Characterizing Real-Valued Multivariate Complex Polynomials and Their Symmetric Tensor Representations

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Abstract

In this paper we study multivariate polynomial functions in complex variables and the corresponding associated symmetric tensor representations. The focus is on finding conditions under which such complex polynomials/tensors always take real values. We introduce the notion of symmetric conjugate forms and general conjugate forms, and present characteristic conditions for such complex polynomials to be real-valued. As applications of our results, we discuss the relation between nonnegative polynomials and sums of squares in the context of complex polynomials. Moreover, new notions of eigenvalues/eigenvectors for complex tensors are introduced, extending properties from the Hermitian matrices. Finally, we discuss an important property for symmetric tensors, which states that the largest absolute value of eigenvalue of a symmetric real tensor is equal to its largest singular value; the result is known as Banach's theorem. We show that a similar result holds in the complex case as well.

Keywords: symmetric complex tensor, conjugate complex polynomial, tensor eigenvalue, tensor eigenvector, nonnegative complex polynomial, Banach's theorem.

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1 Introduction

In this paper we set out to study the functions in multivariate complex variables which however always take real values. Such functions are frequently encountered in engineering applications arising from signal processing [2, 19], electrical engineering [27], and control theory [30]. It is interesting to note that such complex functions are usually not studied by conventional complex analysis, since they are typically not even analytic because the Cauchy-Riemann conditions will never be satisfied unless the function in question is trivial. There has been a surge of research attention to solve optimization models related to such kind of complex functions [2, 27, 28, 13, 14]. Sorber et al. [29] developed a MATLAB toolbox called 'Complex Optimization Toolbox' for optimization problems in complex variables, where the complex function in question is either *pre-assumed* to be always real-valued [27], or it is the modulus/norm of a complex function [2, 28]. An interesting question thus arises: *Can such real-valued complex functions be characterized?* Indeed there does exist a class of special complex functions that always take real values: the Hermitian quadratic form $\boldsymbol{x}^{\mathrm{H}} A \boldsymbol{x}$ where A is a Hermitian matrix. In this case, the quadratic structure plays a key role. This motivates us to search more general complex polynomial functions with the same property. Interestingly, such complex polynomials can be completely characterized, as we will present in this paper.

As is well-known, polynomials can be represented by tensors. The same question can be asked about complex tensors. In fact, there is a considerable amount of recent research attention on the applications of complex tensor optimization. For instance, Hilling and Sudberythe [12] formulated a quantum entanglement problem as a complex multilinear form optimization under the spherical constraint, and Zhang and Qi [33] discussed a quantum eigenvalue problem, which arised from the geometric measure of entanglement of a multipartite symmetric pure state in the complex tensor space. Examples of complex polynomial optimization include Aittomaki and Koivunen [1] who formulated the problem of beam-pattern synthesis in array signal processing as complex quartic polynomial minimization, and Aubry et al. [2] who modeled a radar signal processing problem by complex polynomial optimization. Solution methods for complex polynomial optimization can be found in, e.g. [27, 13, 14]. As mentioned before, polynomials and tensors are known to be related. In particular in the real domain, homogeneous polynomials (or forms) are bijectively related to *symmetric* tensors (aka super-symmetric in some papers in the literature), i.e., the components of the tensor is invariant under the permutation of its indices. This important class of tensors generalizes the concept of symmetric matrices. As the role played by symmetric matrices in matrix theory and quadratic optimization, symmetric tensors have a profound role to play in tensor eigenvalue problems and polynomial optimization. A natural question can be asked about complex tensors: *What is the higher order complex tensor generalization of the Hermitian matrix?* In this paper, we manage to identify two classes of symmetric complex tensors, both of which include Hermitian matrices as a special case when the order of the tensor is two.

In recent years, the eigenvalue of tensor has become a topic of intensive research interest. To the best of our knowledge, a first attempt to generalize eigenvalue decomposition of matrices can be traced back to 2000 when De Lathauwer et al. [7] introduced the so-called higher-order eigenvalue decomposition. Shortly after that, Kofidis and Regalia [15] showed that blind deconvolution can be formulated as a nonlinear eigenproblem. A systematic study of eigenvalues of tensors was pioneered by Lim [18] and Qi [21] independently in 2005. Various applications of tensor eigenvalues and the connections to polynomial optimization problems have been proposed; cf. [22, 20, 33, 8] and the references therein. We refer the interested readers to the survey papers [16, 23] for more details on the spectral theory of tensors and various applications of tensors. Computation of tensor eigenvalues is an important source for polynomial optimization [10, 17]. Essentially the problem is to maximize or minimize a homogeneous polynomial under the spherical constraint, which can also be used to test the (semi)-definiteness of a symmetric tensor. This is closely related to the nonnegativity of a polynomial function, whose history can be traced back to Hilbert [9] in 1888 where the relationship of nonnegative polynomials and sums of squares (SOS) of polynomials was first established.

In this paper we are primarily interested in complex polynomials/tensors that arise in the context of optimization. By nature of optimization, we are interested in the complex polynomials that always take real values. However, it is easy to see that if no *conjugate* term is involved, then the only class of real-valued complex polynomials is the set of real constant functions[1]. Therefore, the conjugate terms are necessary for a complex polynomial to be real-valued. Hermitian quadratic forms mentioned earlier belong to this category, which is an active area of research in optimization; see e.g. [19, 31, 26]. In the aforementioned papers [22, 20, 8] on eigenvalues of complex tensors, the associated complex polynomials however are not real-valued. The aim of this paper is different. We target for a systematic study on the nature of symmetricity for higher order complex tensors which will lead to the property that the associated polynomials always take real values. The main contribution of the paper is to give a full characterization for the real-valued conjugate complex polynomials and to identify two classes of symmetric complex tensors, which have already shown potentials in the algorithms design [2, 13, 14]. We also show that a nonnegative univariate conjugate polynomial is not necessarily an SOS polynomial, in contrast to a well-known property of its real counterpart.

This paper is organized as follows. We start with the preparation of various notations and terminologies in Section 2. In particular, two types of conjugate complex polynomials are defined and their symmetric complex tensor representations are discussed. Section 3 presents the necessary and sufficient condition for real-valued conjugate complex polynomials, based on which two types of symmetric complex tensors are defined, corresponding to the two types of real-valued conjugate complex polynomials. Section 4 discusses the nonnegative properties of certain conjugate polynomials, in particular the relationship between nonnegativity and SOS for univariate complex conjugate polynomials. As an important result in this paper, we then present the definitions and properties of eigenvalues and eigenvectors for two types of symmetric complex tensors in Section 5. Finally in Section 6, we discuss Banach's theorem, which states that the largest absolute value of eigenvalue of a symmetric real tensor is equal to its largest singular value, and extend it to the two new types of symmetric complex tensors.

2 Preparation

Throughout this paper we shall use boldface lowercase letters, capital letters, and calligraphic letters to denote vectors, matrices, and tensors, respectively. For example, a vector $\boldsymbol{x}$, a matrix A, and a tensor $\mathcal{F}$. We use subscripts to denote their components, e.g. x_i being the i-th entry of a vector $\boldsymbol{x}$, A_{ij} being the (i,j)-th entry of a matrix A, and $\mathcal{F}_{ijk}$ being the (i,j,k)-th entry of a third order tensor $\mathcal{F}$. As usual, the field of real numbers and the field of complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively.

For any complex number $z = a + \mathbf{i}b \in \mathbb{C}$ with $a, b \in \mathbb{R}$, its real part and imaginary part are denoted by $\operatorname{Re} z := a$ and $\operatorname{Im} z := b$, respectively. Its modulus is denoted by $|z| := \sqrt{\overline{z}z} = \sqrt{a^2 + b^2}$, where $\overline{z} := a - \mathbf{i}b$ denotes the conjugate of z. For any vector $\boldsymbol{x} \in \mathbb{C}^n$, we denote $\boldsymbol{x}^{\mathrm{H}} := \overline{\boldsymbol{x}}^{\mathrm{T}}$ to be the transpose of its conjugate, similar operation applying to matrices.

A multivariate complex polynomial $f(\boldsymbol{x})$ is a polynomial function of variable $\boldsymbol{x} \in \mathbb{C}^n$ whose

[1] This should be differentiated from the notion of real-symmetric complex polynomial, sometimes also called real-valued complex polynomial in abstract algebra, i.e., $\overline{f(\boldsymbol{x})} = f(\overline{\boldsymbol{x}})$.

coefficients are complex, e.g. $f(x_1,x_2)=x_1+(1-\mathbf{i}){x_2}^2$. A multivariate *conjugate* complex polynomial (sometimes abbreviated by conjugate polynomial in this paper) $f_C(\boldsymbol{x})$ is a polynomial function of variables $\boldsymbol{x},\overline{\boldsymbol{x}}\in\mathbb{C}^n$, which is differentiated by the subscript C, standing for 'conjugate', e.g. $f_C(x_1,x_2)=x_1+\overline{x_2}+\overline{x_1}x_2+(1-\mathbf{i}){x_2}^2$. In particular, a general n-dimensional d-th degree conjugate complex polynomial can be explicitly written as summation of monomials

$$f_C(\boldsymbol{x}):=\sum_{\ell=0}^{d}\sum_{k=0}^{\ell}\sum_{1\le i_1\le\cdots\le i_k\le n,1\le j_1\le\cdots\le j_{\ell-k}\le n} a_{i_1\dots i_k,j_1\dots j_{\ell-k}}\overline{x_{i_1}\dots x_{i_k}}x_{j_1}\dots x_{j_{\ell-k}}.$$

In this definition, it is obvious that complex polynomials are a subclass of conjugate complex polynomials. Remark that a pure complex polynomial can never only take real values unless it is a constant. This observation follows trivially from the basic theorem of algebra.

Given a d-th order complex tensor $\mathcal{F}\in\mathbb{C}^{n_1\times\cdots\times n_d}$, its associated multilinear form is defined as

$$\mathcal{F}(\boldsymbol{x}^1,\dots,\boldsymbol{x}^d):=\sum_{1\le i_1\le n_1,\dots,1\le i_d\le n_d}\mathcal{F}_{i_1\dots i_d}x^1_{i_1}\dots x^d_{i_d},$$

where $\boldsymbol{x}^k\in\mathbb{C}^{n_k}$ for $k=1,\dots,d$. A tensor $\mathcal{F}\in\mathbb{C}^{n_1\times\cdots\times n_d}$ is called *symmetric* if $n_1=\cdots=n_d\,(=n)$ and every component $\mathcal{F}_{i_1\dots i_d}$ is invariant under all permutations of the indices $\{i_1,\dots,i_d\}$. Closely related to a symmetric tensor $\mathcal{F}\in\mathbb{C}^{n^d}$ is a general d-th degree complex homogeneous polynomial function $f(\boldsymbol{x})$ (or complex form) of variable $\boldsymbol{x}\in\mathbb{C}^n$, i.e.,

$$f(\boldsymbol{x}):=\mathcal{F}(\underbrace{\boldsymbol{x},\dots,\boldsymbol{x}}_{d})=\sum_{1\le i_1,\dots,i_d\le n}\mathcal{F}_{i_1\dots i_d}x_{i_1}\dots x_{i_d}. \tag{1}$$

In fact, symmetric tensors (either in the real domain or in the complex domain) are bijectively related to homogeneous polynomials; see [16]. In particular, for any n-dimensional d-th degree complex form

$$f(\boldsymbol{x})=\sum_{1\le i_1\le\cdots\le i_d\le n}a_{i_1\dots i_d}x_{i_1}\dots x_{i_d},$$

there is a uniquely defined n-dimensional d-th order symmetric complex tensor $\mathcal{F}\in\mathbb{C}^{n^d}$ with

$$\mathcal{F}_{i_1\dots i_d}=\frac{a_{i_1\dots i_d}}{|\Pi(i_1\dots i_d)|}\quad\forall\,1\le i_1\le\cdots\le i_d\le n$$

satisfying (1), where $\Pi(i_1\dots i_d)$ is the set of all distinct permutations of the indices $\{i_1,\dots,i_d\}$. On the other hand, in light of formula (1), a complex form $f(\boldsymbol{x})$ is easily obtained from the symmetric multilinear form $\mathcal{F}(\boldsymbol{x}^1,\dots,\boldsymbol{x}^d)$ by letting $\boldsymbol{x}^1=\cdots=\boldsymbol{x}^d=\boldsymbol{x}$.

2.1 Symmetric conjugate forms and their tensor representations

To discuss higher order conjugate complex forms and complex tensors, let us start with the well established properties of the Hermitian matrices. Let $A\in\mathbb{C}^{n^2}$ with $A^{\mathrm{H}}=A$, which is not symmetric in the usual sense because $A^{\mathrm{T}}\neq A$ in general. The following conjugate quadratic form

$$\boldsymbol{x}^{\mathrm{H}}A\boldsymbol{x}=\sum_{1\le i,j\le n}A_{ij}\overline{x_i}x_j$$

always takes real values for any $\boldsymbol{x}\in\mathbb{C}^n$. In particular, we notice that each monomial in the above form is the product of one 'conjugate' variable $\overline{x_i}$ and one usual (non-conjugate) variable x_j.

To extend the above form to higher degrees, let us consider the following special class of conjugate polynomials, to be called *symmetric conjugate forms*:

$$f_S(\boldsymbol{x}) := \sum_{1\le i_1\le\cdots\le i_d\le n, 1\le j_1\le\cdots\le j_d\le n} a_{i_1\dots i_d, j_1\dots j_d}\overline{x_{i_1}\dots x_{i_d}}x_{j_1}\dots x_{j_d}. \qquad (2)$$

Essentially, $f_S(\boldsymbol{x})$ is the summation of all the possible $2d$-th degree monomials that consist of exact d conjugate variables and d usual variables. Here the subscript 'S' stands for 'symmetric'. The following example is a special case of (2).

Example 2.1 *Given a d-th degree complex form $g(\boldsymbol{x}) = \sum_{1\le i_1\le\cdots\le i_d\le n} c_{i_1\dots i_d}x_{i_1}\dots x_{i_d}$, the function*

$$\begin{aligned}|g(\boldsymbol{x})|^2 &= \left(\sum_{1\le i_1\le\cdots\le i_d\le n}\overline{c_{i_1\dots i_d}x_{i_1}\dots x_{i_d}}\right)\left(\sum_{1\le j_1\le\cdots\le j_d\le n} c_{j_1\dots j_d}x_{j_1}\dots x_{j_d}\right)\\ &= \sum_{1\le i_1\le\cdots\le i_d\le n, 1\le j_1\le\cdots\le j_d\le n}\left(\overline{c_{i_1\dots i_d}}\cdot c_{j_1\dots j_d}\right)\overline{x_{i_1}\dots x_{i_d}}x_{j_1}\dots x_{j_d}\end{aligned}$$

is a $2d$-th degree symmetric conjugate form.

Notice that $|g(\boldsymbol{x})|^2$ is actually a real-valued conjugate polynomial. Later in Section 3 we shall show that a symmetric conjugate form $f_S(\boldsymbol{x})$ in (2) always takes real values if and only if the coefficients of any pair of conjugate monomials $\overline{x_{i_1}\dots x_{i_d}}x_{j_1}\dots x_{j_d}$ and $\overline{x_{j_1}\dots x_{j_d}}x_{i_1}\dots x_{i_d}$ are conjugate to each other, i.e.,

$$a_{i_1\dots i_d, j_1\dots j_d} = \overline{a_{j_1\dots j_d, i_1\dots i_d}} \quad \forall\, 1\le i_1\le\cdots\le i_d\le n,\ 1\le j_1\le\cdots\le j_d\le n.$$

As any complex form uniquely defines a symmetric complex tensor and vice versa, we observe a class of tensors representable for symmetric conjugate forms. A $2d$-th order tensor $\mathcal{F}\in\mathbb{C}^{n^{2d}}$ is called *partial-symmetric* if for any $1\le i_1\le\cdots\le i_d\le n$, $1\le i_{d+1}\le\cdots\le i_{2d}\le n$

$$\mathcal{F}_{j_1\dots j_d j_{d+1}\dots j_{2d}} = \mathcal{F}_{i_1\dots i_d i_{d+1}\dots i_{2d}} \quad \forall\,(j_1\dots j_d)\in\Pi(i_1\dots i_d),\ (j_{d+1}\dots j_{2d})\in\Pi(i_{d+1}\dots i_{2d}). \qquad (3)$$

We remark that the so-called partial-symmetricity is a concept first studied in [11, Section 2.1] in the framework of mixed polynomial forms, i.e., for any fixed first d indices of the tensor, it is symmetric with respective to its last d indices, and vise versa. It is clear that partial-symmetricity (3) is weaker than the usual symmetricity for tensors.

Let us formally define the bijection $\mathbf{S}$ (taking the first initial of symmetric conjugate forms) between symmetric conjugate forms and partial-symmetric complex tensors, as follows:

- $\mathbf{S}(\mathcal{F}) = f_S$: Given a partial-symmetric tensor $\mathcal{F}\in\mathbb{C}^{n^{2d}}$ with its associated multilinear form $\mathcal{F}(\boldsymbol{x}^1,\dots,\boldsymbol{x}^{2d})$, the symmetric conjugate form is defined as

$$f_S(\boldsymbol{x}) = \mathcal{F}(\underbrace{\overline{\boldsymbol{x}},\dots,\overline{\boldsymbol{x}}}_{d},\underbrace{\boldsymbol{x},\dots,\boldsymbol{x}}_{d}) = \sum_{1\le i_1,\dots,i_{2d}\le n}\mathcal{F}_{i_1\dots i_d i_{d+1}\dots i_{2d}}\overline{x_{i_1}\dots x_{i_d}}x_{i_{d+1}}\dots x_{i_{2d}}.$$

- $\mathbf{S}^{-1}(f_S) = \mathcal{F}$: Given a symmetric conjugate form f_S (2), the components of the partial-symmetric tensor $\mathcal{F}\in\mathbb{C}^{n^{2d}}$ are defined by

$$\begin{aligned}\mathcal{F}_{j_1\dots j_d j_{d+1}\dots j_{2d}} = \frac{a_{i_1\dots i_d, i_{d+1}\dots i_{2d}}}{|\Pi(i_1\dots i_d)|\cdot|\Pi(i_{d+1}\dots i_{2d})|} \quad &\forall\, 1\le i_1\le\cdots\le i_d\le n,\\ &1\le i_{d+1}\le\cdots\le i_{2d}\le n,\\ &(j_1\dots j_d)\in\Pi(i_1\dots i_d),\\ &(j_{d+1}\dots j_{2d})\in\Pi(i_{d+1}\dots i_{2d}).\end{aligned} \qquad (4)$$

According to the mappings defined above, the following result readily follows.

Lemma 2.2 *The bijection* $\mathbf{S}$ *is well-defined, i.e., any* n*-dimensional* $2d$*-th order partial-symmetric tensor* $\mathcal{F} \in \mathbb{C}^{n^{2d}}$ *uniquely defines an* n*-dimensional* $2d$*-th degree symmetric conjugate form, and vice versa.*

2.2 General conjugate forms and their tensor representations

In (2), for each monomial the numbers of conjugate variables and the original variables are always equal. This restriction can be further removed. We call the following class of conjugate polynomials to be *general conjugate forms*:

$$f_G(\boldsymbol{x}) = \sum_{k=0}^{d} \sum_{1 \le i_1 \le \cdots \le i_k \le n, 1 \le j_1 \le \cdots \le j_{d-k} \le n} a_{i_1 \ldots i_k, j_1 \ldots j_{d-k}} \overline{x_{i_1} \ldots x_{i_k}} x_{j_1} \ldots x_{j_{d-k}}. \tag{5}$$

Essentially, $f_G(\boldsymbol{x})$ is the summation of all the possible d-th degree monomials, allowing any number of conjugate variables as well as the original variables in each monomial. Here the subscript 'G' stands for 'general'. Obviously $f_S(\boldsymbol{x})$ is a special case of $f_G(\boldsymbol{x})$, and $f_G(\boldsymbol{x})$ is a special case of $f_C(\boldsymbol{x})$.

In Section 3 we shall show that a general conjugate form $f_G(\boldsymbol{x})$ which will always take real values for all $\boldsymbol{x}$ if and only if the coefficients of each pair of conjugate monomials are conjugate to each other. To this end, below we shall explicitly treat the conjugate variables as new variables.

- $\mathbf{G}(\mathcal{F}) = f_G$: Given a symmetric tensor $\mathcal{F} \in \mathbb{C}^{(2n)^d}$ with its associated multilinear form $\mathcal{F}(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^d)$, the general conjugate form of $\boldsymbol{x} \in \mathbb{C}^n$ is defined as

$$f_G(\boldsymbol{x}) = \mathcal{F}\Bigg(\underbrace{\begin{pmatrix} \overline{\boldsymbol{x}} \\ \boldsymbol{x} \end{pmatrix}, \ldots, \begin{pmatrix} \overline{\boldsymbol{x}} \\ \boldsymbol{x} \end{pmatrix}}_{d}\Bigg). \tag{6}$$

- $\mathbf{G}^{-1}(f_G) = \mathcal{F}$: Given a general conjugate form f_G of $\boldsymbol{x} \in \mathbb{C}^n$ as (5), the components of the symmetric tensor $\mathcal{F} \in \mathbb{C}^{(2n)^d}$ are defined as follows: for any $1 \le j_1, \ldots, j_d \le 2n$, sort them in a nondecreasing order as $1 \le i_1 \le \cdots \le i_d \le 2n$ and let $k = \max_{1 \le j \le d}\{i_j \le n\}$, then

$$\mathcal{F}_{j_1 \ldots j_d} = \frac{a_{i_1 \ldots i_k, (i_{k+1}-n) \ldots (i_d - n)}}{\Pi(i_1 \ldots i_d)}.$$

Similar as Lemma 2.2, the following is easily verified; we leave its proof to the interested readers.

Lemma 2.3 *The bijection* $\mathbf{G}$ *is well-defined, i.e., any* $2n$*-dimensional* d*-th order symmetric tensor* $\mathcal{F} \in \mathbb{C}^{(2n)^d}$ *uniquely defines an* n*-dimensional* d*-th degree general conjugate form, and vice versa.*

To conclude this section we remark that a partial-symmetric tensor (representation for a symmetric conjugate form) is less restrictive than a symmetric tensor (representation for a general conjugate form), while a symmetric conjugate form is a special case of a general conjugate form. One should note that the dimensions of these two tensor representations are actually different.

3 Real-valued conjugate forms and their tensor representations

In this section, we study the two types of conjugate complex forms introduced in Section 2: symmetric conjugate forms and general conjugate forms.

3.1 Real-valued conjugate polynomials

Let us first focus on polynomials, and present the following general result.

Theorem 3.1 *A conjugate complex polynomial function is real-valued if and only if the coefficients of any pair of its conjugate monomials are conjugate to each other, i.e., any two monomials* $au_C(\boldsymbol{x})$ *and* $bv_C(\boldsymbol{x})$ *with* a *and* b *being their coefficients satisfying* $\overline{u_C(\boldsymbol{x})} = v_C(\boldsymbol{x})$ *must follow that* $\overline{a} = b$.

Specifically, applying Theorem 3.1 to the two classes of conjugate forms that we just introduced, the conditions for them to always take real values can now be characterized:

Corollary 3.2 *A symmetric conjugate form*

$$f_S(\boldsymbol{x}) = \sum_{1\le i_1\le\dots\le i_d\le n, 1\le j_1\le\dots\le j_d\le n} a_{i_1\dots i_d, j_1\dots j_d}\overline{x_{i_1}\dots x_{i_d}}x_{j_1}\dots x_{j_d}$$

is real-valued if and only if

$$a_{i_1\dots i_d, j_1\dots j_d} = \overline{a_{j_1\dots j_d, i_1\dots i_d}} \quad \forall\, 1\le i_1\le\dots\le i_d\le n,\ 1\le j_1\le\dots\le j_d\le n. \tag{7}$$

A general conjugate form

$$f_G(\boldsymbol{x}) = \sum_{k=0}^{d}\sum_{1\le i_1\le\dots\le i_k\le n, 1\le j_1\le\dots\le j_{d-k}\le n} a_{i_1\dots i_k, j_1\dots j_{d-k}}\overline{x_{i_1}\dots x_{i_k}}x_{j_1}\dots x_{j_{d-k}}$$

is real-valued if and only if

$$a_{i_1\dots i_k, j_1\dots j_{d-k}} = \overline{a_{j_1\dots j_{d-k}, i_1\dots i_k}} \quad \forall\, 1\le i_1\le\dots\le i_k\le n,\ 1\le j_1\le\dots\le j_{d-k}\le n,\ 0\le k\le d.$$

Before getting into the technical proof of Theorem 3.1, let us first present an alternative representation of real-valued symmetric conjugate forms, as a consequence of Corollary 3.2.

Proposition 3.3 *A symmetric conjugate form* $f_S(\boldsymbol{x})$ *is real-valued if and only if*

$$f_S(\boldsymbol{x}) = \sum_{k\in K}\alpha_k|g_k(\boldsymbol{x})|^2,$$

where $g_k(\boldsymbol{x})$ *is a complex form and* $\alpha_k\in\mathbb{R}$ *for all* $k\in K$.

Proof. The 'if' part is trivial. Next we prove the 'only if' part of the proposition. If $f_S(\boldsymbol{x})$ is real-valued, by Corollary 3.2 we have (7). Then for any $1\le i_1\le\dots\le i_d\le n$ and $1\le j_1\le\dots\le j_d\le n$, the sum of the conjugate pair satisfies

$$\begin{aligned}
&a_{i_1\dots i_d, j_1\dots j_d}\overline{x_{i_1}\dots x_{i_d}}x_{j_1}\dots x_{j_d} + a_{j_1\dots j_d, i_1\dots i_d}\overline{x_{j_1}\dots x_{j_d}}x_{i_1}\dots x_{i_d}\\
=\ &a_{i_1\dots i_d, j_1\dots j_d}\overline{x_{i_1}\dots x_{i_d}}x_{j_1}\dots x_{j_d} + \overline{a_{i_1\dots i_d, j_1\dots j_d}x_{j_1}\dots x_{j_d}}x_{i_1}\dots x_{i_d}\\
=\ &|x_{i_1}\dots x_{i_d} + a_{i_1\dots i_d, j_1\dots j_d}x_{j_1}\dots x_{j_d}|^2 - |x_{i_1}\dots x_{i_d}|^2 - |a_{i_1\dots i_d j_1\dots j_d}x_{j_1}\dots x_{j_d}|^2.
\end{aligned}$$

Summing up all such pairs, the conclusion follows. □

Similarly we have the following result for general conjugate forms.

Proposition 3.4 *A general conjugate form $f_G(\boldsymbol{x})$ is real-valued if and only if*

$$f_G(\boldsymbol{x}) = \sum_{k\in K} \alpha_k |g_k(\boldsymbol{x})|^2,$$

where $g_k(\boldsymbol{x})$ is a complex polynomial and $\alpha_k \in \mathbb{R}$ for all $k \in K$.

Let us now turn to proving Theorem 3.1. We first show the 'if' part of the theorem, which is quite straightforward. To see this, for any pair of conjugate monomials (including self-conjugate monomial as a special case) of a conjugate complex polynomial: $au_C(\boldsymbol{x})$ and $bv_C(\boldsymbol{x})$ with $a, b \in \mathbb{C}$ being their coefficients and $\overline{u_C(\boldsymbol{x})} = v_C(\boldsymbol{x})$, if $\overline{a} = b$, then

$$\overline{au_C(\boldsymbol{x}) + bv_C(\boldsymbol{x})} = \overline{au_C(\boldsymbol{x}) + \overline{a}\overline{u_C(\boldsymbol{x})}} = au_C(\boldsymbol{x}) + \overline{a}\overline{u_C(\boldsymbol{x})} = au_C(\boldsymbol{x}) + bv_C(\boldsymbol{x}),$$

implying that $au_C(\boldsymbol{x}) + bv_C(\boldsymbol{x})$ is real-valued. Since all the conjugate monomials of a conjugate complex polynomial can be partitioned by conjugate pairs and self-conjugate monomials, the result follows immediately.

To proceed to the 'only if' part of the theorem, let us consider an easier case of the univariate conjugate polynomials.

Lemma 3.5 *A univariate conjugate complex polynomial $\sum_{\ell=0}^{d}\sum_{k=0}^{\ell} b_{k,\ell-k}\overline{x}^k x^{\ell-k} = 0$ for all $x \in \mathbb{C}$ if and only if all its coefficients are zeros, i.e., $b_{k,\ell-k} = 0$ for all $0 \le \ell \le d$ and $0 \le k \le \ell$.*

Proof. Let $x = \rho e^{\mathbf{i}\theta}$, the identity can be rewritten as

$$\sum_{\ell=0}^{d}\left(\sum_{k=0}^{\ell} b_{k,\ell-k} e^{\mathbf{i}(\ell-2k)\theta}\right)\rho^\ell = 0 \quad \forall\, \rho \in (0,\infty),\ \theta \in [0, 2\pi). \tag{8}$$

For any fixed θ, the function can be viewed as a polynomial with respect to ρ. Therefore the coefficient of the highest degree monomial ρ^d must be zero, i.e.,

$$\sum_{k=0}^{d} b_{k,d-k} e^{\mathbf{i}(d-2k)\theta} = 0 \quad \forall\, \theta \in [0, 2\pi).$$

Consequently we have for any $\theta \in [0, 2\pi)$,

$$\sum_{k=0}^{d} \mathrm{Re}\,(b_{k,d-k}) \cos((d-2k)\theta) - \sum_{k=0}^{d} \mathrm{Im}\,(b_{k,d-k}) \sin((d-2k)\theta) = 0 \tag{9}$$

and

$$\sum_{k=0}^{d} \mathrm{Im}\,(b_{k,d-k}) \cos((d-2k)\theta) + \sum_{k=0}^{d} \mathrm{Re}\,(b_{k,d-k}) \sin((d-2k)\theta) = 0. \tag{10}$$

The first and second parts of (9) can be respectively simplified as

$$\sum_{k=0}^{d} \mathrm{Re}\,(b_{k,d-k}) \cos((d-2k)\theta) = \begin{cases} \sum_{k=0}^{\frac{d-1}{2}} \mathrm{Re}\,(b_{k,d-k} + b_{d-k,k}) \cos((d-2k)\theta) & d \text{ is odd} \\ \sum_{k=0}^{\frac{d-2}{2}} \mathrm{Re}\,(b_{k,d-k} + b_{d-k,k}) \cos((d-2k)\theta) + \mathrm{Re}\,(b_{d/2,d/2}) & d \text{ is even} \end{cases}$$

and

$$\sum_{k=0}^{d} \mathrm{Im}\,(b_{k,d-k}) \sin((d-2k)\theta) = \sum_{k=0}^{\lfloor\frac{d-1}{2}\rfloor} \mathrm{Im}\,(b_{k,d-k} - b_{d-k,k}) \sin((d-2k)\theta).$$

By the orthogonality of the trigonometric functions, it further leads to

$$\mathrm{Re}\,(b_{k,d-k}+b_{d-k,k}) = \mathrm{Im}\,(b_{k,d-k}-b_{d-k,k}) = 0 \quad \forall\, k=0,1,\dots,d.$$

Similarly, (10) implies

$$\mathrm{Re}\,(b_{k,d-k}-b_{d-k,k}) = \mathrm{Im}\,(b_{k,d-k}+b_{d-k,k}) = 0 \quad \forall\, k=0,1,\dots,d.$$

Combining the above two sets of identities yields

$$b_{k,d-k}=0 \quad \forall\, k=0,1,\dots,d.$$

The degree of the function in (8) (in terms of ρ) is then reduced by 1. The desired result can thus be proven by induction. □

Let us now extend Lemma 3.5 to the general multivariate conjugate polynomials.

Lemma 3.6 *An n-dimensional d-th degree conjugate complex polynomial*

$$\sum_{\ell=0}^{d}\sum_{k=0}^{\ell}\sum_{1\le i_1\le\dots\le i_k\le n,1\le j_1\le\dots\le j_{\ell-k}\le n} b_{i_1\dots i_k,j_1\dots j_{\ell-k}}\overline{x_{i_1}\dots x_{i_k}}x_{j_1}\dots x_{j_{\ell-k}}=0$$

for all $\boldsymbol{x}\in\mathbb{C}^n$ if and only if all its coefficients are zeros, i.e., $b_{i_1\dots i_k,j_1\dots j_{\ell-k}}=0$ for all $0\le\ell\le d$, $0\le k\le\ell$, $1\le i_1\le\dots\le i_k\le n$, $1\le j_1\le\dots\le j_{d-k}\le n$.

Proof. We shall prove the result by induction on the dimension n. The case $n=1$ is already shown in Lemma 3.5. Suppose the claim holds for all positive integers no more than $n-1$. Then for the dimension n, the conjugate polynomial $f_C(\boldsymbol{x})$ can be rewritten according to the degrees of $\overline{x_1}$ and x_1 as

$$f_C(\boldsymbol{x})=\sum_{\ell=0}^{d}\sum_{k=0}^{\ell}\overline{x_1}^{k}{x_1}^{\ell-k}g_C^{\ell k}(x_2,\dots,x_n).$$

For any given $x_2,\dots,x_n\in\mathbb{C}$, taking f_C as a univariate conjugate polynomial of x_1, by Lemma 3.5 we have

$$g_C^{\ell k}(x_2,\dots,x_n)=0 \quad \forall\, 0\le\ell\le d,\ 0\le k\le\ell.$$

For any given (ℓ,k), as $g_C^{\ell k}(x_2,\dots,x_n)$ is a conjugate polynomial of dimension at most $n-1$, by the induction hypothesis all the coefficients of $g_C^{\ell k}$ are zeros. Observing that all the coefficients of f_C are distributed in the coefficients of $g_C^{\ell k}$ for all (ℓ,k), the claimed result is proven for the dimension n. The proof is thus complete by induction. □

With Lemma 3.6 at hand, we can finally complete the 'only if' part of Theorem 3.1. Suppose a conjugate polynomial $f(\boldsymbol{x})$ is real-valued for all $\boldsymbol{x}\in\mathbb{C}^n$. Clearly we have $f(\boldsymbol{x})-\overline{f(\boldsymbol{x})}=0$ for all $\boldsymbol{x}\in\mathbb{C}^n$, i.e.,

$$\sum_{\ell=0}^{d}\sum_{k=0}^{\ell}\sum_{1\le i_1\le\dots\le i_k\le n,1\le j_1\le\dots\le j_{\ell-k}\le n}\left(b_{i_1\dots i_k,j_1\dots j_{\ell-k}}-\overline{b_{j_1\dots j_{\ell-k},i_1\dots i_k}}\right)\overline{x_{i_1}\dots x_{i_k}}x_{j_1}\dots x_{j_{\ell-k}}.$$

By Lemma 3.6 it follows that $b_{i_1\dots i_k,j_1\dots j_{\ell-k}}-\overline{b_{j_1\dots j_{\ell-k},i_1\dots i_k}}=0$ for all $0\le\ell\le d$, $0\le k\le\ell$, $1\le i_1\le\dots\le i_k\le n$, $1\le j_1\le\dots\le j_{d-k}\le n$, proving the 'only if' part of Theorem 3.1.

With Theorem 3.1, in particular Corollary 3.2, we are in a position to characterize the tensor representations for real-valued conjugate forms.

3.2 Conjugate partial-symmetric tensors

As any symmetric conjugate form uniquely defines a partial-symmetric tensor (Lemma 2.2), it is interesting to see more structured tensor representations for real-valued symmetric conjugate forms.

Definition 3.7 *A 2d-th order tensor $\mathcal{F} \in \mathbb{C}^{n^{2d}}$ is called conjugate partial-symmetric if*
(1) $\mathcal{F}_{i_1 \dots i_d i_{d+1} \dots i_{2d}} = \mathcal{F}_{j_1 \dots j_d j_{d+1} \dots j_{2d}}\ \forall\, (j_1 \dots j_d) \in \Pi(i_1 \dots i_d),\ (j_{d+1} \dots j_{2d}) \in \Pi(i_{d+1} \dots i_{2d})$, and
(2) $\mathcal{F}_{i_1 \dots i_d i_{d+1} \dots i_{2d}} = \overline{\mathcal{F}_{i_{d+1} \dots i_{2d} i_1 \dots i_d}}$
hold for any $1 \le i_1 \le \dots \le i_d \le n$, $1 \le i_{d+1} \le \dots \le i_{2d} \le n$.

We remark that when $d = 1$, a conjugate partial-symmetric tensor is simply a Hermitian matrix. The conjugate partial-symmetric tensors and the real-valued symmetric conjugate forms are connected as follows.

Lemma 3.8 *Any n-dimensional $2d$-th order conjugate partial-symmetric tensor $\mathcal{F} \in \mathbb{C}^{n^{2d}}$ uniquely defines (under $\mathbf{S}$) an n-dimensional $2d$-th degree real-valued symmetric conjugate form, and vice versa (under $\mathbf{S}^{-1}$).*

Proof. For any conjugate partial-symmetric tensor $\mathcal{F}$, $f_S = \mathbf{S}(\mathcal{F})$ satisfies

$$\begin{aligned}\overline{f_S(\boldsymbol{x})} = \overline{\mathcal{F}(\underbrace{\overline{\boldsymbol{x}}, \dots, \overline{\boldsymbol{x}}}_{d}, \underbrace{\boldsymbol{x}, \dots, \boldsymbol{x}}_{d})} &= \sum_{1 \le i_1, \dots, i_{2d} \le n} \overline{\mathcal{F}_{i_1 \dots i_d i_{d+1} \dots i_{2d}} \overline{x_{i_1} \dots x_{i_d}} x_{i_{d+1}} \dots x_{i_{2d}}} \\ &= \sum_{1 \le i_1, \dots, i_{2d} \le n} \overline{\mathcal{F}_{i_1 \dots i_d i_{d+1} \dots i_{2d}}} x_{i_1} \dots x_{i_d} \overline{x_{i_{d+1}} \dots x_{i_{2d}}} \\ &= \sum_{1 \le i_1, \dots, i_{2d} \le n} \mathcal{F}_{i_{d+1} \dots i_{2d} i_1 \dots i_d} \overline{x_{i_{d+1}} \dots x_{i_{2d}}} x_{i_1} \dots x_{i_d} \\ &= f_S(\boldsymbol{x}),\end{aligned}$$

implying that f_S is real-valued.

On the other hand, for any real-valued symmetric conjugate form $f_S(\boldsymbol{x})$ in (2), it follows from Corollary 3.2 that $a_{i_1 \dots i_d, j_1 \dots j_d} = \overline{a_{j_1 \dots j_d, i_1 \dots i_d}}$ for all possible $(i_1, \dots, i_d, j_1, \dots, j_d)$. By (4), its tensor representation $\mathcal{F} = \mathbf{S}^{-1}(f_S)$ with

$$\mathcal{F}_{i_1 \dots i_d i_{d+1} \dots i_{2d}} = \frac{a_{i_1 \dots i_d, i_{d+1} \dots i_{2d}}}{|\Pi(i_1 \dots i_d)| \cdot |\Pi(i_{d+1} \dots i_{2d})|}$$

satisfies the 2nd condition in Definition 3.7, proving the conjugate partial-symmetricity of $\mathcal{F}$. □

Below is a useful property of the conjugate partial-symmetric tensors.

Lemma 3.9 *For any conjugate partial-symmetric tensor $\mathcal{F} \in \mathbb{C}^{n^{2d}}$ and vectors $\boldsymbol{x}^1, \dots, \boldsymbol{x}^{d-1} \in \mathbb{C}^n$,*

$$f_C(\boldsymbol{z}) := \mathcal{F}(\overline{\boldsymbol{z}}, \overline{\boldsymbol{x}^1}, \dots, \overline{\boldsymbol{x}^{d-1}}, \boldsymbol{z}, \boldsymbol{x}^1, \dots, \boldsymbol{x}^{d-1})$$

is a Hermitian quadratic form of the variable $\boldsymbol{z} \in \mathbb{C}^n$, i.e., the matrix

$$Q := \mathcal{F}(\bullet, \overline{\boldsymbol{x}^1}, \dots, \overline{\boldsymbol{x}^{d-1}}, \bullet, \boldsymbol{x}^1, \dots, \boldsymbol{x}^{d-1})$$

is a Hermitian matrix.

Proof. To prove $Q_{ij} = \overline{Q_{ji}}$ for all $1 \le i, j \le n$ we only need to show

$$\mathcal{F}(\boldsymbol{e}^i, \overline{\boldsymbol{x}^1}, \dots, \overline{\boldsymbol{x}^{d-1}}, \boldsymbol{e}^j, \boldsymbol{x}^1, \dots, \boldsymbol{x}^{d-1}) = \overline{\mathcal{F}(\boldsymbol{e}^j, \overline{\boldsymbol{x}^1}, \dots, \overline{\boldsymbol{x}^{d-1}}, \boldsymbol{e}^i, \boldsymbol{x}^1, \dots, \boldsymbol{x}^{d-1})},$$

where $\boldsymbol{e}^i$ denotes the vector whose i-th component is 1 and others are zeros for $i = 1, \dots, n$. In fact, according to Definition 3.7

$$\begin{aligned}
\mathcal{F}(\boldsymbol{e}^i, \overline{\boldsymbol{x}^1}, \dots, \overline{\boldsymbol{x}^{d-1}}, \boldsymbol{e}^j, \boldsymbol{x}^1, \dots, \boldsymbol{x}^{d-1}) &= \sum_{1 \le i_1, \dots, i_{d-1}, j_1, \dots, j_{d-1} \le n} \mathcal{F}_{i i_1 \dots i_{d-1} j j_1 \dots j_{d-1}} \overline{\boldsymbol{x}^1_{i_1}} \dots \overline{\boldsymbol{x}^{d-1}_{i_{d-1}}} \boldsymbol{x}^1_{j_1} \dots \boldsymbol{x}^{d-1}_{j_{d-1}} \\
&= \sum_{1 \le j_1, \dots, j_{d-1}, i_1, \dots, i_{d-1} \le n} \overline{\mathcal{F}_{j j_1 \dots j_{d-1} i i_1 \dots i_{d-1}}} \boldsymbol{x}^1_{j_1} \dots \boldsymbol{x}^{d-1}_{j_{d-1}} \overline{\boldsymbol{x}^1_{i_1}} \dots \overline{\boldsymbol{x}^{d-1}_{i_{d-1}}} \\
&= \sum_{1 \le j_1, \dots, j_{d-1}, i_1, \dots, i_{d-1} \le n} \overline{\mathcal{F}_{j j_1 \dots j_{d-1} i i_1 \dots i_{d-1}} \overline{\boldsymbol{x}^1_{j_1}} \dots \overline{\boldsymbol{x}^{d-1}_{j_{d-1}}} \boldsymbol{x}^1_{i_1} \dots \boldsymbol{x}^{d-1}_{i_{d-1}}} \\
&= \overline{\mathcal{F}(\boldsymbol{e}^j, \overline{\boldsymbol{x}^1}, \dots, \overline{\boldsymbol{x}^{d-1}}, \boldsymbol{e}^i, \boldsymbol{x}^1, \dots, \boldsymbol{x}^{d-1})}.
\end{aligned}$$

□

In general, it can be shown for any conjugate partial-symmetric tensor $\mathcal{F} \in \mathbb{C}^{n^{2d}}$ and vectors $\boldsymbol{x}^1, \dots, \boldsymbol{x}^t \in \mathbb{C}^n$ with $1 \le t < d$ that

$$\mathcal{F}(\underbrace{\bullet, \dots, \bullet}_{d-t}, \overline{\boldsymbol{x}^1}, \dots, \overline{\boldsymbol{x}^t}, \underbrace{\bullet, \dots, \bullet}_{d-t}, \boldsymbol{x}^1, \dots, \boldsymbol{x}^t)$$

is a conjugate partial-symmetric tensor in $\mathbb{C}^{n^{2d-2t}}$.

3.3 Conjugate super-symmetric tensors

Similar as for the real-valued symmetric conjugate forms, we have the following tensor representations for the real-valued general conjugate forms.

Definition 3.10 *A $2n$-dimensional tensor $\mathcal{F} \in \mathbb{C}^{(2n)^d}$ is called conjugate super-symmetric if*
(1) $\mathcal{F}$ is symmetric, i.e., $\mathcal{F}_{i_1 \dots i_d} = \mathcal{F}_{j_1 \dots j_d}$ $\forall (j_1 \dots j_d) \in \Pi(i_1 \dots i_d)$, and
(2) $\mathcal{F}_{j_1 \dots j_d} = \overline{\mathcal{F}_{i_1 \dots i_d}}$ if $|i_k - j_k| = n$ holds for all $1 \le k \le d$
hold for any $1 \le i_1 \le \dots \le i_d \le n$.

Clearly, the conjugate super-symmetricity is stronger than the ordinary symmetricity for complex tensors. Under the mapping $\mathbf{G}$ defined in Section 2.2, we have the following tensor representations for the real-valued general conjugate forms.

Proposition 3.11 *Any $2n$-dimensional d-th order conjugate super-symmetric tensor $\mathcal{F} \in \mathbb{C}^{(2n)^d}$ uniquely defines (under $\mathbf{G}$) an n-dimensional d-th degree real-valued general conjugate form, and vice versa (under $\mathbf{G}^{-1}$).*

4 Nonnegative real-valued conjugate polynomials

An important aspect of polynomials is the theory of nonnegativity. However, most existing results only apply for polynomials in real variables, for the reason that such polynomials are real-valued. Since we have introduced several classes of complex polynomials which are real-valued, the question about their nonnegativity naturally arises. In particular, in this section, we study the relationship between nonnegativity and sums of squares (SOS) for univariate conjugate polynomials. In the real domain, this problem was completely solved by Hilbert [9] in 1888, where the only three general classes of real polynomials whose nonnegativity is equivalent to SOS are: (1) univariate polynomials, (2) quadratic polynomials, and (3) bivariate quartic polynomials. However, relationship between nonnegative complex polynomials and SOS has not been established explicitly in the literature as far as we know. This section aims to fill in this gap, using the notion of conjugate polynomials. To begin with, let us start with the definitions of nonnegativity and SOS for a conjugate complex polynomial.

Definition 4.1 *A conjugate complex polynomial $f_C(\boldsymbol{x})$ is nonnegative if*

$$\operatorname{Re} f_C(\boldsymbol{x}) \geq 0 \quad \forall\, \boldsymbol{x} \in \mathbb{C}^n.$$

In particular, a real-valued conjugate polynomial $f_C(\boldsymbol{x})$ is nonnegative if $f_C(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \mathbb{C}^n$.

Definition 4.2 *A conjugate complex polynomial $f_C(\boldsymbol{x})$ is SOS if there exist conjugate complex polynomials $g_C^1, \dots, g_C^m$ such that*

$$\operatorname{Re} f_C(\boldsymbol{x}) = \sum_{i=1}^{m} |g_C^i(\boldsymbol{x})|^2.$$

In particular, a real-valued conjugate polynomial $f_C(\boldsymbol{x})$ is SOS if $f_C(\boldsymbol{x}) = \sum_{i=1}^m |g_C^i(\boldsymbol{x})|^2$.

Our next proposition states that it is sufficient to focus on the real-valued conjugate polynomials as long as the nonnegativity is concerned.

Proposition 4.3 *Any conjugate complex polynomial $f_C(\boldsymbol{x})$ can be uniquely written as $g_C(\boldsymbol{x}) + \mathbf{i} h_C(\boldsymbol{x})$ where both g_C and h_C are real-valued conjugate polynomials.*

Proof. As any conjugate complex polynomial can be partitioned by pairs of conjugate monomials and self-conjugate monomials, it suffices to rewrite the summation of a pair of conjugate monomials (a self-conjugate monomial can be split into two halves and then taken as a pair). Let $au_C(\boldsymbol{x})$ and $bv_C(\boldsymbol{x})$ be a pair of conjugate monomials of f_C with $a, b \in \mathbb{C}$ being their coefficients and $\overline{u_C(\boldsymbol{x})} = v_C(\boldsymbol{x})$. Denote

$$(\alpha, \beta) = \left(\frac{a + \overline{b}}{2}, \frac{a - \overline{b}}{2\mathbf{i}} \right),$$

and so

$$au_C(\boldsymbol{x}) + bv_C(\boldsymbol{x}) = (\alpha u_C(\boldsymbol{x}) + \overline{\alpha} v_C(\boldsymbol{x})) + \mathbf{i} \left(\beta u_C(\boldsymbol{x}) + \overline{\beta} v_C(\boldsymbol{x}) \right),$$

where $\alpha u_C(\boldsymbol{x}) + \overline{\alpha} v_C(\boldsymbol{x})$ and $\beta u_C(\boldsymbol{x}) + \overline{\beta} v_C(\boldsymbol{x})$ are both real-valued by Theorem 3.1. Therefore the decomposition of f_C as real and complex parts is constructed. The uniqueness of the decomposition is obvious. $\square$

Let us now focus on univariate conjugate polynomials, i.e.,

$$h_C(x) = \sum_{\ell=0}^{d} \sum_{k=0}^{\ell} a_{k,\ell-k} \overline{x}^k x^{\ell-k}. \tag{11}$$

It is obvious that SOS implies nonnegativity, but the other way round implication is the topic of study in this section. Under certain circumstance, the equivalence between SOS and nonnegativity can be achieved, such as Hilbert's classical results in the real domain as we mentioned earlier. In particular, in the real domain a univariate polynomial is nonnegative if and only if it is SOS representable, while this equivalence does not hold anymore for bivariate polynomials and beyond. Another example is the so-called Riesz-Féjer theorem (see e.g. [34]):

Theorem 4.4 (Riesz-Féjer) *A univariate complex polynomial* $\operatorname{Re} h(x) \geq 0$ *for all* $|x| = 1$ *if and only if there exist* $c_0, c_1, \ldots, c_d \in \mathbb{C}$ *such that*

$$\operatorname{Re} h(x) = \left| \sum_{k=0}^{d} c_k x^k \right|^2.$$

This states a special class of univariate complex polynomials (also a special class of univariate conjugate polynomials) whose nonnegativity is equivalent to SOS if the variable $x \in \mathbb{C}$ lies on the unit circle of the complex plane. Although the real part of any univariate complex polynomial of x can be viewed as a special real polynomial with two variables $(\operatorname{Re} x, \operatorname{Im} x)$, the relationship between nonnegativity and SOS remains unclear under this light. To further study this problem for the univariate conjugate polynomial (11), let us first generalize Theorem 4.4.

Proposition 4.5 *A univariate conjugate polynomial* $\operatorname{Re} h_C(x) \geq 0$ *for all* $|x| = 1$ *if and only if there exist* $c_0, c_1, \ldots, c_d \in \mathbb{C}$ *such that*

$$\operatorname{Re} h_C(x) = \left| \sum_{k=0}^{d} c_k x^k \right|^2.$$

Proof. The 'if' part is trivial. Let us prove the 'only if' part. Since $|x| = 1$ we have $\overline{x} = x^{-1}$ and then $\overline{x}^k x^{\ell-k} = x^{\ell-2k}$. Consequently,

$$\operatorname{Re} h_C(x) = \operatorname{Re} \left(\sum_{\ell=0}^{d} \sum_{k=0}^{\ell} a_{k,\ell-k} x^{\ell-2k} \right) = \operatorname{Re} \left(\sum_{\ell=-d}^{d} b_\ell x^\ell \right) = \sum_{\ell=1}^{d} \operatorname{Re} \left((b_\ell + \overline{b_{-\ell}}) x^\ell \right) + \operatorname{Re} b_0,$$

where

$$b_\ell = \begin{cases} \sum_{k=0}^{\lfloor \frac{d-\ell}{2} \rfloor} a_{k,\ell+k} & \ell \geq 0, \\ \sum_{k=0}^{\lfloor \frac{d+\ell}{2} \rfloor} a_{k-\ell,k} & \ell < 0. \end{cases}$$

Let us define $g(x) = \sum_{\ell=1}^{d} (b_\ell + \overline{b_{-\ell}}) x^\ell + b_0$, which is a univariate complex polynomial. If $\operatorname{Re} g(x) = \operatorname{Re} h_C(x) \geq 0$ for all $|x| = 1$, by applying Riesz-Féjer theorem (Theorem 4.4) on $g(x)$, there exist $c_0, c_1, \ldots, c_d \in \mathbb{C}$ such that

$$\operatorname{Re} h_C(x) = \operatorname{Re} g(x) = \left| \sum_{k=0}^{d} c_k x^k \right|^2,$$

proving the 'only if' part. □

However if we do drop the constraint $|x| = 1$ in Proposition 4.5, the equivalence does not hold for univariate conjugate polynomials in general.

Theorem 4.6 *For a univariate conjugate polynomial* $h_C(x)$ *defined by* (11), $\operatorname{Re} h_C(x) \geq 0$ *for all* $x \in \mathbb{C}$ *does not imply that it is SOS representable.*

Proof. Suppose on the contrary, for any nonnegative univariate conjugate polynomial $h_C(x)$, there exist univariate conjugate polynomials $g_C^1, \dots, g_C^m$ such that

$$\operatorname{Re} h_C(x) = \sum_{i=1}^m |g_C^i(x)|^2.$$

Let us consider a general nonnegative bivariate real polynomial

$$p(y,z) = \sum_{\ell=0}^d \sum_{k=0}^\ell p_{k,\ell-k} y^k z^{\ell-k} \ge 0 \quad \forall\, y,z \in \mathbb{R}.$$

Denote $x = (y + \mathbf{i}z)/2$, and so $y = x + \overline{x}$ and $z = -\mathbf{i}(x - \overline{x})$. Consequently $p(y,z)$ can be written as

$$p(y,z) = \sum_{\ell=0}^d \sum_{k=0}^\ell p_{k,\ell-k} (x+\overline{x})^k \left(-\mathbf{i}x + \mathbf{i}\overline{x}\right)^{\ell-k} := h_C(x).$$

It is obvious that $h_C(x)$ is a real-valued nonnegative univariate conjugate polynomial.

By the assumption, there exist univariate conjugate polynomials $g_C^1, \dots, g_C^m$ such that

$$p(y,z) = h_C(x) = \operatorname{Re} h_C(x) = \sum_{i=1}^m |g_C^i(x)|^2 = \sum_{i=1}^m \left((\operatorname{Re} g_C^i(x))^2 + (\operatorname{Im} g_C^i(x)^2) \right).$$

Notice that both $\operatorname{Re} g_C^i(x)$ and $\operatorname{Im} g_C^i(x)$ can be rewritten as real polynomials of the real variables (y,z) by replacing x with $(y+\mathbf{i}z)/2$. Therefore we get an SOS representation of any nonnegative bivariate real polynomial $p(y,z)$, which contradicts the fact that nonnegativity of a bivariate real polynomial is not necessarily SOS. □

To conclude this section, we present the following result extended from the real case, which states that any real convex form is nonnegative (see e.g. [25]).

Proposition 4.7 *Any real-valued convex general conjugate form $f_G(\boldsymbol{x})$ (including symmetric conjugate form $f_S(\boldsymbol{x})$ as a special case) is nonnegative.*

Proof. Suppose real-valued $f_G(\boldsymbol{x})$ is convex at any $\boldsymbol{x} \in \mathbb{C}^n$. Define $\hat{f}_{\boldsymbol{x},\boldsymbol{y}} : \mathbb{R} \to \mathbb{R}$ with $\hat{f}_{\boldsymbol{x},\boldsymbol{y}}(t) = f_G(\boldsymbol{x} + t\boldsymbol{y})$. It is well know in convex analysis that $\hat{f}_{\boldsymbol{x},\boldsymbol{y}}(t)$ is a convex function of $t \in \mathbb{R}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n$. Let the tensor representation of f_G, $\mathbf{G}^{-1}(f_G) = \mathcal{F}$, and by (6) we have

$$\hat{f}_{\boldsymbol{x},\boldsymbol{y}}(t) = \mathcal{F}\Bigg(\underbrace{\begin{pmatrix} \overline{\boldsymbol{x}+t\boldsymbol{y}} \\ \boldsymbol{x}+t\boldsymbol{y} \end{pmatrix}, \dots, \begin{pmatrix} \overline{\boldsymbol{x}+t\boldsymbol{y}} \\ \boldsymbol{x}+t\boldsymbol{y} \end{pmatrix}}_{d} \Bigg).$$

As $\mathcal{F}$ is symmetric, direct computation shows that

$$\hat{f}'_{\boldsymbol{x},\boldsymbol{y}}(t) = d\mathcal{F}\Bigg(\begin{pmatrix} \overline{\boldsymbol{y}} \\ \boldsymbol{y} \end{pmatrix}, \underbrace{\begin{pmatrix} \overline{\boldsymbol{x}+t\boldsymbol{y}} \\ \boldsymbol{x}+t\boldsymbol{y} \end{pmatrix}, \dots, \begin{pmatrix} \overline{\boldsymbol{x}+t\boldsymbol{y}} \\ \boldsymbol{x}+t\boldsymbol{y} \end{pmatrix}}_{d-1} \Bigg),$$

and furthermore

$$\hat{f}''_{\boldsymbol{x},\boldsymbol{y}}(t) = (d-1)d\mathcal{F}\Bigg(\begin{pmatrix} \overline{\boldsymbol{y}} \\ \boldsymbol{y} \end{pmatrix}, \begin{pmatrix} \overline{\boldsymbol{y}} \\ \boldsymbol{y} \end{pmatrix}, \underbrace{\begin{pmatrix} \overline{\boldsymbol{x}+t\boldsymbol{y}} \\ \boldsymbol{x}+t\boldsymbol{y} \end{pmatrix}, \dots, \begin{pmatrix} \overline{\boldsymbol{x}+t\boldsymbol{y}} \\ \boldsymbol{x}+t\boldsymbol{y} \end{pmatrix}}_{d-2} \Bigg) \ge 0$$

for all $t \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n$. In particular, by letting $t = 0$ and $\boldsymbol{y} = \boldsymbol{x}$ we get

$$\hat{f}''_{\boldsymbol{x},\boldsymbol{x}}(0) = (d-1)d\mathcal{F}\Bigg(\underbrace{\begin{pmatrix}\overline{\boldsymbol{x}}\\ \boldsymbol{x}\end{pmatrix}, \ldots, \begin{pmatrix}\overline{\boldsymbol{x}}\\ \boldsymbol{x}\end{pmatrix}}_{d}\Bigg) = (d-1)d\, f_G(\boldsymbol{x}) \geq 0$$

for all $\boldsymbol{x} \in \mathbb{C}^n$, proving the nonnegativity of $f_G(\boldsymbol{x})$. □

5 Eigenvalues and eigenvectors of complex tensors

As mentioned earlier, Lim [18] and Qi [21] independently proposed to systematically study the eigenvalues and eigenvectors for real tensors, though in their prior works De Lathauwer et al. [7] and Kofidis and Regalia [15] discussed various aspects of tensor eigenproblems already. Subsequently, the topic has attracted much attention due to the potential applications in magnetic resonance imaging, polynomial optimization theory, quantum physics, statistical data analysis, higher order Markov chains, and so on. After that, this study was also extended to complex tensors [22, 20, 8] without considering the conjugate variables. Zhang and Qi in [33] proposed the so-called Q-eigenvalues of complex tensors:

Definition 5.1 (Zhang and Qi [33]) *A scalar λ is called a Q-eigenvalue of a symmetric complex tensor $\mathcal{H}$, if there exists a vector $\boldsymbol{x}$ called Q-eigenvector, such that*

$$\begin{cases} \mathcal{H}(\bullet, \underbrace{\boldsymbol{x}, \ldots, \boldsymbol{x}}_{d-1}) = \lambda \overline{\boldsymbol{x}} \\ \boldsymbol{x}^{\mathrm{H}}\boldsymbol{x} = 1 \\ \lambda \in \mathbb{R}. \end{cases} \tag{12}$$

However, as the corresponding complex tensor does not have conjugate-type symmetricity, the eigenvalue defined above does not specialize to the classical eigenvalues of Hermitian matrices. In particular, $\lambda \in \mathbb{R}$ is put in the system (12). Now with all the new notions introduced in the previous sections—in particular the bijection between conjugate partial-symmetric tensors and real-valued symmetric conjugate forms, and the bijection between conjugate super-symmetric tensors and real-valued general conjugate forms—we are able to present new definitions and properties of eigenvalues for complex tensors, which are more naturally related to that of the Hermitian matrices.

5.1 Definitions and properties of eigenvalues

Let us first introduce two types of eigenvalues for conjugate partial-symmetric tensors and conjugate super-symmetric tensors.

Definition 5.2 *$\lambda \in \mathbb{C}$ is called C-eigenvalue of a conjugate partial-symmetric tensor $\mathcal{F}$, if there exists a vector $\boldsymbol{x} \in \mathbb{C}^n$ called C-eigenvector, such that*

$$\begin{cases} \mathcal{F}(\bullet, \underbrace{\overline{\boldsymbol{x}}, \ldots, \overline{\boldsymbol{x}}}_{d-1}, \underbrace{\boldsymbol{x}, \ldots, \boldsymbol{x}}_{d}) = \lambda \boldsymbol{x} \\ \boldsymbol{x}^{\mathrm{H}}\boldsymbol{x} = 1. \end{cases} \tag{13}$$

Definition 5.3 *$\lambda \in \mathbb{C}$ is called G-eigenvalue of a conjugate super-symmetric tensor $\mathcal{F}$, if there exists a vector $\boldsymbol{x} \in \mathbb{C}^n$ called G-eigenvector, such that*

$$\begin{cases} \mathcal{F}\left(\begin{pmatrix} \bullet \\ \bullet \end{pmatrix}, \underbrace{\begin{pmatrix} \overline{\boldsymbol{x}} \\ \boldsymbol{x} \end{pmatrix}, \dots, \begin{pmatrix} \overline{\boldsymbol{x}} \\ \boldsymbol{x} \end{pmatrix}}_{d-1} \right) = \lambda \begin{pmatrix} \boldsymbol{x} \\ \overline{\boldsymbol{x}} \end{pmatrix} \\ \boldsymbol{x}^{\mathrm{H}} \boldsymbol{x} = 1. \end{cases} \tag{14}$$

In fact, these two types of eigenvalues defined above are always real, although they are defined in the complex domain. This property generalizes the well-known property of the Hermitian matrices. In particular, Definition 5.2 includes eigenvalues of Hermitian matrices as a special case when $d = 1$.

Proposition 5.4 *Any C-eigenvalue of a conjugate partial-symmetric tensor is always real; so is any G-eigenvalue of a conjugate super-symmetric tensor.*

Proof. Suppose $(\lambda, \boldsymbol{x})$ is a C-eigenvalue and C-eigenvector pair of a conjugate partial-symmetric tensor $\mathcal{F}$. Multiplying $\overline{\boldsymbol{x}}$ on both sides of the first equation in (13), we get

$$\mathcal{F}(\underbrace{\overline{\boldsymbol{x}}, \dots, \overline{\boldsymbol{x}}}_{d}, \underbrace{\boldsymbol{x}, \dots, \boldsymbol{x}}_{d}) = \lambda \overline{\boldsymbol{x}}^{\mathrm{T}} \boldsymbol{x} = \lambda.$$

As $\mathcal{F}$ is conjugate partial-symmetric, the symmetric conjugate form $\mathcal{F}(\underbrace{\overline{\boldsymbol{x}}, \dots, \overline{\boldsymbol{x}}}_{d}, \underbrace{\boldsymbol{x}, \dots, \boldsymbol{x}}_{d})$ is real-valued, and so is λ.

Next, suppose $(\lambda, \boldsymbol{x})$ is a G-eigenvalue and G-eigenvector pair of a conjugate super-symmetric tensor $\mathcal{F}$. Multiplying $\binom{\overline{\boldsymbol{x}}}{\boldsymbol{x}}$ on both sides of the first equation in (14) yields

$$\mathcal{F}\left(\underbrace{\begin{pmatrix} \overline{\boldsymbol{x}} \\ \boldsymbol{x} \end{pmatrix}, \dots, \begin{pmatrix} \overline{\boldsymbol{x}} \\ \boldsymbol{x} \end{pmatrix}}_{d} \right) = \lambda \begin{pmatrix} \overline{\boldsymbol{x}} \\ \boldsymbol{x} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \boldsymbol{x} \\ \overline{\boldsymbol{x}} \end{pmatrix} = 2\lambda \boldsymbol{x}^{\mathrm{H}} \boldsymbol{x} = 2\lambda.$$

As $\mathcal{F}$ is conjugate super-symmetric, the general conjugate form $\mathcal{F}\left(\underbrace{\begin{pmatrix} \overline{\boldsymbol{x}} \\ \boldsymbol{x} \end{pmatrix}, \dots, \begin{pmatrix} \overline{\boldsymbol{x}} \\ \boldsymbol{x} \end{pmatrix}}_{d} \right)$ is real-valued, and so is λ. □

As a consequence of Proposition 5.4, one can define the C-eigenvalue $\lambda \in \mathbb{R}$ and its corresponding C-eigenvector $\boldsymbol{x} \in \mathbb{C}^n$ for a conjugate partial-symmetric tensor $\mathcal{F}$ equivalently as follows.

Proposition 5.5 *$\lambda \in \mathbb{C}$ is a C-eigenvalue of a conjugate partial-symmetric tensor $\mathcal{F}$, if and only if there exists a vector $\boldsymbol{x} \in \mathbb{C}^n$, such that*

$$\begin{cases} \mathcal{F}(\underbrace{\overline{\boldsymbol{x}}, \dots, \overline{\boldsymbol{x}}}_{d}, \underbrace{\boldsymbol{x}, \dots, \boldsymbol{x}}_{d-1}, \bullet) = \lambda \overline{\boldsymbol{x}} \\ \boldsymbol{x}^{\mathrm{H}} \boldsymbol{x} = 1. \end{cases} \tag{15}$$

Proof. Suppose a C-eigenvalue λ and C-eigenvector $\boldsymbol{x}$ satisfy (13). By Proposition 5.4 we know that $\lambda \in \mathbb{R}$. Therefore

$$\lambda \overline{\boldsymbol{x}} = \overline{\lambda \boldsymbol{x}} = \overline{\mathcal{F}(\bullet, \underbrace{\overline{\boldsymbol{x}}, \dots, \overline{\boldsymbol{x}}}_{d-1}, \underbrace{\boldsymbol{x}, \dots, \boldsymbol{x}}_{d})} = \overline{\mathcal{F}}(\bullet, \underbrace{\boldsymbol{x}, \dots, \boldsymbol{x}}_{d-1}, \underbrace{\overline{\boldsymbol{x}}, \dots, \overline{\boldsymbol{x}}}_{d}) = \mathcal{F}(\underbrace{\overline{\boldsymbol{x}}, \dots, \overline{\boldsymbol{x}}}_{d}, \underbrace{\boldsymbol{x}, \dots, \boldsymbol{x}}_{d-1}, \bullet),$$

where the last equality is due to the conjugate partial-symmetricity of $\mathcal{F}$. Finally, the converse can be proven similarly. $\square$

One important property of the Z-eigenvalues for real symmetric tensors is that they can be fully characterized by the KKT solutions of a certain optimization problem [18, 21]. At a first glance, this property may not hold for C-eigenvalues and G-eigenvalues since the real-valued complex functions are not analytic. Therefore, direct extension of the KKT condition of an optimization problem with such objective function may not be valid. However, this class of functions is indeed analytic if we treat the complex variables and their conjugates as a whole due to the so-called Wirtinger calculus [24] in German literature, developed in the early 20th century. In the optimization context, without noticing the Wirtinger calculus, Brandwood [4] first proposed the notion of complex gradient. In particular, the gradient of a real-valued complex function can be taken as $\left(\frac{\partial}{\partial \boldsymbol{x}}, \frac{\partial}{\partial \overline{\boldsymbol{x}}}\right)$. Interested readers are referred to [27] for more discussions on the Wirtinger calculus in optimization with complex variables.

With the help of Wirtinger calculus, we are able to characterize C-eigenvalues and C-eigenvectors in terms of the KKT solutions. Therefore many optimization techniques can be applied to find the C-eigenvalues/eigenvectors for a conjugate partial-symmetric tensor.

Proposition 5.6 *$\boldsymbol{x} \in \mathbb{C}^n$ is a C-eigenvector associated with a C-eigenvalue $\lambda \in \mathbb{R}$ for a conjugate partial-symmetric tensor $\mathcal{F}$ if and only if $\boldsymbol{x}$ is a KKT point of the optimization problem*

$$\max_{\boldsymbol{x}^{\mathrm{H}}\boldsymbol{x}=1} \mathcal{F}(\underbrace{\overline{\boldsymbol{x}}, \dots, \overline{\boldsymbol{x}}}_{d}, \underbrace{\boldsymbol{x}, \dots, \boldsymbol{x}}_{d})$$

with Lagrange multiplier being $d\lambda$ and the corresponding objective value being λ.

Proof. Denote μ to be the Lagrange multiplier associated with the constraint $\boldsymbol{x}^{\mathrm{H}}\boldsymbol{x} = 1$. The KKT condition gives rise to the equations

$$\begin{cases} d\mathcal{F}(\cdot, \underbrace{\overline{\boldsymbol{x}}, \dots, \overline{\boldsymbol{x}}}_{d-1}, \underbrace{\boldsymbol{x}, \dots, \boldsymbol{x}}_{d}) - \mu \boldsymbol{x} = 0 \\ d\mathcal{F}(\underbrace{\overline{\boldsymbol{x}}, \dots, \overline{\boldsymbol{x}}}_{d}, \cdot, \underbrace{\boldsymbol{x}, \dots, \boldsymbol{x}}_{d-1}) - \mu \overline{\boldsymbol{x}} = 0 \\ \boldsymbol{x}^{\mathrm{H}}\boldsymbol{x} = 1. \end{cases}$$

The conclusion follows immediately by comparing the above with (13) and (15). $\square$

Similarly, we have the following characterization.

Proposition 5.7 *$\boldsymbol{x} \in \mathbb{C}^n$ is a G-eigenvector associated with a G-eigenvalue $\lambda \in \mathbb{R}$ for a conjugate super-symmetric tensor $\mathcal{F}$ if and only if $\boldsymbol{x}$ is a KKT point of the optimization problem*

$$\max_{\boldsymbol{x}^{\mathrm{H}}\boldsymbol{x}=1} \mathcal{F}\Bigg(\underbrace{\begin{pmatrix}\overline{\boldsymbol{x}} \\ \boldsymbol{x}\end{pmatrix}, \dots, \begin{pmatrix}\overline{\boldsymbol{x}} \\ \boldsymbol{x}\end{pmatrix}}_{d}\Bigg)$$

with Lagrange multiplier being $d\lambda$ and the corresponding objective value being λ.

5.2 Eigenvalues of complex tensors and their relations

Although the definitions of the C-eigenvalue, the G-eigenvalue, and the previously defined Q-eigenvalue involve different tensor spaces, they are indeed closely related. Our main result in this section essentially states that the Q-eigenvalue is a special case of the C-eigenvalue, and the C-eigenvalue is a special case of the G-eigenvalue.

Theorem 5.8 *Denote $\mathcal{H} \in \mathbb{C}^{n^d}$ to be a complex tensor and define $\mathcal{F} = \overline{\mathcal{H}} \otimes \mathcal{H} \in \mathbb{C}^{n^{2d}}$. It holds that*
(i) $\mathcal{H}$ is symmetric if and only if $\mathcal{F}$ is conjugate partial-symmetric;
(ii) All the C-eigenvalues of $\mathcal{F}$ are nonnegative;
(iii) λ^2 is a C-eigenvalue of $\mathcal{F}$ if and only if λ is a Q-eigenvalue of $\mathcal{H}$.

Proof. (i) This equivalence can be easily verified by the definition of conjugate partial-symmetricity (Definition 3.7).

(ii) Let $\boldsymbol{x} \in \mathbb{C}^n$ be a C-eigenvector associated with a C-eigenvalue $\lambda \in \mathbb{R}$ of $\mathcal{F}$. By multiplying $\boldsymbol{x}$ on both sides of the first equation in (13), we obtain

$$\begin{aligned}
\lambda &= \mathcal{F}(\underbrace{\overline{\boldsymbol{x}}, \dots, \overline{\boldsymbol{x}}}_{d}, \underbrace{\boldsymbol{x}, \dots, \boldsymbol{x}}_{d}) \\
&= (\overline{\mathcal{H}} \otimes \mathcal{H})(\underbrace{\overline{\boldsymbol{x}}, \dots, \overline{\boldsymbol{x}}}_{d}, \underbrace{\boldsymbol{x}, \dots, \boldsymbol{x}}_{d}) \\
&= \overline{\mathcal{H}}(\underbrace{\overline{\boldsymbol{x}}, \dots, \overline{\boldsymbol{x}}}_{d}) \cdot \mathcal{H}(\underbrace{\boldsymbol{x}, \dots, \boldsymbol{x}}_{d}) \\
&= \overline{\mathcal{H}(\underbrace{\boldsymbol{x}, \dots, \boldsymbol{x}}_{d})} \cdot \mathcal{H}(\underbrace{\boldsymbol{x}, \dots, \boldsymbol{x}}_{d}) \\
&= |\mathcal{H}(\underbrace{\boldsymbol{x}, \dots, \boldsymbol{x}}_{d})|^2 \geq 0.
\end{aligned}$$

(iii) Suppose $\boldsymbol{x} \in \mathbb{C}^n$ is a Q-eigenvector associated with a Q-eigenvalue $\lambda \in \mathbb{R}$ of $\mathcal{H}$. By the definition (12) we have $\boldsymbol{x}^{\mathrm{H}}\boldsymbol{x} = 1$ and $\mathcal{H}(\bullet, \underbrace{\boldsymbol{x}, \dots, \boldsymbol{x}}_{d-1}) = \lambda \overline{\boldsymbol{x}}$, and so

$$\mathcal{H}(\underbrace{\boldsymbol{x}, \dots, \boldsymbol{x}}_{d}) = \lambda \boldsymbol{x}^{\mathrm{T}} \overline{\boldsymbol{x}} = \lambda.$$

By the similar derivation in the proof of (ii), we get

$$\mathcal{F}(\bullet, \underbrace{\overline{\boldsymbol{x}}, \dots, \overline{\boldsymbol{x}}}_{d-1}, \underbrace{\boldsymbol{x}, \dots, \boldsymbol{x}}_{d}) = \overline{\mathcal{H}}(\bullet, \underbrace{\overline{\boldsymbol{x}}, \dots, \overline{\boldsymbol{x}}}_{d-1}) \cdot \mathcal{H}(\underbrace{\boldsymbol{x}, \dots, \boldsymbol{x}}_{d}) = \overline{\mathcal{H}(\bullet, \underbrace{\boldsymbol{x}, \dots, \boldsymbol{x}}_{d-1})} \cdot \lambda = \overline{\lambda \overline{\boldsymbol{x}}} \cdot \lambda = \lambda^2 \boldsymbol{x},$$

implying that λ^2 is a C-eigenvalue of $\mathcal{F}$.

On the other hand, suppose $\boldsymbol{x} \in \mathbb{C}^n$ is a C-eigenvector associated with a nonnegative C-eigenvalue λ^2 of $\mathcal{F}$. Then by (15) we have $\boldsymbol{x}^{\mathrm{H}}\boldsymbol{x} = 1$ and

$$\overline{\mathcal{H}(\underbrace{\boldsymbol{x}, \dots, \boldsymbol{x}}_{d})} \cdot \mathcal{H}(\bullet, \underbrace{\boldsymbol{x}, \dots, \boldsymbol{x}}_{d-1}) = \overline{\mathcal{H}}(\underbrace{\overline{\boldsymbol{x}}, \dots, \overline{\boldsymbol{x}}}_{d}) \cdot \mathcal{H}(\underbrace{\boldsymbol{x}, \dots, \boldsymbol{x}}_{d-1}, \bullet) = \mathcal{F}(\underbrace{\overline{\boldsymbol{x}}, \dots, \overline{\boldsymbol{x}}}_{d}, \underbrace{\boldsymbol{x}, \dots, \boldsymbol{x}}_{d-1}, \bullet) = \lambda^2 \overline{\boldsymbol{x}}, \quad (16)$$

where the first equality is due to the symmetricity of $\mathcal{H}$. This leads to $|\mathcal{H}(\underbrace{\boldsymbol{x},\ldots,\boldsymbol{x}}_{d})|^2=\lambda^2$. Let $\mathcal{H}(\underbrace{\boldsymbol{x},\ldots,\boldsymbol{x}}_{d})=\lambda e^{\mathbf{i}\theta}$ with $\theta\in[0,2\pi)$, and further define $\boldsymbol{y}=\boldsymbol{x}e^{-\mathbf{i}\theta/d}$. We then get

$$\mathcal{H}(\underbrace{\boldsymbol{y},\ldots,\boldsymbol{y}}_{d})=\mathcal{H}(\underbrace{\boldsymbol{x}e^{-\mathbf{i}\theta/d},\ldots,\boldsymbol{x}e^{-\mathbf{i}\theta/d}}_{d})=(e^{-\mathbf{i}\theta/d})^d\mathcal{H}(\underbrace{\boldsymbol{x},\ldots,\boldsymbol{x}}_{d})=e^{-\mathbf{i}\theta}\lambda e^{\mathbf{i}\theta}=\lambda.$$

Now we are able to verify that $\boldsymbol{y}$ is a Q-eigenvector associated with Q-eigenvalue λ of $\mathcal{H}$. First $\boldsymbol{y}^{\mathrm{H}}\boldsymbol{y}=(\boldsymbol{x}e^{-\mathbf{i}\theta/d})^{\mathrm{H}}\boldsymbol{x}e^{-\mathbf{i}\theta/d}=1$, and second by (16)

$$\lambda^2\overline{\boldsymbol{x}}=\overline{\mathcal{H}(\underbrace{\boldsymbol{x},\ldots,\boldsymbol{x}}_{d})}\cdot\mathcal{H}(\bullet,\underbrace{\boldsymbol{x},\ldots,\boldsymbol{x}}_{d-1})=\overline{\lambda e^{\mathbf{i}\theta}}\mathcal{H}(\bullet,\underbrace{\boldsymbol{y}e^{\mathbf{i}\theta/d},\ldots,\boldsymbol{y}e^{\mathbf{i}\theta/d}}_{d-1})=\lambda e^{-\mathbf{i}\theta}(e^{\mathbf{i}\theta/d})^{d-1}\mathcal{H}(\bullet,\underbrace{\boldsymbol{y},\ldots,\boldsymbol{y}}_{d-1}),$$

we finally get

$$\mathcal{H}(\bullet,\underbrace{\boldsymbol{y},\ldots,\boldsymbol{y}}_{d-1})=\lambda\overline{\boldsymbol{x}}e^{\mathbf{i}\theta/d}=\lambda\overline{\boldsymbol{y}e^{\mathbf{i}\theta/d}}e^{\mathbf{i}\theta/d}=\lambda\overline{\boldsymbol{y}}.$$

□

In Section 2, by definition a symmetric conjugate form is a special general conjugate form. Hence in terms of their tensor representations, a conjugate partial-symmetric tensor is a special case of conjugate super-symmetric tensor, although they live in different tensor spaces. To study the relationship between the C-eigenvalues and the G-eigenvalues, let us introduce an embedded conjugate partial-symmetric tensor $\mathcal{F}\in\mathbb{C}^{n^{2d}}$ to the space of $\mathbb{C}^{(2n)^{2d}}$. The conjugate super-symmetric tensor $\mathcal{G}\in\mathbb{C}^{(2n)^{2d}}$ corresponding to $\mathcal{F}$ is then defined by

$$\mathcal{G}_{j_1\ldots j_{2d}}=\begin{cases}\mathcal{F}_{i_1\ldots i_{2d}}/\binom{2d}{d} & (j_1\ldots j_{2d})\in\Pi(i_1,\ldots,i_d,i_{d+1}+n,\ldots,i_{2d}+n)\\ 0 & \text{otherwise.}\end{cases}\tag{17}$$

For example when $d=1$, a conjugate partial-symmetric tensor is simply a Hermitian matrix $A\in\mathbb{C}^{n^2}$. Then its embedded conjugate super-symmetric tensor is $\begin{pmatrix} O & A/2\\ A^{\mathrm{T}}/2 & O\end{pmatrix}\in\mathbb{C}^{(2n)^2}$, and clearly we have

$$\overline{\boldsymbol{x}}^{\mathrm{T}}A\boldsymbol{x}=\begin{pmatrix}\overline{\boldsymbol{x}}\\ \boldsymbol{x}\end{pmatrix}^{\mathrm{T}}\begin{pmatrix} O & A/2\\ A^{\mathrm{T}}/2 & O\end{pmatrix}\begin{pmatrix}\overline{\boldsymbol{x}}\\ \boldsymbol{x}\end{pmatrix}.$$

In general it is straightforward to verify that $\mathcal{F}(\underbrace{\overline{\boldsymbol{x}},\ldots,\overline{\boldsymbol{x}}}_{d},\underbrace{\boldsymbol{x},\ldots,\boldsymbol{x}}_{d})=\mathcal{G}\Bigg(\underbrace{\begin{pmatrix}\overline{\boldsymbol{x}}\\ \boldsymbol{x}\end{pmatrix},\ldots,\begin{pmatrix}\overline{\boldsymbol{x}}\\ \boldsymbol{x}\end{pmatrix}}_{2d}\Bigg)$. Based on this, we are led to the following relationship between the C-eigenvalues and the G-eigenvalues, whose proof is similar to that of Theorem 5.8.

Theorem 5.9 *If $\mathcal{G}\in\mathbb{C}^{(2n)^{2d}}$ is a conjugate super-symmetric tensor induced by a conjugate partial-symmetric tensor $\mathcal{F}\in\mathbb{C}^{n^{2d}}$ according to (17), then λ is a C-eigenvalue of $\mathcal{F}$ if and only if $\lambda/2$ is a G-eigenvalue of $\mathcal{G}$.*

6 Extending Banach's theorem to the real-valued conjugate forms

A classical result originally due to Banach [3] states that if $\mathcal{L}(\boldsymbol{x}^1,\ldots,\boldsymbol{x}^d)$ is a continuous symmetric d-linear form, then

$$\sup\{|\mathcal{L}(\boldsymbol{x}^1,\ldots,\boldsymbol{x}^d)| \mid \|\boldsymbol{x}^1\|\le 1,\ldots,\|\boldsymbol{x}^d\|\le 1\} = \sup\{|\mathcal{L}(\underbrace{\boldsymbol{x},\ldots,\boldsymbol{x}}_{d})| \mid \|\boldsymbol{x}\|\le 1\}. \tag{18}$$

In the real tensor setting, where $\boldsymbol{x}\in\mathbb{R}^n$ and $\mathcal{L}$ is a symmetric multilinear form associated with a real symmetric tensor $\mathcal{L}\in\mathbb{R}^{n^d}$, (18) states that the largest singular value [18] of $\mathcal{L}$ is equal to the largest absolute value of eigenvalue [21] of $\mathcal{L}$, i.e.,

$$\max_{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}=1,\,\boldsymbol{x}\in\mathbb{R}^n} |\mathcal{L}(\underbrace{\boldsymbol{x},\ldots,\boldsymbol{x}}_{d})| = \max_{(\boldsymbol{x}^i)^{\mathrm{T}}\boldsymbol{x}^i=1,\,\boldsymbol{x}^i\in\mathbb{R}^n,\,i=1,\ldots,d} \mathcal{L}(\boldsymbol{x}^1,\ldots,\boldsymbol{x}^d). \tag{19}$$

Alternatively, (19) is essentially equivalent to the fact that the best rank-one approximation of a real symmetric tensor can be obtained at a symmetric rank-one tensor [5, 32]. A recent development on this topic for special classes of real symmetric tensors can be found in [6]. In this section, we shall extend the Banach's theorem to the symmetric conjugate forms (the conjugate partial-symmetric tensors) and the general conjugate forms (the conjugate super-symmetric tensors).

6.1 Equivalence for conjugate super-symmetric tensors

Let us start with the conjugate super-symmetric tensors, which are a generalization of conjugate partial-symmetric tensors. A key result led to the equivalence (Theorem 6.2) is the following.

Lemma 6.1 *For a given real tensor $\mathcal{F}\in\mathbb{R}^{n^d}$, if $\mathcal{F}(\boldsymbol{x}^1,\ldots,\boldsymbol{x}^d)=\mathcal{F}(\boldsymbol{x}^{\pi(1)},\ldots,\boldsymbol{x}^{\pi(d)})$ for any $\boldsymbol{x}^1,\ldots,\boldsymbol{x}^d\in\mathbb{R}^n$ and any permutation π of $\{1,\ldots,d\}$, then $\mathcal{F}$ is symmetric.*

Proof. Denote $\boldsymbol{e}^i$ to be the vector whose i-th component is 1 and others are zeros for $i=1,\ldots,n$. For any permutation π of $\{i_1,\ldots,i_d\}$, we have

$$\mathcal{F}_{\pi(i_1)\ldots\pi(i_d)} = \mathcal{F}(\boldsymbol{e}^{\pi(i_1)},\ldots,\boldsymbol{e}^{\pi(i_d)}) = \mathcal{F}(\boldsymbol{e}^{i_1},\ldots,\boldsymbol{e}^{i_d}) = \mathcal{F}_{i_1\ldots i_d},$$

and the conclusion follows. □

Our first result in this section extends (19) to any conjugate super-symmetric tensors in the complex domain.

Theorem 6.2 *For any conjugate super-symmetric tensor $\mathcal{G}\in\mathbb{C}^{(2n)^d}$, we have*

$$\max_{\boldsymbol{x}^{\mathrm{H}}\boldsymbol{x}=1} \left|\mathcal{G}\Bigg(\underbrace{\begin{pmatrix}\overline{\boldsymbol{x}}\\ \boldsymbol{x}\end{pmatrix},\ldots,\begin{pmatrix}\overline{\boldsymbol{x}}\\ \boldsymbol{x}\end{pmatrix}}_{d}\Bigg)\right| = \max_{(\boldsymbol{x}^i)^{\mathrm{H}}\boldsymbol{x}^i=1,\,i=1,\ldots,d} \operatorname{Re}\mathcal{G}\Bigg(\begin{pmatrix}\overline{\boldsymbol{x}^1}\\ \boldsymbol{x}^1\end{pmatrix},\ldots,\begin{pmatrix}\overline{\boldsymbol{x}^d}\\ \boldsymbol{x}^d\end{pmatrix}\Bigg). \tag{20}$$

Proof. Let $\boldsymbol{y}^i=\begin{pmatrix}\operatorname{Re}\boldsymbol{x}^i\\ \operatorname{Im}\boldsymbol{x}^i\end{pmatrix}\in\mathbb{R}^{2n}$ for $i=1,\ldots,d$. We observe that $\operatorname{Re}\mathcal{G}\Bigg(\begin{pmatrix}\overline{\boldsymbol{x}^1}\\ \boldsymbol{x}^1\end{pmatrix},\ldots,\begin{pmatrix}\overline{\boldsymbol{x}^d}\\ \boldsymbol{x}^d\end{pmatrix}\Bigg)$ is also a multilinear form with respect to $\boldsymbol{y}^1,\ldots,\boldsymbol{y}^d$. As a result, we are able to find a real tensor $\mathcal{F}\in\mathbb{R}^{(2n)^d}$ such that

$$\mathcal{F}(\boldsymbol{y}^1,\ldots,\boldsymbol{y}^d) = \operatorname{Re}\mathcal{G}\Bigg(\begin{pmatrix}\overline{\boldsymbol{x}^1}\\ \boldsymbol{x}^1\end{pmatrix},\ldots,\begin{pmatrix}\overline{\boldsymbol{x}^d}\\ \boldsymbol{x}^d\end{pmatrix}\Bigg). \tag{21}$$

As $\mathcal{G}$ is conjugate super-symmetric, for any $\boldsymbol{y}^1, \dots, \boldsymbol{y}^d \in \mathbb{R}^{2n}$ and any permutation π of $\{1, \dots, d\}$, one has

$$\mathcal{F}(\boldsymbol{y}^1, \dots, \boldsymbol{y}^d) = \operatorname{Re} \mathcal{G}\left(\begin{pmatrix} \overline{\boldsymbol{x}^1} \\ \boldsymbol{x}^1 \end{pmatrix}, \dots, \begin{pmatrix} \overline{\boldsymbol{x}^d} \\ \boldsymbol{x}^d \end{pmatrix} \right) = \operatorname{Re} \mathcal{G}\left(\begin{pmatrix} \overline{\boldsymbol{x}^{\pi(1)}} \\ \boldsymbol{x}^{\pi(1)} \end{pmatrix}, \dots, \begin{pmatrix} \overline{\boldsymbol{x}^{\pi(d)}} \\ \boldsymbol{x}^{\pi(d)} \end{pmatrix} \right) = \mathcal{F}(\boldsymbol{y}^{\pi(1)}, \dots, \boldsymbol{y}^{\pi(d)}).$$

By Lemma 6.1 we have that the real tensor $\mathcal{F}$ is symmetric. Finally, noticing that $(\boldsymbol{y}^i)^{\mathrm{T}} \boldsymbol{y}^i = (\boldsymbol{x}^i)^{\mathrm{H}} \boldsymbol{x}^i$ for $i = 1, \dots, d$, the conclusion follows immediately by applying (19) to $\mathcal{F}$ and then using the equality (21). $\square$

6.2 Equivalence for conjugate partial-symmetric tensors

For a conjugate partial-symmetric tensor $\mathcal{F} \in \mathbb{C}^{n^{2d}}$, as it is a special case of conjugate super-symmetric tensors, one can certainly embed $\mathcal{F}$ into a super-symmetric structure $\mathcal{G} \in \mathbb{C}^{(2n)^{2d}}$ using (17). By applying Theorem 6.2 to $\mathcal{G}$ and rewrite its associated form in terms of $\mathcal{F}$ we get an equivalent expression as (20). However, this expression is not succinct. Taking the case $d = 2$ (degree 4) for example, this would lead to

$$\max_{\boldsymbol{x}^{\mathrm{H}} \boldsymbol{x} = 1} |\mathcal{F}(\overline{\boldsymbol{x}}, \overline{\boldsymbol{x}}, \boldsymbol{x}, \boldsymbol{x})| = \max_{(\boldsymbol{x}^i)^{\mathrm{H}} \boldsymbol{x}^i = 1,\, i=1,2,3,4} f(\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3, \boldsymbol{x}^4),$$

where

$$\begin{aligned} f(\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3, \boldsymbol{x}^4) := \frac{1}{6} \Big(& \mathcal{F}(\overline{\boldsymbol{x}^1}, \overline{\boldsymbol{x}^2}, \boldsymbol{x}^3, \boldsymbol{x}^4) + \mathcal{F}(\overline{\boldsymbol{x}^1}, \overline{\boldsymbol{x}^3}, \boldsymbol{x}^2, \boldsymbol{x}^4) + \mathcal{F}(\overline{\boldsymbol{x}^1}, \overline{\boldsymbol{x}^4}, \boldsymbol{x}^2, \boldsymbol{x}^3) \\ & + \mathcal{F}(\overline{\boldsymbol{x}^2}, \overline{\boldsymbol{x}^3}, \boldsymbol{x}^1, \boldsymbol{x}^4) + \mathcal{F}(\overline{\boldsymbol{x}^2}, \overline{\boldsymbol{x}^4}, \boldsymbol{x}^1, \boldsymbol{x}^3) + \mathcal{F}(\overline{\boldsymbol{x}^3}, \overline{\boldsymbol{x}^4}, \boldsymbol{x}^1, \boldsymbol{x}^2) \Big). \end{aligned}$$

Instead, one would hope to get

$$\max_{\boldsymbol{x}^{\mathrm{H}} \boldsymbol{x} = 1} |\mathcal{F}(\underbrace{\overline{\boldsymbol{x}}, \dots, \overline{\boldsymbol{x}}}_{d}, \underbrace{\boldsymbol{x}, \dots, \boldsymbol{x}}_{d})| = \max_{(\boldsymbol{x}^i)^{\mathrm{H}} \boldsymbol{x}^i = 1,\, i=1,\dots,2d} \operatorname{Re} \mathcal{F}(\overline{\boldsymbol{x}^1}, \dots, \overline{\boldsymbol{x}^d}, \boldsymbol{x}^{d+1}, \dots, \boldsymbol{x}^{2d}). \tag{22}$$

However, this does not hold in general. The main reason is that

$$\mathcal{G}\left(\begin{pmatrix} \overline{\boldsymbol{x}^1} \\ \boldsymbol{x}^1 \end{pmatrix}, \dots, \begin{pmatrix} \overline{\boldsymbol{x}^{2d}} \\ \boldsymbol{x}^{2d} \end{pmatrix} \right) \neq \mathcal{F}(\overline{\boldsymbol{x}^1}, \dots, \overline{\boldsymbol{x}^d}, \boldsymbol{x}^{d+1}, \dots, \boldsymbol{x}^{2d}),$$

which is easily observed since its left hand side is invariant under the permutation of $(\boldsymbol{x}^1, \dots, \boldsymbol{x}^{2d})$ while its right hand side is not. In particular, (22) only holds for $d = 1$, i.e, Hermitian matrices; see the following result and Example 6.4.

Proposition 6.3 *For any Hermitian matrix $Q \in \mathbb{C}^{n \times n}$, it holds that*

$$(L) \quad \max_{\boldsymbol{z}^{\mathrm{H}} \boldsymbol{z} = 1} |\boldsymbol{z}^{\mathrm{H}} Q \boldsymbol{z}| = \max_{\boldsymbol{x}^{\mathrm{H}} \boldsymbol{x} = \boldsymbol{y}^{\mathrm{H}} \boldsymbol{y} = 1} \operatorname{Re} \boldsymbol{x}^{\mathrm{T}} Q \boldsymbol{y}. \quad (R)$$

Furthermore, for any optimal solution $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ of (R) with $\overline{\boldsymbol{x}^*} + \boldsymbol{y}^* \neq 0$, $(\overline{\boldsymbol{x}^*} + \boldsymbol{y}^*)/\|\overline{\boldsymbol{x}^*} + \boldsymbol{y}^*\|$ is an optimal solution of (L) as well.

Proof. Denote $v(L)$ and $v(R)$ to be the optimal values of (L) and (R), respectively. Noticing that $\mathrm{Re}\,\boldsymbol{x}^{\mathrm{T}}Q\boldsymbol{y} = \frac{1}{2}(\boldsymbol{x}^{\mathrm{T}}Q\boldsymbol{y} + \overline{\boldsymbol{x}}^{\mathrm{T}}\overline{Q}\overline{\boldsymbol{y}})$, by the optimality condition of (R) we have that

$$\begin{cases} Q\boldsymbol{y}^* - 2\lambda\overline{\boldsymbol{x}^*} = 0 \\ \overline{Q}\overline{\boldsymbol{y}^*} - 2\lambda\boldsymbol{x}^* = 0 \\ \overline{Q}\boldsymbol{x}^* - 2\mu\overline{\boldsymbol{y}^*} = 0 \\ Q\overline{\boldsymbol{x}^*} - 2\mu\boldsymbol{y}^* = 0 \\ (\boldsymbol{x}^*)^{\mathrm{H}}\boldsymbol{x}^* = 1 \\ (\boldsymbol{y}^*)^{\mathrm{H}}\boldsymbol{y}^* = 1, \end{cases} \tag{23}$$

where λ and μ are the lagrangian multipliers of the constraints $\boldsymbol{x}^{\mathrm{H}}\boldsymbol{x} = 1$ and $\boldsymbol{y}^{\mathrm{H}}\boldsymbol{y} = 1$, respectively.

The summation of the first two equations in (23) leads to

$$2\mathrm{Re}\,(\boldsymbol{x}^*)^{\mathrm{T}}Q\boldsymbol{y}^* = (\boldsymbol{x}^*)^{\mathrm{T}}Q\boldsymbol{y}^* + (\overline{\boldsymbol{x}^*})^{\mathrm{T}}\overline{Q}\overline{\boldsymbol{y}^*} = 2\lambda(\boldsymbol{x}^*)^{\mathrm{T}}\overline{\boldsymbol{x}^*} + 2\lambda(\overline{\boldsymbol{x}^*})^{\mathrm{T}}\boldsymbol{x}^* = 4\lambda(\boldsymbol{x}^*)^{\mathrm{H}}\boldsymbol{x}^* = 4\lambda.$$

Similarly, the summation of the third and fourth equations in (23) leads to

$$2\mathrm{Re}\,(\boldsymbol{x}^*)^{\mathrm{T}}Q\boldsymbol{y}^* = 4\mu,$$

which further leads to

$$v(R) = \mathrm{Re}\,(\boldsymbol{x}^*)^{\mathrm{T}}Q\boldsymbol{y}^* = 2\lambda = 2\mu. \tag{24}$$

Moreover, the summation of the first and third equations in (23) yields

$$Q(\boldsymbol{y}^* + \overline{\boldsymbol{x}^*}) - 2\lambda(\overline{\boldsymbol{x}^*} + \boldsymbol{y}^*) = 0,$$

which further leads to

$$(\boldsymbol{y}^* + \overline{\boldsymbol{x}^*})^{\mathrm{H}}Q(\boldsymbol{y}^* + \overline{\boldsymbol{x}^*}) = 2\lambda(\boldsymbol{y}^* + \overline{\boldsymbol{x}^*})^{\mathrm{H}}(\overline{\boldsymbol{x}^*} + \boldsymbol{y}^*) = 2\lambda\|\overline{\boldsymbol{x}^*} + \boldsymbol{y}^*\|^2.$$

Let $\boldsymbol{z}^* = (\overline{\boldsymbol{x}^*} + \boldsymbol{y}^*)/\|\overline{\boldsymbol{x}^*} + \boldsymbol{y}^*\|$. Clearly $\boldsymbol{z}^*$ is a feasible solution of (L). By (24) we have that

$$(\boldsymbol{z}^*)^{\mathrm{H}}Q\boldsymbol{z}^* = 2\lambda = \mathrm{Re}\,(\boldsymbol{x}^*)^{\mathrm{T}}Q\boldsymbol{y}^* = v(R).$$

This implies that $v(L) \geq v(R)$. Notice that (R) is a relaxation of (L) and hence $v(L) \leq v(R)$. Therefore we conclude that $v(R) = v(L)$, and an optimal solution $\boldsymbol{z}^*$ of (L) is constructed from an optimal solution $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ of (R). □

Example 6.4 *Let $\mathcal{F} \in \mathbb{C}^{2^4}$ with $\mathcal{F}_{1122} = \mathcal{F}_{2211} = 1$ and other entries being zeros. Clearly $\mathcal{F}$ is conjugate partial-symmetric. We have (22) fail to hold since:*

- $|\mathcal{F}(\overline{\boldsymbol{x}}, \overline{\boldsymbol{x}}, \boldsymbol{x}, \boldsymbol{x})| = |\overline{x_1}^2 x_2{}^2 + \overline{x_2}^2 x_1{}^2| \leq 2|x_1|^2|x_2|^2 \leq \frac{1}{2}(|x_1|^2 + |x_2|^2)^2 = \frac{1}{2}$ *for any $\boldsymbol{x} \in \mathbb{C}^2$ with $\boldsymbol{x}^{\mathrm{H}}\boldsymbol{x} = 1$.*
- $\mathcal{F}(\overline{\boldsymbol{x}}, \overline{\boldsymbol{y}}, \boldsymbol{z}, \boldsymbol{w}) = \overline{x_1 y_1} z_2 w_2 + \overline{x_2 y_2} z_1 w_1 = 1$ *for $\boldsymbol{x} = \boldsymbol{y} = (1, 0)^{\mathrm{T}}$ and $\boldsymbol{z} = \boldsymbol{w} = (0, 1)^{\mathrm{T}}$.*

Although (22) does not hold, we have a relaxed version of the general equivalence result for the conjugate partial-symmetric tensors. By Lemma 3.9, $\mathcal{F}(\overline{\boldsymbol{x}^1}, \ldots, \overline{\boldsymbol{x}^d}, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^d)$ always takes real values, and we have the following result.

Theorem 6.5 *For any conjugate partial-symmetric tensor $\mathcal{F} \in \mathbb{C}^{n^{2d}}$, we have*

$$\max_{\boldsymbol{x}^{\mathrm{H}}\boldsymbol{x}=1} \mathcal{F}(\underbrace{\overline{\boldsymbol{x}}, \ldots, \overline{\boldsymbol{x}}}_{d}, \underbrace{\boldsymbol{x}, \ldots, \boldsymbol{x}}_{d}) = \max_{(\boldsymbol{x}^i)^{\mathrm{H}}\boldsymbol{x}^i=1,\, i=1,\ldots,d} \mathcal{F}(\overline{\boldsymbol{x}^1}, \ldots, \overline{\boldsymbol{x}^d}, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^d)$$

To prove this theorem, one cannot directly apply any existing result of Banach's type, since $\mathcal{F}(\overline{\boldsymbol{x}^1},\dots,\overline{\boldsymbol{x}^d},\boldsymbol{x}^1,\dots,\boldsymbol{x}^d)$ is not even a multilinear form; rather, it is a multi-quadratic form. However, it is straightforward to apply the same technique in proving Banach's result for the real tensor case (e.g. Theorem 4.1 in [5]) to prove Theorem 6.5. We leave it to the interested readers. One key step in this proof is the following result, in the same vein as Proposition 6.3.

Proposition 6.6 *For any symmetric complex matrix $Q \in \mathbb{C}^{n^2}$, i.e., $Q^{\mathrm{T}} = Q$, it holds that*

$$(L') \qquad \max_{\boldsymbol{z}^{\mathrm{H}}\boldsymbol{z}=1} \operatorname{Re} \boldsymbol{z}^{\mathrm{T}} Q \boldsymbol{z} = \max_{\boldsymbol{x}^{\mathrm{H}}\boldsymbol{x}=\boldsymbol{y}^{\mathrm{H}}\boldsymbol{y}=1} \operatorname{Re} \boldsymbol{x}^{\mathrm{T}} Q \boldsymbol{y}. \quad (R')$$

Furthermore, for any optimal solution $(\boldsymbol{x}^*,\boldsymbol{y}^*)$ of (R') with $\boldsymbol{x}^* \pm \boldsymbol{y}^* \neq 0$, $(\boldsymbol{x}^* \pm \boldsymbol{y}^*)/\|\boldsymbol{x}^* \pm \boldsymbol{y}^*\|$ is an optimal solution of (L') as well.

Since the proof can be constructed almost identically to that of Proposition 6.3, we omit the details here.

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