boldsymbol{y}_m^2) \ge 0 \quad \forall\, \boldsymbol{y}_1,\boldsymbol{y}_2,\ldots,\boldsymbol{y}_m \in \mathbb{R}^n$$

and is therefore co-quadratic nonnegative. This completes the induction step (4.3), which concludes the whole proof. $\square$

As co-quadratic M-quasiconvexity is more general than M-quasiconvexity, here we provide a stronger statement.

Theorem 4.3. *If F is a co-quadratic nonnegative form associated with a $2m$-th order symmetric tensor $\mathcal{F} \in \mathbb{S}^{n^{2m}}$, then it is also M-quasiconvex, i.e.,*

$$F(\boldsymbol{y}_1^2\boldsymbol{y}_2^2\ldots\boldsymbol{y}_m^2) \ge 0\ \forall\, \boldsymbol{y}_i \in \mathbb{R}^n \Longrightarrow F(\boldsymbol{x}_1\boldsymbol{x}_2\ldots\boldsymbol{x}_{2m}) \le \max_{1\le i\le 2m}\{F(\boldsymbol{x}_i^{2m})\}\ \forall\, \boldsymbol{x}_i \in \mathbb{R}^n.$$

In other words, $\mathbb{S}_{2+}^{n^{2m}} \subseteq \mathbb{M}^{n^{2m}}$.

Proof. Let $\xi_1,\xi_2,\ldots,\xi_m$ be Benoulli random variables, each taking values 1 and -1 with equal probability, satisfying $\prod_{i=1}^m \xi_i = -1$. For any $\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_{2m} \in \mathbb{R}^n$, since F is co-quadratic nonnegative, we observe that

$$\begin{aligned} 0 &\le \mathsf{E}\left[F\left((\boldsymbol{x}_1+\xi_1\boldsymbol{x}_2)^2(\boldsymbol{x}_3+\xi_2\boldsymbol{x}_4)^2\ldots(\boldsymbol{x}_{2m-1}+\xi_m\boldsymbol{x}_{2m})^2\right)\right] \\ &= \sum_{i_1=1}^{2}\sum_{i_2=3}^{4}\cdots\sum_{i_m=2m-1}^{2m} F(\boldsymbol{x}_{i_1}^2\boldsymbol{x}_{i_2}^2\ldots\boldsymbol{x}_{i_m}^2) - 2^m F(\boldsymbol{x}_1\boldsymbol{x}_2\ldots\boldsymbol{x}_{2m}). \end{aligned} \tag{4.4}$$

This is because the function F is multilinear, and the expectation of the coefficient for any term other than those in the right hand side of (4.4) is zero.

Since F is co-quadratic nonnegative, by Theorem 4.2 it is co-quadratic M-quasiconvex, and we have

$$F(\boldsymbol{x}_{i_1}^2\boldsymbol{x}_{i_2}^2\ldots\boldsymbol{x}_{i_m}^2) \le \max\left\{F(\boldsymbol{x}_{i_1}^{2m}), F(\boldsymbol{x}_{i_2}^{2m}),\ldots,F(\boldsymbol{x}_{i_m}^{2m})\right\} \le \max_{1\le i\le 2m}\{F(\boldsymbol{x}_i^{2m})\}$$

for all $i_1 \in \{1,2\}, i_2 \in \{3,4\},\ldots, i_m \in \{2m-1,2m\}$. Therefore (4.4) further leads to

$$2^m F(\boldsymbol{x}_1\boldsymbol{x}_2\ldots\boldsymbol{x}_{2m}) \le \sum_{i_1=1}^{2}\sum_{i_2=3}^{4}\cdots\sum_{i_m=2m-1}^{2m} F(\boldsymbol{x}_{i_1}^2\boldsymbol{x}_{i_2}^2\ldots\boldsymbol{x}_{i_m}^2) \le 2^m \max_{1\le i\le 2m}\{F(\boldsymbol{x}_i^{2m})\}.$$

proving that F is M-quasiconvex. □

We conclude this section with the following result, which is an immediate consequence of Theorem 4.3 and Definitions 2.6 and 2.7.

COROLLARY 4.4. *A co-quadratic nonnegative tensor is also M-quasiconvex, and an M-quasiconvex tensor is also co-quadratic M-quasiconvex, i.e.,*

$$\mathbb{S}_{2+}^{n^{2m}} \subseteq \mathbb{M}^{n^{2m}} \subseteq \mathbb{M}_2^{n^{2m}}.$$

5. Proper containments . The analysis in the previous section triggers the question about the further relationships among the sets $\mathbb{S}_{2+}^{n^{2m}}$, $\mathbb{M}^{n^{2m}}$ and $\mathbb{M}_2^{n^{2m}}$. Our first result in this section generalizes the case $m = 1$, which shows that these three sets are indeed the same even when $m = 2$; i.e., $\mathbb{S}_{2+}^{n^4} = \mathbb{M}^{n^4} = \mathbb{M}_2^{n^4}$.

THEOREM 5.1. *If F is a nonnegative form associated with a 4th order symmetric tensor $\mathcal{F} \in \mathbb{S}^{n^4}$, then the following three statements are equivalent:*

1. *F is co-quadratic nonnegative, i.e., $F(\boldsymbol{x}_1^2\boldsymbol{x}_2^2) \ge 0\ \forall\, \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$;*
2. *F is M-quasiconvex, i.e., $F(\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4) \le \max_{1\le i\le 4}\{F(\boldsymbol{x}_i^4)\}\ \forall\, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4 \in \mathbb{R}^n$;*
3. *F is co-quadratic M-quasiconvex, i.e., $F(\boldsymbol{x}_1^2\boldsymbol{x}_2^2) \le \max_{1\le i\le 2}\{F(\boldsymbol{x}_i^4)\}\ \forall\, \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$.*

Proof. According to Corollary 4.4, we only need to show that co-quadratic M-quasiconvexity implies co-quadratic nonnegativity for a quartic form F; i.e., Statement 3 implies Statement 1 in Theorem 5.1. Indeed, by the equivalent property in Lemma 3.3, F is co-quadratic M-quasiconvex is equivalent to

$$F(\boldsymbol{x}^2\boldsymbol{y}^2) \le \frac{1}{2}\left(F(\boldsymbol{x}^4) + F(\boldsymbol{y}^4)\right) \quad \forall\, \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n.$$

It leads to

$$F\left((\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{x}+\boldsymbol{y})^2\right) = F(\boldsymbol{x}^4) + F(\boldsymbol{y}^4) - 2F(\boldsymbol{x}^2\boldsymbol{y}^2) \ge 0 \quad \forall\, \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n.$$

Therefore F is co-quadratic nonnegative, completing the proof. □

As checking the convexity of a quartic form is in general NP-hard [1], Theorem 5.1 implies that checking the M-quasiconvexity or co-quadratic M-quasiconvexity for a quartic form is also NP-hard. However, the task of investigating the relationships among $\mathbb{S}_{2+}^{n^{2m}}$, $\mathbb{M}^{n^{2m}}$ and $\mathbb{M}_2^{n^{2m}}$ for $m \ge 3$ becomes more complicated in general. Our second result asserts that at least two of these three cones are distinct when $m \ge 3$.

EXAMPLE 5.2. *Let $G(\boldsymbol{x}^6) := 3x_1^6 + 3x_2^6 + 15x_1^4x_2^2$ be a bivariate homogeneous polynomial associated with a 6th order symmetric tensor $\mathcal{G} \in \mathbb{S}^{2^6}$, i.e., $\mathcal{G}_{111111} = \mathcal{G}_{222222} = 3$, $\mathcal{G}_{111122} = \mathcal{G}_{111212} = \cdots = 1$, and all the rest entries are zeros. We have:*

1. *$\mathcal{G}$ is nonnegative;*
2. *$\mathcal{G}$ is not co-quadratic nonnegative;*
3. *$G(\boldsymbol{x}^2\boldsymbol{y}^4) \ge 0\ \forall\, \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$;*
4. *$G(\boldsymbol{x}^2\boldsymbol{y}^4) \le \max\{G(\boldsymbol{x}^6), G(\boldsymbol{y}^6)\}\ \forall\, \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$;*
5. *$G(\boldsymbol{x}^2\boldsymbol{y}^2\boldsymbol{z}^2) \le \max\{G(\boldsymbol{x}^6), G(\boldsymbol{y}^6), G(\boldsymbol{z}^6)\}\ \forall\, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n$ ($\mathcal{G}$ is co-quadratic M-quasiconvex).*

Toward proving these statements, we first notice that $G(\boldsymbol{x}^6) = 3\,x_1^6 + 3\,x_2^6 + 15\,x_1^4x_2^2 \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^2$, implying that $\mathcal{G}$ is nonnegative. Next by letting $\boldsymbol{x} = (15,1)^{\mathrm{T}}$, $\boldsymbol{y} = (-3,-13)^{\mathrm{T}}$ and $\boldsymbol{z} = (-5,10)^{\mathrm{T}}$ we have $G(\boldsymbol{x}^2\boldsymbol{y}^2\boldsymbol{z}^2) = -367575$, implying that $\mathcal{G}$ is not co-quadratic nonnegative.

For the third statement, we notice that for $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^2$

$$
\begin{aligned}
G(\boldsymbol{x}^2\boldsymbol{y}^4) &= 3x_1^2y_1^4 + 3x_2^2y_2^4 + x_2^2y_1^4 + 6x_1^2y_1^2y_2^2 + 8x_1x_2y_1^3y_2 \\
&= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} 3y_1^4 + 6y_1^2y_2^2 & 4y_1^3y_2 \\ 4y_1^3y_2 & y_1^4 + 3y_2^4 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
\end{aligned}
$$

Denote $Q = \begin{bmatrix} 3y_1^4 + 6y_1^2y_2^2 & 4y_1^3y_2 \\ 4y_1^3y_2 & y_1^4 + 3y_2^4 \end{bmatrix}$. By noticing that $Q_{11} \geq 0$, $Q_{22} \geq 0$ and

$$
\det(Q) = 3y_1^8 + 9y_1^4y_2^4 + 18y_1^2y_2^6 - 10y_1^6y_2^2 = \frac{(3y_1^4 - 5y_1^2y_2^2)^2}{3} + \frac{2y_1^4y_2^4}{3} + 18y_1^2y_2^6 \geq 0,
$$

we have that Q is positive semidefinite for any $\boldsymbol{y} \in \mathbb{R}^2$, proving the third statement. This is in fact equivalent to that $G(\boldsymbol{x}^6)$ is a convex function.

For the fourth statement, by Lemma 3.3 it suffices to show

$$
3G(\boldsymbol{x}^2\boldsymbol{y}^4) \leq G(\boldsymbol{x}^6) + 2G(\boldsymbol{y}^6) \quad \forall\, \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^2.
$$

Direct computation shows that

$$
\begin{aligned}
g_1(\boldsymbol{x},\boldsymbol{y}) &:= G(\boldsymbol{x}^6) + 2G(\boldsymbol{y}^6) - 3G(\boldsymbol{x}^2\boldsymbol{y}^4) \\
&= x_1^6 + x_2^6 + 5x_1^4x_2^2 + 2y_1^6 + 2y_2^6 + 10y_1^4y_2^2 \\
&\quad - (3x_1^2y_1^4 + 3x_2^2y_2^4 + x_2^2y_1^4 + 6x_1^2y_1^2y_2^2 + 8x_1x_2y_1^3y_2).
\end{aligned}
$$

Establishing $g_1(\boldsymbol{x},\boldsymbol{y}) \geq 0$ is not very easy. Instead, we shall prove this inequality by numerical optimization: we call GloptiPoly 3 [12] to solve the unconstrained polynomial optimization problem

$$
\min_{\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^2} g_1(\boldsymbol{x},\boldsymbol{y}).
$$

When the relaxation order is set to be 3, the global optimality is guaranteed for this problem, whose optimal value is zero with the optimal solution being $\boldsymbol{x} = \boldsymbol{y} = (0,0)^{\mathrm{T}}$. The fourth statement is thus verified.

For the last statement of Example 5.2, similar method for the fourth statement is applied. By Lemma 3.3 we need to show that

$$
3G(\boldsymbol{x}^2\boldsymbol{y}^2\boldsymbol{z}^2) \leq G(\boldsymbol{x}^6) + G(\boldsymbol{y}^6) + G(\boldsymbol{z}^6) \quad \forall\, \boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in \mathbb{R}^2,
$$

which is

$$
\begin{aligned}
&x_1^6 + x_2^6 + 5x_1^4x_2^2 + y_1^6 + y_2^6 + 5y_1^4y_2^2 + z_1^6 + z_2^6 + 5z_1^4z_2^2 - (3x_1^2y_1^2z_1^2 + 3x_2^2y_2^2z_2^2 \\
&\quad + x_1^2y_1^2z_2^2 + x_1^2y_2^2z_1^2 + x_2^2y_1^2z_1^2 + 4x_1x_2y_1y_2z_1^2 + 4x_1x_2y_1^2z_1z_2 + 4x_1^2y_1y_2z_1z_2) \\
=:&g_2(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \geq 0.
\end{aligned}
$$

Applying Gloptipoly 3 to solve the problem

$$
\min_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in \mathbb{R}^2} g_2(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})
$$

and setting the relaxation order to be 4, the global optimality is also certified. The optimal value for this problem is zero, with the optimal solution being $\boldsymbol{x} = \boldsymbol{y} = \boldsymbol{z} = (0,0)^{\mathrm{T}}$. This proves the last statement, i.e., $\mathcal{G}$ is co-quadratic M-quasiconvex.

Example 5.2 clearly differentiates the co-quadratic nonnegativity and the co-quadratic M-quasiconvexity for sixth order symmetric tensors, which is not the case for quadratic and quartic tensors.

THEOREM 5.3. *For $m \ge 3$ and $n \ge 2$, it holds that $\mathbb{S}_{2+}^{n^{2m}} \subset \mathbb{M}_2^{n^{2m}}$.*

One may certainly wonder about the status of $\mathbb{M}^{n^{2m}}$ in the chain of containing relationship $\mathbb{S}_{2+}^{n^{2m}} \subseteq \mathbb{M}^{n^{2m}} \subseteq \mathbb{M}_2^{n^{2m}}$. We are however unable to completely settle this question at this point, though we can be sure that at least one of the containment is proper as Theorem 5.3 shows. Besides, it is currently computationally impossible to verify whether $\mathcal{G}$ in Example 5.2 is M-quasiconvex or not, whose answer should tell which proper containment in this chain is true. However, we believe that all these containing relationships are proper when $m \ge 3$, which leads to the following conjecture:

CONJECTURE 5.4. *For $m \ge 3$ and $n \ge 2$, it holds that $\mathbb{S}_{2+}^{n^{2m}} \subset \mathbb{M}^{n^{2m}} \subset \mathbb{M}_2^{n^{2m}}$.*

6. Applications. Polynomial and tensor optimization is a rapidly expanding field in recent years. In this section we show how the study of these new classes of nonnegative tensors can be useful. To begin, let us consider the following general constrained homogeneous polynomial optimization model

$$\begin{array}{ll} \max & f(\boldsymbol{x}) = F(\underbrace{\boldsymbol{x}, \boldsymbol{x}, \ldots, \boldsymbol{x}}_{2m}) \\ \text{s.t.} & \boldsymbol{x} \in S \end{array}$$

where the constraint set $S \subseteq \mathbb{R}^n$ is compact. Essentially, the model is to find the largest eigenvalue of a symmetric tensor $\mathcal{F}$ over S. Based on the relationships among $\mathbb{S}_{2+}^{n^{2m}}$, $\mathbb{M}^{n^{2m}}$ and $\mathbb{M}_2^{n^{2m}}$ discussed in previous sections, we have the following result.

THEOREM 6.1. *If $\mathcal{F} \in \mathbb{S}^{n^{2m}}$ is M-quasiconvex, then for any $S \subseteq \mathbb{R}^n$*

$$\begin{aligned} (L) \quad \max_{\boldsymbol{x} \in S} F(\underbrace{\boldsymbol{x}, \boldsymbol{x}, \ldots, \boldsymbol{x}}_{2m}) &= \max_{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m \in S} F(\boldsymbol{x}_1, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m, \boldsymbol{x}_m) \quad (M) \\ &= \max_{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{2m} \in S} F(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{2m}). \quad (R) \end{aligned}$$

Proof. Denote the optimal value of problems (L), (M) and (R) to be $v(L)$, $v(M)$ and $v(R)$, respectively. As (R) is a relaxation of (M) and (M) is a relaxation of (L), we have that $v(L) \le v(M) \le v(R)$. It suffices to show $v(L) \ge v(R)$.

In fact, since $\mathcal{F} \in \mathbb{S}^{n^{2m}}$ is M-quasiconvex, i.e.,

$$F(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{2m}) \le \max_{1 \le i \le 2m} \{F(\underbrace{\boldsymbol{x}_i, \boldsymbol{x}_i, \ldots, \boldsymbol{x}_i}_{2m})\} \quad \forall \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{2m} \in S.$$

Therefore $v(R) \le v(L)$. This proves the theorem. $\square$

Theorem 6.1 establishes equivalence between homogeneous polynomial optimization and its multilinear form relaxation problem over *any* constraint set, given that $\mathcal{F}$ is an M-quasiconvex tensor. Hence, as stated in Section 4 of [7], Theorem 6.1 suggests an alternative way to deal with homogeneous polynomial optimization model, say (L). The procedure can be divided into two steps. The first step is to relax (L) to a multiquadratic form optimization (M) or multilinear form optimization (R). One choice for solving (M) or (R) is to implement the MBI method [7], or block coordinate decent method (BCD) [22]. The second step is to construct a stationary solution (or an optimal solution) for the original problem. In this circumstance, suppose that $(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{2m})$ is a stationary solution (or, an optimal solution) for (R), then we can directly find the stationary solution (or, the optimal solution) $\boldsymbol{x}_{i^*}$ for (L), where

$$i^* = \arg \max_{1 \le i \le 2m} F(\underbrace{\boldsymbol{x}_i, \boldsymbol{x}_i, \ldots, \boldsymbol{x}_i}_{2m}),$$

as the proof of Theorem 6.1 suggested. The best solution for (L) is already among the solutions $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{2m}$. The procedure is much simpler as compared to Algorithm KKT proposed in Section 4.2 of [7], thanks to the properties of M-quasiconvexity.

An immediate consequence of Theorem 6.1 is the following:

Corollary 6.2. *If $\mathcal{F} \in \mathbb{S}^{n^{2m}}$ is M-quasiconvex and integers $\lambda_i \ge 0$ $(i = 1, 2, \ldots, s)$ with $\sum_{i=1}^{s} \lambda_i = 2m$, then for any $S \subseteq \mathbb{R}^n$*

$$\max_{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_s \in S} F(\underbrace{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_1}_{\lambda_1}, \underbrace{\boldsymbol{x}_2, \ldots, \boldsymbol{x}_2}_{\lambda_2}, \ldots, \underbrace{\boldsymbol{x}_s, \ldots, \boldsymbol{x}_s}_{\lambda_s}) = \max_{\boldsymbol{x} \in S} F(\underbrace{\boldsymbol{x}, \boldsymbol{x}, \ldots, \boldsymbol{x}}_{2m}).$$

As the constraint set S can be chosen arbitrarily, it leaves us a lot of room to solve numerous potential optimization problems, including discrete polynomial optimization models, some of which will be discussed in the following subsections.

We remark that it is in general hard to check the M-quasiconvexity of a symmetric tensor which is a condition of Theorem 6.1 and Corollary 6.2. However, according to Theorem 4.3, we may take the co-quadratic nonnegativity as an alternative which is relatively easier to handle.

6.1. Tensor eigenvalue problem. The concept of eigenvalues/eigenvectors of tensors was proposed by Lim [20] and Qi [29] independently in 2005. In particular, finding the largest eigenvalue of a symmetric tensor [29, 30] is exactly the following spherical constrained homogeneous polynomial optimization problem:

$$\begin{array}{lll} (H) & \max & f(\boldsymbol{x}) = F(\underbrace{\boldsymbol{x}, \boldsymbol{x}, \ldots, \boldsymbol{x}}_{d}) \\ & \text{s.t.} & \|\boldsymbol{x}\| = 1,\, \boldsymbol{x} \in \mathbb{R}^n, \end{array}$$

where F is a multilinear form associated with a symmetric tensor $\mathcal{F} \in \mathbb{R}^{n^d}$. This problem has received much attention lately, not only for its fundamental properties but also for its wide applications, including numerical linear algebra, solid mechanic, signal processing, and quantum physics. It is also equivalent to the best rank-one approximation of a symmetric tensor; see e.g. [19].

A tensor relaxation method (relaxing to a multilinear form optimization model) for (H) is the following:

$$\begin{array}{lll} (T) & \max & F(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_d) \\ & \text{s.t.} & \|\boldsymbol{x}_i\| = 1,\, \boldsymbol{x}_i \in \mathbb{R}^n,\, i = 1, 2, \ldots, d. \end{array}$$

This relaxation is first proposed in [10] where relationship between (H) and (T) as well as their approximate methods are discussed. Later, Chen et al. [7] and Zhang et al. [33] further explored this idea to solve (H). In particular, Chen et al. [7] designed an efficient algorithm to find a stationary solution of (H) from a stationary solution of (T), by lifting the objective function of (H) to be a nonnegative form for an even order symmetric tensor.

The study of the new classes of nonnegative tensors however gives us a much simpler way to deal with the model (H) when the degree of the objective is even. Rather than relaxing (H) to (T) directly, we first add a constant form $\alpha(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x})^m = \alpha$ to the objective function, where $m = d/2$ in (H). By choosing α large enough we can ensure that the new objective function $f_\alpha(\boldsymbol{x}) := f(\boldsymbol{x}) + \alpha(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x})^m$ is co-quadratic nonnegative, hence M-quasiconvex, which is guaranteed by the following result.

THEOREM 6.3. *Denote $\mathcal{H} \in \mathbb{S}^{n^{2m}}$ to be the symmetric tensor associated with the homogeneous polynomial function $(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x})^m$. Then $\mathcal{H} \in \operatorname{int}\mathbb{S}^{n^{2m}}_{2+}$ (the interior of $\mathbb{S}^{n^{2m}}_{2+}$).*

Proof. Denote $\mathbb{K}^{n^{2m}}$ to be the convex hull of the set of symmetric rank-one tensors, i.e.,

$$\mathbb{K}^{n^{2m}} = \operatorname{conv}\left\{ \mathcal{F} \in \mathbb{S}^{n^{2m}} \,\middle|\, \mathcal{F} = \underbrace{\boldsymbol{a} \circ \boldsymbol{a} \circ \cdots \circ \boldsymbol{a}}_{2m},\ \boldsymbol{a} \in \mathbb{R}^n \right\}.$$

The proof is essentially based on the result that $\mathcal{H} \in \operatorname{int}\mathbb{K}^{n^{2m}}$, which is stated as Theorem 8.15 in [31]. By noticing from Example 2.4 that any even order symmetric rank-one tensor is co-quadratic nonnegative, we have that any tensor in $\mathbb{K}^{n^{2m}}$ is co-quadratic nonnegative, and hence $\mathbb{K}^{n^{2m}} \subseteq \mathbb{S}^{n^{2m}}_{2+}$. This proves $\mathcal{H} \in \operatorname{int}\mathbb{S}^{n^{2m}}_{2+}$. ☐

After lifting the objective function of (H) by α, we can then equivalently reformulate the new homogeneous polynomial $f_\alpha(\boldsymbol{x})$ to its multilinear relaxation like (T), and apply the MBI method to get a stationary (which is often optimal) solution. This equivalence is guaranteed by Theorem 6.1 since co-quadratic nonnegativity implies M-quasiconvexity. Finally, to return a solution to the original problem (H), we only need to find the best i^* satisfying $i^* = \arg\max_{1\le i\le 2m} F_\alpha(\underbrace{\boldsymbol{x}_i, \boldsymbol{x}_i, \ldots, \boldsymbol{x}_i}_{2m})$. This is because of the M-quasiconvexity of $\mathcal{F}_\alpha$, which is guaranteed by its co-quadratic nonnegativity (Theorem 4.3).

Indeed by Theorem 6.1, we can also equivalently transfer (H) to a multiquadratic form optimization after its objective function is lifted to $f_\alpha(\boldsymbol{x})$

$$\begin{array}{lll} (Q) & \max & F_\alpha(\boldsymbol{x}_1, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m, \boldsymbol{x}_m) \\ & \text{s.t.} & \|\boldsymbol{x}_i\| = 1,\ \boldsymbol{x}_i \in \mathbb{R}^n,\ i = 1, 2, \ldots, m. \end{array}$$

This relaxation can also be solved by the MBI or BCD method as its subproblem (fixing $m-1$ blocks and optimizing one block) is a matrix eigenvalue problem. Moreover, after getting the solution of (Q), we are able to return a solution of (H) by choosing $i^* = \arg\max_{1\le i\le m} F_\alpha(\underbrace{\boldsymbol{x}_i, \boldsymbol{x}_i, \ldots, \boldsymbol{x}_i}_{2m})$ because of the co-quadratic M-quasiconvexity of $\mathcal{F}_\alpha$, which is also guaranteed by its co-quadratic nonnegativity (Theorem 4.2).

6.2. Bi-quadratic model in circuit design. The quadratic assignment problem (QAP) is known as one of the most challenging problems in combinatorial optimization. Recently, there have been attempts to solve the QAP via semidefinite programming relaxations as a lower bounding procedure; see, e.g. [9, 8, 28]. The biquadratic assignment problem (BiQAP) is a generalization of the QAP, which is to minimize a quartic polynomial of an assignment matrix:

$$\begin{array}{ll} \min & \sum_{1\le i,j,k,\ell,s,t,u,v\le n} \mathcal{A}_{ijk\ell}\mathcal{B}_{stuv}X_{is}X_{jt}X_{ku}X_{\ell v} \\ \text{s.t.} & \sum_{j=1}^n X_{ij}=1,\ i=1,2,\ldots,n \\ & \sum_{i=1}^n X_{ij}=1,\ j=1,2,\ldots,n \\ & X_{ij}\in\{0,1\},\ i,j=1,2,\ldots,n \\ & X\in\mathbb{R}^{n\times n}, \end{array}$$

where $\mathcal{A},\mathcal{B}\in\mathbb{R}^{n\times n\times n\times n}$. Motivated by a practical application in the very large scale integrated synthesis problem, BiQAP was first introduced and studied by Burkard et al. [6]. Later on, due to its difficulty, several heuristics for the BiQAP were developed by Burkard and Cela [5] and Mavridou et al. [25].

The objective function of the BiQAP is a fourth degree polynomial function of the variables X_{ij}'s, where X is taken as an n^2-dimensional vector. In particular, by denoting $\boldsymbol{x}:=\operatorname{vec}(X)\in\mathbb{R}^{n^2}$, we can find a quartic form F associated with a symmetric quartic tensor $\mathcal{F}:=\operatorname{sym}(-\mathcal{A}\otimes\mathcal{B})\in\mathbb{S}^{(n^2)^4}$ where the notion 'sym' symmetrizes a tensor, such that

$$F(\boldsymbol{x},\boldsymbol{x},\boldsymbol{x},\boldsymbol{x}):=\sum_{1\le i,j,k,\ell,s,t,u,v\le n} -\mathcal{A}_{ijk\ell}\mathcal{B}_{stuv}X_{is}X_{jt}X_{ku}X_{\ell v}.$$

For the constraints, one finds vectors $\boldsymbol{a}_1,\boldsymbol{a}_2,\ldots,\boldsymbol{a}_n,\boldsymbol{b}_1,\boldsymbol{b}_2,\ldots,\boldsymbol{b}_n\in\mathbb{R}^{n^2}$ in such a way that

$$\begin{aligned} \boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x} &= \sum_{j=1}^n X_{ij}=1,\ i=1,2,\ldots,n, \\ \boldsymbol{b}_j^{\mathrm{T}}\boldsymbol{x} &= \sum_{i=1}^n X_{ij}=1,\ j=1,2,\ldots,n. \end{aligned}$$

We have the equivalent formulation of the BiQAP

$$\begin{array}{ll} \max & F(\boldsymbol{x},\boldsymbol{x},\boldsymbol{x},\boldsymbol{x}) \\ \text{s.t.} & \boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x}=1,\ i=1,2,\ldots,n \\ & \boldsymbol{b}_j^{\mathrm{T}}\boldsymbol{x}=1,\ j=1,2,\ldots,n \\ & \boldsymbol{x}\in\{0,1\}^{n^2}. \end{array}$$

An important observation is that $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}=n$ as X is an assignment matrix. Thus the objective function of the above model can be lifted up to a co-quadratic nonnegative form by adding a constant term $\alpha(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x})^2$ according to Theorem 6.3. In fact, it is not hard to verify that setting $\alpha=3\|\mathcal{F}\|$ is enough to guarantee the co-quadratic nonnegativity of $F_\alpha(\boldsymbol{x},\boldsymbol{x},\boldsymbol{x},\boldsymbol{x}):=F(\boldsymbol{x},\boldsymbol{x},\boldsymbol{x},\boldsymbol{x})+\alpha(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x})^2$, which implies that it is M-quasiconvex. By Theorem 6.1, the BiQAP can then be reformulated in the multilinear form model:

$$\begin{array}{ll} \max & F_\alpha(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{w}) \\ \text{s.t.} & \boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x}=\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{y}=\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{z}=\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{w}=1,\ i=1,2,\ldots,n \\ & \boldsymbol{b}_j^{\mathrm{T}}\boldsymbol{x}=\boldsymbol{b}_j^{\mathrm{T}}\boldsymbol{y}=\boldsymbol{b}_j^{\mathrm{T}}\boldsymbol{z}=\boldsymbol{b}_j^{\mathrm{T}}\boldsymbol{w}=1,\ j=1,2,\ldots,n \\ & \boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{w}\in\{0,1\}^{n^2}. \end{array}$$

Note that the subproblem of the above optimization model (fixing 3 blocks and optimizing one block) is linear assignment, which is easily solvable. Therefore the MBI method introduced in [7] can be used to solve this model and get a stationary solution $(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{z}^*, \boldsymbol{w}^*)$. Finally, $\arg\max\{F_\alpha(\boldsymbol{x}^*, \boldsymbol{x}^*, \boldsymbol{x}^*, \boldsymbol{x}^*), F_\alpha(\boldsymbol{y}^*, \boldsymbol{y}^*, \boldsymbol{y}^*, \boldsymbol{y}^*), F_\alpha(\boldsymbol{z}^*, \boldsymbol{z}^*, \boldsymbol{z}^*, \boldsymbol{z}^*), F_\alpha(\boldsymbol{w}^*, \boldsymbol{w}^*, \boldsymbol{w}^*, \boldsymbol{w}^*)\}$ is a local stationary solution to the original BiQAP, as guaranteed by the M-quasiconvexity of $\mathcal{F}_\alpha$.

6.3. Portfolio selection with higher moments. In modern portfolio management, the celebrated mean-variance model was originally introduced by Markowitz [24] back in 1952, where the portfolio selection problem is modeled by minimizing the variance of the return of the portfolio for a given level of its expected return, as follows:

$$\begin{array}{ll} \min & \boldsymbol{x}^{\mathrm{T}}\Sigma\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{x} = \mu_0 \\ & \boldsymbol{e}^{\mathrm{T}}\boldsymbol{x} = 1 \\ & \boldsymbol{x} \in \mathbb{R}^n, \end{array}$$

where $\boldsymbol{\mu}$ and Σ are the mean vector and co-variance matrix of n given assets respectively, and $\boldsymbol{e}$ is the all-one vector. Much of the mean-variance theory has been focusing on the first two moments of the return of the portfolio. Recently, the framework of mean-variance has been extended to include the skewness and kurtosis information; see, e.g. Jondeau and Rockinger [16], Kleniati et al. [17], Maringer and Parpas [23] and the references therein.

Let us now consider a similar model proposed in [26] which minimizes the kurtosis under the constraints of the mean and variance of the portfolio, as follows:

$$\begin{array}{lll} (K) & \min & F(\boldsymbol{x}, \boldsymbol{x}, \boldsymbol{x}, \boldsymbol{x}) \\ & \text{s.t.} & \boldsymbol{x}^{\mathrm{T}}\Sigma\boldsymbol{x} = \sigma^2 \\ & & \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{x} \geq \mu_0 \\ & & \boldsymbol{e}^{\mathrm{T}}\boldsymbol{x} = 1 \\ & & \boldsymbol{x} \in S, \end{array}$$

where S can be $\mathbb{R}^n$ or $\mathbb{R}^n_+$ depending on if short selling is allowed or not. Toward the solution method, we may let $\boldsymbol{y} = \frac{1}{\sigma}\Sigma^{\frac{1}{2}}\boldsymbol{x}$ since the co-variance matrix Σ is positive semidefinite. Hence $\boldsymbol{x} = \sigma\Sigma^{-\frac{1}{2}}\boldsymbol{y}$ (use the Moore-Penrose inverse if Σ is not positive definite). We can reformulate (K) to the following equivalent model:

$$\begin{array}{lll} (K') & \max & F'(\boldsymbol{y}, \boldsymbol{y}, \boldsymbol{y}, \boldsymbol{y}) = -\sigma^4 F\left(\Sigma^{-\frac{1}{2}}\boldsymbol{y}, \Sigma^{-\frac{1}{2}}\boldsymbol{y}, \Sigma^{-\frac{1}{2}}\boldsymbol{y}, \Sigma^{-\frac{1}{2}}\boldsymbol{y}\right) \\ & \text{s.t.} & \boldsymbol{y}^{\mathrm{T}}\boldsymbol{y} = 1 \\ & & \sigma\boldsymbol{\mu}^{\mathrm{T}}\Sigma^{-\frac{1}{2}}\boldsymbol{y} \geq \mu_0 \\ & & \sigma\boldsymbol{e}^{\mathrm{T}}\Sigma^{-\frac{1}{2}}\boldsymbol{y} = 1 \\ & & \boldsymbol{y} \in S', \end{array}$$

where S' is either $\mathbb{R}^n$ or some linear inequalities $\Sigma^{-\frac{1}{2}}\boldsymbol{y} \in \mathbb{R}^n_+$. Noticing that $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{y} = 1$ and using a similar lift method introduced in the previous subsections, we can change the objective function of (K') to a co-quadratic nonnegative form $F'(\boldsymbol{y}, \boldsymbol{y}, \boldsymbol{y}, \boldsymbol{y}) + \alpha(\boldsymbol{y}^{\mathrm{T}}\boldsymbol{y})^2$ by letting $\alpha = 3\|\mathcal{F}'\|$. Then we can reformulate it to a multilinear form optimization model as Theorem 6.1 stipulated. The solution of (K') can be obtained by applying the MBI or BCD method to solve the multilinear model and comparing the objective F' among the four blocks of variables in the solution of the multilinear model, as the M-quasiconvexity of the form $F'(\boldsymbol{y}, \boldsymbol{y}, \boldsymbol{y}, \boldsymbol{y}) + \alpha(\boldsymbol{y}^{\mathrm{T}}\boldsymbol{y})^2$ enables us to relaxed the coupled block variables to an easier-looking but equivalent block optimization model.

7. Concluding Remarks. In this paper we study three new classes of nonnegative tensors: co-quadratic nonnegativity, M-quasiconvexity, and co-quadratic M-quasiconvexity. We also discuss the relationships among them. We demonstrate that M-quasiconvexity plays an important role in linking homogeneous polynomial optimization problems to their tensor relaxation counterparts. As a result, this makes it possible to solve polynomial optimization models using block coordinate search methods. Furthermore, it is natural to extend the notion of co-quadratic nonnegativity and M-quasiconvexity to a more general setting.

DEFINITION 7.1. *Suppose F is a multilinear form associated with a tm-th order symmetric tensor $\mathcal{F} \in \mathbb{S}^{n^{tm}}$ where t and m are two positive integers. The tensor $\mathcal{F}$ (or the form F) is called co-nonnegative of order t over $S \subseteq \mathbb{R}^n$ if*

$$F(\underbrace{\boldsymbol{x}_1, \dots, \boldsymbol{x}_1}_{t}, \underbrace{\boldsymbol{x}_2, \dots, \boldsymbol{x}_2}_{t}, \dots, \underbrace{\boldsymbol{x}_m, \dots, \boldsymbol{x}_m}_{t}) \geq 0 \quad \forall\, \boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_m \in S.$$

DEFINITION 7.2. *Suppose F is a multilinear form associated with a tm-th order symmetric tensor $\mathcal{F} \in \mathbb{S}^{n^{tm}}$ where t and m are two positive integers. The tensor $\mathcal{F}$ is called M-quasiconvex of order t over $S \subseteq \mathbb{R}^n$ if*

$$\begin{aligned} & F(\underbrace{\boldsymbol{x}_1, \dots, \boldsymbol{x}_1}_{t}, \underbrace{\boldsymbol{x}_2, \dots, \boldsymbol{x}_2}_{t}, \dots, \underbrace{\boldsymbol{x}_m, \dots, \boldsymbol{x}_m}_{t}) \\ \leq & \max_{1 \leq i \leq m} \{F(\underbrace{\boldsymbol{x}_i, \boldsymbol{x}_i, \dots, \boldsymbol{x}_i}_{tm})\} \qquad \forall\, \boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_m \in S. \end{aligned}$$

Therefore, the co-quadratic nonnegativity is co-nonnegativity of order 2 over $\mathbb{R}^n$, and the M-quasiconvexity and the co-quadratic M-quasiconvexity are M-quasiconvexity of order 1 and order 2, respectively. For any two positive integers $t_1 < t_2$ with t_2 a multiple of t_1, it is easy to see that M-quasiconvexity of order t_1 implies M-quasiconvexity of order t_2 for a symmetric tensor $\mathcal{F} \in \mathbb{S}^{n^{t_2 m}}$. To conclude this paper, we present the following result which generalizes Theorem 4.2. Its proof is similar to that of Theorem 4.2 and is hence omitted here.

THEOREM 7.3. *If S is a linear subspace of $\mathbb{R}^n$, then co-nonnegativity of order t over S implies M-quasiconvexity of order t over S.*

REFERENCES

[1] A. A. Ahmadi, A. Olshevsky, P. A. Parrilo, and J. N. Tsitsiklis, *NP-hardness of deciding convexity of quartic polynomials and related problems*, Mathematical Programming, 137, 453–476, 2013.

[2] S. Banach, *Über homogene Polynome in* (L^2), Studia Mathematica, 7, 36–44, 1938.

[3] A. Barvinok, *A Course in Convexity*, Graduate Studies in Mathematics, 54, AMS, Providence, RI, 2002.

[4] G. Blekherman, P. Parrilo, and R. Thomas, *Semidefinite Optimization and Convex Algebraic Geometry*, MOS-SIAM Series on Optimization, SIAM, Philadelphia, PA, 2012.

[5] R. E. Burkard and E. Cela, *Heuristis for biqudratic assignment problems and their computational comparison*, European Journal of Operational Research, 83, 283–300, 1995.

[6] R. E. Burkard, E. Cela, and B. Klinz, *On the biquadratic assignment problem.* In P. M. Pardalos and H. Wolkowicz (eds.), *Quadratic Assignment and Related Problems*, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Volumn 16, 117–146, AMS, Providence, RI, 1994.

[7] B. Chen, S. He, Z. Li, and S. Zhang, *Maximum block improvement and polynomial optimization*, SIAM Journal on Optimization, 22, 87–107, 2012.

[8] E. de Klerk and R. Sotirov, *Exploiting group symmetry in semidefinite programming relaxations of the quadratic assignment problem*, Mathematical Programming, 122, 225–246, 2010.
[9] Y. Ding and H. Wolkowicz, *A low-dimensional semidefinite relaxation for the quadratic assignment problem*, Mathematics of Operations Research, 34, 1008–1022, 2009.
[10] S. He, Z. Li, and S. Zhang, *Approximation algorithms for homogeneous polynomial optimization with quadratic constraints*, Mathematical Programming, 125, 353–383, 2010.
[11] J. W. Helton and J. Nie, *Semidefinite representation of convex sets*, Mathematical Programming, 122, 21–64, 2010.
[12] D. Henrion, J. B. Lasserre, and J. Loefberg, *GloptiPoly 3: Moments, optimization and semidefinite programming*, Optimization Methods and Software, 24, 761–779, 2009.
[13] D. Hilbert, *Über die Darstellung definiter Formen als Summe von Formenquadraten*, Mathematische Annalen, 32, 342–350, 1888.
[14] B. Jiang, S. He, Z. Li, and S. Zhang *Moments tensors, Hilbert's identity, and k-wise uncorrelated random variables*. Mathematics of Operations Research, 39, 775–788, 2014.
[15] B. Jiang, Z. Li, and S. Zhang, *On cones of nonnegative quartic forms*, Technical Report, Department of Industrial and Systems Engineering, University of Minnesota, Minneapolis, Minnesota, 2011.
[16] E. Jondeau and M. Rockinger, *Optimal portfolio allocation under higher moments*, European Financial Management, 12, 29–55, 2006.
[17] P.-M. Kleniati, P. Parpas, and B. Rustem, *Partitioning procedure for polynomial optimization: Application to portfolio decisions with higher order moments*, COMISEF Working Papers Series, WPS-023, 2009.
[18] T. G. Kolda and B. W. Bader, *Tensor decompositions and applications*, SIAM Review, 51, 455–500, 2009.
[19] Z. Li, S. He, and S. Zhang, *Approximation Methods for Polynomial Optimization: Models, Algorithms, and Applications*, SpringerBriefs in Optimization, Springer, New York, NY, 2012.
[20] L.-H. Lim, *Singular values and eigenvalues of tensors: A variational approach*, Proceedings of the 1st IEEE International Workshop on Computational Advances of Multi-tensor Adaptive Processing, 129–132, 2005.
[21] Z.-Q. Luo, J. F. Sturm, and S. Zhang, *Multivariate nonnegative quadratic mappings*, SIAM Journal on Optimization, 14, 1140–1162, 2004.
[22] Z.-Q. Luo and P. Tseng, *On the convergence of the coordinate descent method for convex differentiable minimization*, Journal of Optimization Theory and Applications, 72, 7–35, 1992.
[23] D. Maringer and P. Parpas, *Global optimization of higher order moments in portfolio selection*, Journal of Global Optimization, 43, 219–230, 2009.
[24] H. M. Markowitz, *Portfolio selection*, Journal of Finance, 7, 79–91, 1952.
[25] T. Mavridou, P. M. Pardalos, L. S. Pitsoulis, and M. G. C. Resende, *A GRASP for the biquadratic assignment problem*, European Journal of Operational Research, 105, 613–621, 1998.
[26] M. Mhiri and J.-L. Prigent, *International portfolio optimization with higher moments*, International Journal of Economics and Finance, 2, 157–169, 2010.
[27] A. Pappas, Y. Sarantopoulos, and A. Tonge, *Norm attaining polynomials*, Bulletin of the London Mathematical Society, 39, 255–264, 2007.
[28] J. Peng, H. Mittelmann, and X. Li, *A new relaxation framework for quadratic assignment problems based on matrix splitting*, Mathematical Programming Computation, 2, 59–77, 2010.
[29] L. Qi, *Eigenvalues of a real supersymmetric tensor*, Journal of Symbolic Computation, 40, 1302–1324, 2005.
[30] L. Qi, *Eigenvalues and invariants of tensors*, Journal of Mathematical Analysis and Applications, 325, 1363–1377, 2007.
[31] B. Reznick, *Sums of Even Powers of Real Linear Forms*, Memoirs of the American Mathematical Society, Volume 96, Number 463, 1992.
[32] J. F. Sturm and S. Zhang, *On cones of nonnegative quadratic functions*, Mathematics of Operations Research, 28, 246–267, 2003.
[33] X. Zhang, C. Ling, and L. Qi, *The best rank-1 approximation of a symmetric tensor and related spherical optimization problems*, SIAM Journal on Matrix Analysis and Applications, 33, 806–821, 2012.